Maxwell

A TREATISE ON ELECTRICITY & MAGNETISM

Vol.1

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# James Clerk Maxwell

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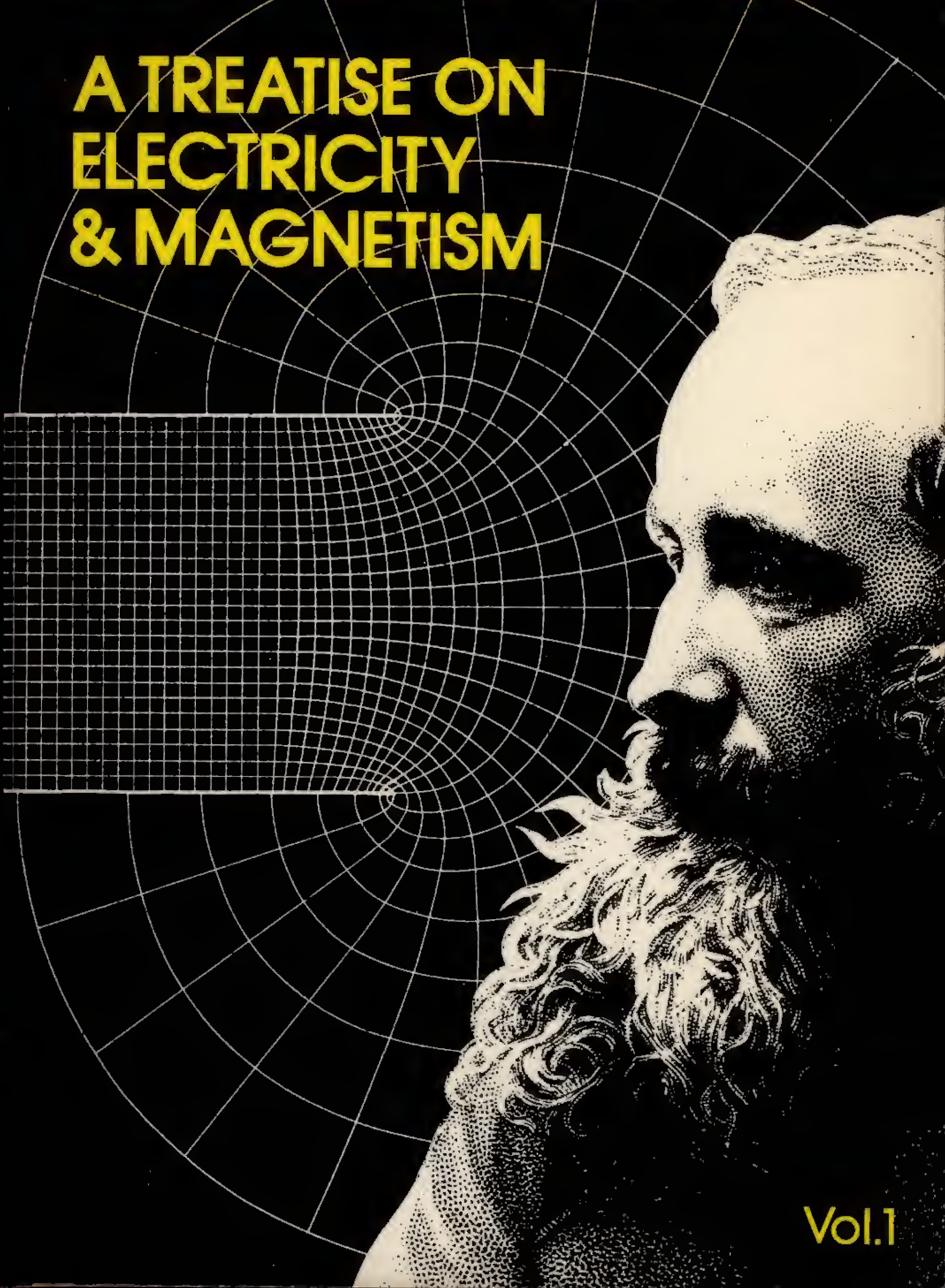

Vol.1

A TREATISE ON

# ELECTRICITY AND MAGNETISM

By

JAMES CLERK MAXWELL

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THIRD EDITION

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# PREFACE TO THE FIRST EDITION

THE fact that certain bodies, after being rubbed, appear to attract other bodies, was known to the ancients. In modern times, a great variety of other phenomena have been observed, and have been found to be related to these phenomena of attraction. They have been classed under the name of *Electric* phenomena, amber, ἤλεκτρον, having been the substance in which they were first described.

Other bodies, particularly the loadstone, and pieces of iron and steel which have been subjected to certain processes, have also been long known to exhibit phenomena of action at a distance. These phenomena, with others related to them, were found to differ from the electric phenomena, and have been classed under the name of *Magnetic* phenomena, the loadstone, μάγνης, being found in the Thessalian Magnesia.

These two classes of phenomena have since been found to be related to each other, and the relations between the various phenomena of both classes, so far as they are known, constitute the science of Electromagnetism.

In the following Treatise I propose to describe the most important of these phenomena, to shew how they may be subjected to measurement, and to trace the mathematical connexions of the quantities measured. Having thus obtained the data for a mathematical theory of electromagnetism, and having shewn how this theory may be applied to the calculation of phenomena, I shall endeavour to place in as clear a light as I can the relations between the mathematical form of this theory and that of the fundamental science of Dynamics, in order that we may be in some degree prepared to determine the kind of dynamical phenomena among which we are to look for illustrations or explanations of the electromagnetic phenomena.

In describing the phenomena, I shall select those which most clearly illustrate the fundamental ideas of the theory, omitting others, or reserving them till the reader is more advanced.

The most important aspect of any phenomenon from a mathematical point of view is that of a measurable quantity. I shall therefore consider electrical phenomena chiefly with a view to their measurement, describing the methods of measurement, and defining the standards on which they depend.

In the application of mathematics to the calculation of electrical quantities, I shall endeavour in the first place to deduce the most general conclusions from the data at our disposal, and in the next place to apply the results to the simplest cases that can be chosen. I shall avoid, as much as I can, those questions which, though they have elicited the skill of mathematicians, have not enlarged our knowledge of science.

The internal relations of the different branches of the science which we have to study are more numerous and complex than those of any other science hitherto developed. Its external relations, on the one hand to dynamics, and on the other to heat, light, chemical action, and the constitution of bodies, seem to indicate the special importance of electrical science as an aid to the interpretation of nature.

It appears to me, therefore, that the study of electromagnetism in all its extent has now become of the first importance as a means of promoting the progress of science.

The mathematical laws of the different classes of phenomena have been to a great extent satisfactorily made out.

The connexions between the different classes of phenomena have also been investigated, and the probability of the rigorous exactness of the experimental laws have been greatly strengthened by a more extended knowledge of their relations to each other.

Finally, some progress has been made in the reduction of electromagnetism to a dynamical science, by shewing that no electromagnetic phenomenon is contradictory to the supposition that it depends on purely dynamical action.

What has been hitherto done, however, has by no means exhausted the field of electrical research. It has rather opened up that field, by pointing out subjects of enquiry, and furnishing us with means of investigation.

It is hardly necessary to enlarge upon the beneficial results of magnetic research on navigation, and the importance of a knowledge of the true direction of the compass, and of the effect of the iron in a ship. But the labours of those who have endeavoured to render navigation more secure by means of magnetic observations have at the same time greatly advanced the progress of pure science.

Gauss, as a member of the German Magnetic Union, brought his powerful intellect to bear on the theory of magnetism, and on the methods of observing it, and he not only added greatly to our knowledge of the theory of attractions, but reconstructed the whole of magnetic science as regards the instruments used, the methods of observation, and the calculation of the results, so that his memoirs on Terrestrial Magnetism may be taken as models of physical research by all those who are engaged in the measurement of any of the forces in nature.

The important applications of electromagnetism to telegraphy have also reacted on pure science by giving a commercial value to accurate electrical measurements, and by affording to electricians the use of apparatus on a scale which greatly transcends that of any ordinary laboratory. The consequences of this demand for electrical knowledge, and of these experimental opportunities for acquiring it, have been already very great, both in stimulating the energies of advanced electricians, and in diffusing among practical men a degree of accurate knowledge which is likely to conduce to the general scientific progress of the whole engineering profession.

There are several treatises in which electrical and magnetic phenomena are described in a popular way. These, however, are not what is wanted by those who have been brought face to face with quantities to be measured, and whose minds do not rest satisfied with lecture-room experiments.

There is also a considerable mass of mathematical memoirs which are of great importance in electrical science, but they lie concealed in the bulky Transactions of learned societies; they do not form a connected system; they are of very unequal merit, and they are for the most part beyond the comprehension of any but professed mathematicians.

I have therefore thought that a treatise would be useful which should have for its principal object to take up the whole subject in a methodical manner, and which should also indicate how each part of the subject is brought within the reach of methods of verification by actual measurement.

The general complexion of the treatise differs considerably from that of several excellent electrical works, published, most of them, in Germany, and it may appear that scant justice is done to the speculations of several eminent electricians and mathematicians. One reason of this is that before I began the study of electricity I resolved to read no mathematics on the subject till I had first read through Faraday's *Experimental Researches in Electricity*. I was aware that there was supposed to be a difference between Faraday's way of conceiving phenomena and that of the mathematicians, so that neither he nor they were satisfied with each other's language. I had also the conviction that this discrepancy did not arise from either party being wrong. I was first convinced of this by Sir William Thomson *, to whose advice and assistance, as

* I take this opportunity of acknowledging my obligations to Sir W. Thomson and to Professor Tait for many valuable suggestions made during the printing of this work.

well as to his published papers, I owe most of what I have learned on the subject.

As I proceeded with the study of Faraday, I perceived that his method of conceiving the phenomena was also a mathematical one, though not exhibited in the conventional form of mathematical symbols. I also found that these methods were capable of being expressed in the ordinary mathematical forms, and thus compared with those of the professed mathematicians.

For instance, Faraday, in his mind's eye, saw lines of force traversing all space where the mathematicians saw centres of force attracting at a distance: Faraday saw a medium where they saw nothing but distance: Faraday sought the seat of the phenomena in real actions going on in the medium, they were satisfied that they had found it in a power of action at a distance impressed on the electric fluids.

When I had translated what I considered to be Faraday's ideas into a mathematical form, I found that in general the results of the two methods coincided, so that the same phenomena were accounted for, and the same laws of action deduced by both methods, but that Faraday's methods resembled those in which we begin with the whole and arrive at the parts by analysis, while the ordinary mathematical methods were founded on the principle of beginning with the parts and building up the whole by synthesis.

I also found that several of the most fertile methods of research discovered by the mathematicians could be expressed much better in terms of ideas derived from Faraday than in their original form.

The whole theory, for instance, of the potential, considered as a quantity which satisfies a certain partial differential equation, belongs essentially to the method which I have called that of Faraday. According to the other method, the potential, if it is to be considered at all, must be regarded as the result of a summation of the electrified particles divided each by its distance from a given point. Hence many of the mathematical discoveries of Laplace, Poisson, Green and Gauss find their proper place in this treatise, and their appropriate expressions in terms of conceptions mainly derived from Faraday.

Great progress has been made in electrical science, chiefly in Germany, by cultivators of the theory of action at a distance. The valuable electrical measurements of W. Weber are interpreted by him according to this theory, and the electromagnetic speculation which was originated by Gauss, and carried on by Weber, Riemann, J. and C. Neumann, Lorenz, &c., is founded on the theory of action at a distance, but depending either directly on the relative velocity of the particles, or on the gradual propagation of something, whether potential or force, from the one particle to the other. The great success which these eminent men have attained in the application of mathematics to electrical phenomena, gives, as is natural, additional weight to their theoretical speculations, so that those who, as students of electricity, turn to them as the greatest authorities in mathematical electricity, would probably imbibe, along with their mathematical methods, their physical hypotheses.

These physical hypotheses, however, are entirely alien from the way of looking at things which I adopt, and one object which I have in view is that some of those who wish to study electricity may, by reading this treatise, come to see that there is another way of treating the subject, which is no less fitted to explain the phenomena, and which, though in some parts it may appear less definite, corresponds, as I think, more faithfully with our actual knowledge, both in what it affirms and in what it leaves undecided.

In a philosophical point of view, moreover, it is exceedingly important that two methods should be compared, both of which have succeeded in explaining the principal electromagnetic phenomena, and both of which have attempted to explain the propagation of light as an electromagnetic phenomenon and have actually calculated its velocity, while at the same time the fundamental conceptions of what actually takes place, as well as most of the secondary conceptions of the quantities concerned, are radically different.

I have therefore taken the part of an advocate rather than that of a judge, and have rather exemplified one method than attempted to give an impartial description of both. I have no doubt that the method which I have called the German one will also find its supporters, and will be expounded with a skill worthy of its ingenuity.

I have not attempted an exhaustive account of electrical phenomena, experiments, and apparatus. The student who desires to read all that is known on these subjects will find great assistance from the *Traité d'Electricité* of Professor A. de la Rive, and from several German treatises, such as Wiedemann's *Galvanismus*, Riess' *Reibungselektricität*, Beer's *Einleitung in die Elektrostatik*, &c.

I have confined myself almost entirely to the mathematical treatment of the subject, but I would recommend the student, after he has learned, experimentally if possible, what are the phenomena to be observed, to read carefully Faraday's *Experimental Researches in Electricity.* He will there find a strictly contemporary historical account of some of the greatest electrical discoveries and investigations, carried on in an order and succession which could hardly have been improved if the results had been known from the first, and expressed in the language of a man who devoted much of his attention to the methods of accurately describing scientific operations and their results *.

It is of great advantage to the student of any subject to read the original memoirs on that subject, for science is always most completely assimilated when it is in the nascent state, and in the case of Faraday's *Researches* this is comparatively easy, as they are published in a separate form, and may be read consecutively. If by anything I have here written I may assist any student in understanding Faraday's modes of thought and expression, I shall regard it as the accomplishment of one of my principal aims—to communicate to others the same delight which I have found myself in reading Faraday's *Researches.*

* *Life and Letters of Faraday*, vol. i. p. 395.

The description of the phenomena, and the elementary parts of the theory of each subject, will be found in the earlier chapters of each of the four Parts into which this treatise is divided. The student will find in these chapters enough to give him an elementary acquaintance with the whole science.

The remaining chapters of each Part are occupied with the higher parts of the theory, the processes of numerical calculation, and the instruments and methods of experimental research.

The relations between electromagnetic phenomena and those of radiation, the theory of molecular electric currents, and the results of speculation on the nature of action at a distance, are treated of in the last four chapters of the second volume.

JAMES CLERK MAXWELL.

*Feb.* 1, 1873.

# PREFACE TO THE SECOND EDITION

WHEN I was asked to read the proof-sheets of the second edition of the *Electricity and Magnetism* the work of printing had already reached the ninth chapter, the greater part of which had been revised by the author.

Those who are familiar with the first edition will see from a comparison with the present how extensive were the changes intended by Professor Maxwell both in the substance and in the treatment of the subject, and how much this edition has suffered from his premature death. The first nine chapters were in some cases entirely rewritten, much new matter being added and the former contents rearranged and simplified.

From the ninth chapter onwards the present edition is little more than a reprint. The only liberties I have taken have been in the insertion here and there of a step in the mathematical reasoning where it seemed to be an advantage to the reader and of a few foot-notes on parts of the subject which my own experience or that of pupils attending my classes shewed to require further elucidation. These foot-notes are in square brackets.

There were two parts of the subject in the treatment of which it was known to me that the Professor contemplated considerable changes: viz. the mathematical theory of the conduction of electricity in a network of wires, and the determination of coefficients of induction in coils of wire. In these subjects I have not found myself in a position to add, from the Professor's notes, anything substantial to the work as it stood in the former edition, with the exception of a numerical table, printed in vol. ii, pp. 317–319. This table will be found very useful in calculating coefficients of induction in circular coils of wire.

In a work so original, and containing so many details of new results, it was impossible but that there should be a few errors in the first edition. I trust that in the present edition most of these will be found to have been corrected. I have the greater confidence in expressing this hope as, in reading some of the proofs, I have had the assistance of various friends conversant with the work, among whom I may mention particularly my brother Professor Charles Niven, and Mr. J. J. Thomson, Fellow of Trinity College, Cambridge.

W. D. NIVEN.

TRINITY COLLEGE, CAMBRIDGE,
*Oct.* 1, 1881.

# PREFACE TO THE THIRD EDITION

I UNDERTOOK the task of reading the proofs of this Edition at the request of the Delegates of the Clarendon Press, by whom I was informed, to my great regret, that Mr. W. D. Niven found that the pressure of his official duties prevented him from seeing another edition of this work through the Press.

The readers of Maxwell's writings owe so much to the untiring labour which Mr. Niven has spent upon them, that I am sure they will regret as keenly as I do myself that anything should have intervened to prevent this Edition from receiving the benefit of his care.

It is now nearly twenty years since this book was written, and during that time the sciences of Electricity and Magnetism have advanced with a rapidity almost unparalleled in their previous history; this is in no small degree due to the views introduced into these sciences by this book: many of its paragraphs have served as the starting-points of important investigations. When I began to revise this Edition it was my intention to give in foot-notes some account of the advances made since the publication of the first edition, not only because I thought it might be of service to the students of Electricity, but also because all recent investigations have tended to confirm in the most remarkable way the views advanced by Maxwell. I soon found, however, that the progress made in the science had been so great that it was impossible to carry out this intention without disfiguring the book by a disproportionate quantity of foot-notes. I therefore decided to throw these notes into a slightly more consecutive form and to publish them separately. They are now almost ready for press, and will I hope appear in a few months. This volume of notes is the one referred to as the 'Supplementary Volume.' A few foot-notes relating to isolated points which could be dealt with briefly are given. All the matter added to this Edition is enclosed within { } brackets.

I have endeavoured to add something in explanation of the argument in those passages in which I have found from my experience as a teacher that nearly all students find considerable difficulties; to have added an explanation of all passages in which I have known students find difficulties would have required more volumes than were at my disposal.

I have attempted to verify the results which Maxwell gives without proof; I have not in all instances succeeded in arriving at the result given by him, and in such cases I have indicated the difference in a foot-note.

I have reprinted from his paper on the Dynamical Theory of the Electromagnetic Field, Maxwell's method of determining the self-induction of a coil. The omission of this from previous editions has caused the method to be frequently attributed to other authors.

In preparing this edition I have received the greatest possible assistance from Mr. Charles Chree, Fellow of King's College, Cambridge. Mr. Chree has read the whole of the proof sheets, and his suggestions have been invaluable. I have also received help from Mr. Larmor, Fellow of St. John's College, Mr. Wilberforce, Demonstrator at the Cavendish Laboratory, and Mr. G. T. Walker, Fellow of Trinity College.

J. J. THOMSON.

CAVENDISH LABORATORY:
*Dec.* 5, 1891.

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# ELECTRICITY AND MAGNETISM.

## PRELIMINARY.

### ON THE MEASUREMENT OF QUANTITIES.

1.] EVERY expression of a Quantity consists of two factors or components. One of these is the name of a certain known quantity of the same kind as the quantity to be expressed, which is taken as a standard of reference. The other component is the number of times the standard is to be taken in order to make up the required quantity. The standard quantity is technically called the Unit, and the number is called the Numerical Value of the quantity.

There must be as many different units as there are different kinds of quantities to be measured, but in all dynamical sciences it is possible to define these units in terms of the three fundamental units of Length, Time, and Mass. Thus the units of area and of volume are defined respectively as the square and the cube whose sides are the unit of length.

Sometimes, however, we find several units of the same kind founded on independent considerations. Thus the gallon, or the volume of ten pounds of water, is used as a unit of capacity as well as the cubic foot. The gallon may be a convenient measure in some cases, but it is not a systematic one, since its numerical relation to the cubic foot is not a round integral number.

2.] In framing a mathematical system we suppose the fundamental units of length, time, and mass to be given, and deduce all the derivative units from these by the simplest attainable definitions.

The formulae at which we arrive must be such that a person of any nation, by substituting for the different symbols the numerical values of the quantities as measured by his own national units, would arrive at a true result.

Hence, in all scientific studies it is of the greatest importance to employ units belonging to a properly defined system, and to know the relations of these units to the fundamental units, so that we may be able at once to transform our results from one system to another.

This is most conveniently done by ascertaining the *dimensions* of every unit in terms of the three fundamental units. When a given unit varies as the $n$th power of one of these units, it is said to be of *n dimensions* as regards that unit.

For instance, the scientific unit of volume is always the cube whose side is the unit of length. If the unit of length varies, the unit of volume will vary as its third power, and the unit of volume is said to be of three dimensions with respect to the unit of length.

A knowledge of the dimensions of units furnishes a test which ought to be applied to the equations resulting from any lengthened investigation. The dimensions of every term of such an equation, with respect to each of the three fundamental units, must be the same. If not, the equation is absurd, and contains some error, as its interpretation would be different according to the arbitrary system of units which we adopt*.

### *The Three Fundamental Units.*

3.] (1) *Length.* The standard of length for scientific purposes in this country is one foot, which is the third part of the standard yard preserved in the Exchequer Chambers.

In France, and other countries which have adopted the metric system, it is the mètre. The mètre is theoretically the ten millionth part of the length of a meridian of the earth measured from the pole to the equator; but practically it is the length of a standard preserved in Paris, which was constructed by Borda to correspond, when at the temperature of melting ice, with the value of the preceding length as measured by Delambre. The mètre has not been altered to correspond with new and more accurate measurements of the earth, but the arc of the meridian is estimated in terms of the original mètre.

* The theory of dimensions was first stated by Fourier, *Théorie de Chaleur*, § 160.

In astronomy the mean distance of the earth from the sun is sometimes taken as a unit of length.

In the present state of science the most universal standard of length which we could assume would be the wave length in vacuum of a particular kind of light, emitted by some widely diffused substance such as sodium, which has well-defined lines in its spectrum. Such a standard would be independent of any changes in the dimensions of the earth, and should be adopted by those who expect their writings to be more permanent than that body.

In treating of the dimensions of units we shall call the unit of length $[L]$. If $l$ is the numerical value of a length, it is understood to be expressed in terms of the concrete unit $[L]$, so that the actual length would be fully expressed by $l\,[L]$.

4.] (2) *Time*. The standard unit of time in all civilized countries is deduced from the time of rotation of the earth about its axis. The sidereal day, or the true period of rotation of the earth, can be ascertained with great exactness by the ordinary observations of astronomers; and the mean solar day can be deduced from this by our knowledge of the length of the year.

The unit of time adopted in all physical researches is one second of mean solar time.

In astronomy a year is sometimes used as a unit of time. A more universal unit of time might be found by taking the periodic time of vibration of the particular kind of light whose wave length is the unit of length.

We shall call the concrete unit of time $[T]$, and the numerical measure of time $t$.

5.] (3) *Mass*. The standard unit of mass is in this country the avoirdupois pound preserved in the Exchequer Chambers. The grain, which is often used as a unit, is defined to be the 7000th part of this pound.

In the metrical system it is the gramme, which is theoretically the mass of a cubic centimètre of distilled water at standard temperature and pressure, but practically it is the thousandth part of the standard kilogramme preserved in Paris.

The accuracy with which the masses of bodies can be compared by weighing is far greater than that hitherto attained in the measurement of lengths, so that all masses ought, if possible, to be compared directly with the standard, and not deduced from experiments on water.

In descriptive astronomy the mass of the sun or that of the earth is sometimes taken as a unit, but in the dynamical theory of astronomy the unit of mass is deduced from the units of time and length, combined with the fact of universal gravitation. The astronomical unit of mass is that mass which attracts another body placed at the unit of distance so as to pıcduce in that body the unit of acceleration.

In framing a universal system of units we may either deduce the unit of mass in this way from those of length and time already defined, and this we can do to a rough approximation in the present state of science; or, if we expect * soon to be able to determine the mass of a single molecule of a standard substance, we may wait for this determination before fixing a universal standard of mass.

We shall denote the concrete unit of mass by the symbol $[M]$ in treating of the dimensions of other units. The unit of mass will be taken as one of the three fundamental units. When, as in the French system, a particular substance, water, is taken as a standard of density, then the unit of mass is no longer independent, but varies as the unit of volume, or as $[L^3]$.

If, as in the astronomical system, the unit of mass is defined with respect to its attractive power, the dimensions of $[M]$ are $[L^3 T^{-2}]$.

For the acceleration due to the attraction of a mass $m$ at a distance $r$ is by the Newtonian Law $\frac{m}{r^2}$. Suppose this attraction to act for a very small time $t$ on a body originally at rest, and to cause it to describe a space $s$, then by the formula of Galileo,

$$s = \tfrac{1}{2} f t^2 = \tfrac{1}{2} \frac{m}{r^2} t^2;$$

whence $m = 2\frac{r^2 s}{t^2}$. Since $r$ and $s$ are both lengths, and $t$ is a time, this equation cannot be true unless the dimensions of $m$ are $[L^3 T^{-2}]$. The same can be shewn from any astronomical equa-

* See Prof. J. Loschmidt, 'Zur Grösse der Luftmolecule,' *Academy of Vienna*, Oct. 12, 1865: G. J. Stoney on 'The Internal Motions of Gases,' *Phil. Mag.*, Aug. 1868; and Sir W. Thomson on 'The Size of Atoms,' *Nature*, March 31, 1870.
{See also Sir W. Thomson on 'The Size of Atoms,' *Nature*, v. 28, pp. 203, 250, 274.}

tion in which the mass of a body appears in some but not in all of the terms *.

## *Derived Units.*

6.] The unit of Velocity is that velocity in which unit of length is described in unit of time. Its dimensions are $[LT^{-1}]$.

If we adopt the units of length and time derived from the vibrations of light, then the unit of velocity is the velocity of light.

The unit of Acceleration is that acceleration in which the velocity increases by unity in unit of time. Its dimensions are $[LT^{-2}]$.

The unit of Density is the density of a substance which contains unit of mass in unit of volume. Its dimensions are $[ML^{-3}]$.

The unit of Momentum is the momentum of unit of mass moving with unit of velocity. Its dimensions are $[MLT^{-1}]$.

The unit of Force is the force which produces unit of momentum in unit of time. Its dimensions are $[MLT^{-2}]$.

This is the absolute unit of force, and this definition of it is implied in every equation in Dynamics. Nevertheless, in many text books in which these equations are given, a different unit of force is adopted, namely, the weight of the national unit of mass; and then, in order to satisfy the equations, the national unit of mass is itself abandoned, and an artificial unit is adopted as the dynamical unit, equal to the national unit divided by the numerical value of the intensity of gravity at the place. In this way both the unit of force and the unit of mass are made to depend on the value of the intensity of gravity, which varies from place to place, so that statements involving these quantities are not complete without a knowledge of the intensity of gravity in the places where these statements were found to be true.

The abolition, for all scientific purposes, of this method of measuring forces is mainly due to the introduction by Gauss of

* If a centimetre and a second are taken as units, the astronomical unit of mass would be about $1{\cdot}537 \times 10^7$ grammes, or 15·37 tonnes, according to Baily's repetition of Cavendish's experiment. Baily adopts 5·6604 as the mean result of all his experiments for the mean density of the earth, and this, with the values used by Baily for the dimensions of the earth and the intensity of gravity at its surface, gives the above value as the direct result of his experiments.

{Cornu's recalculation of Baily's results gives 5·55 as the mean density of the earth, and therefore $1{\cdot}50 \times 10^7$ grammes as the astronomical unit of mass; while Cornu's own experiments give 5·50 as the mean density of the earth, and $1{\cdot}49 \times 10^7$ grammes as the astronomical unit of mass.}

a general system of making observations of magnetic force in countries in which the intensity of gravity is different. All such forces are now measured according to the strictly dynamical method deduced from our definitions, and the numerical results are the same in whatever country the experiments are made.

The unit of Work is the work done by the unit of force acting through the unit of length measured in its own direction. Its dimensions are $[ML^2T^{-2}]$.

The Energy of a system, being its capacity of performing work, is measured by the work which the system is capable of performing by the expenditure of its whole energy.

The definitions of other quantities, and of the units to which they are referred, will be given when we require them.

In transforming the values of physical quantities determined in terms of one unit, so as to express them in terms of any other unit of the same kind, we have only to remember that every expression for the quantity consists of two factors, the unit and the numerical part which expresses how often the unit is to be taken. Hence the numerical part of the expression varies inversely as the magnitude of the unit, that is, inversely as the various powers of the fundamental units which are indicated by the dimensions of the derived unit.

### *On Physical Continuity and Discontinuity.*

7.] A quantity is said to vary continuously if, when it passes from one value to another, it assumes all the intermediate values.

We may obtain the conception of continuity from a consideration of the continuous existence of a particle of matter in time and space. Such a particle cannot pass from one position to another without describing a continuous line in space, and the coordinates of its position must be continuous functions of the time.

In the so-called 'equation of continuity,' as given in treatises on Hydrodynamics, the fact expressed is that matter cannot appear in or disappear from an element of volume without passing in or out through the sides of that element.

A quantity is said to be a continuous function of its variables if, when the variables alter continuously, the quantity itself alters continuously.

Thus, if $u$ is a function of $x$, and if, while $x$ passes continuously from $x_0$ to $x_1$, $u$ passes continuously from $u_0$ to $u_1$, but when $x$ passes from $x_1$ to $x_2$, $u$ passes from $u_1'$ to $u_2$, $u_1'$ being different from $u_1$, then $u$ is said to have a discontinuity in its variation with respect to $x$ for the value $x = x_1$, because it passes abruptly from $u_1$ to $u_1'$ while $x$ passes continuously through $x_1$.

If we consider the differential coefficient of $u$ with respect to $x$ for the value $x = x_1$ as the limit of the fraction

$$\frac{u_2 - u_0}{x_2 - x_0},$$

when $x_2$ and $x_0$ are both made to approach $x_1$ without limit, then, if $x_0$ and $x_2$ are always on opposite sides of $x_1$, the ultimate value of the numerator will be $u_1' - u_1$, and that of the denominator will be zero. If $u$ is a quantity physically continuous, the discontinuity can exist only with respect to particular values of the variable $x$. We must in this case admit that it has an infinite differential coefficient when $x = x_1$. If $u$ is not physically continuous, it cannot be differentiated at all.

It is possible in physical questions to get rid of the idea of discontinuity without sensibly altering the conditions of the case. If $x_0$ is a very little less than $x_1$, and $x_2$ a very little greater than $x_1$, then $u_0$ will be very nearly equal to $u_1$ and $u_2$ to $u_1'$. We may now suppose $u$ to vary in any arbitrary but continuous manner from $u_0$ to $u_2$ between the limits $x_0$ and $x_2$. In many physical questions we may begin with a hypothesis of this kind, and then investigate the result when the values of $x_0$ and $x_2$ are made to approach that of $x_1$ and ultimately to reach it. If the result is independent of the arbitrary manner in which we have supposed $u$ to vary between the limits, we may assume that it is true when $u$ is discontinuous.

### *Discontinuity of a Function of more than One Variable.*

8.] If we suppose the values of all the variables except $x$ to be constant, the discontinuity of the function will occur for particular values of $x$, and these will be connected with the values of the other variables by an equation which we may write

$$\phi = \phi(x, y, z, \&c.) = 0.$$

The discontinuity will occur when $\phi = 0$. When $\phi$ is positive the function will have the form $F_2\ (x, y, z, \&c.)$. When $\phi$ is negative it will have the form $F_1(x, y, z, \&c.)$. There need be no necessary relation between the forms $F_1$ and $F_2$.

To express this discontinuity in a mathematical form, let one of the variables, say $x$, be expressed as a function of $\phi$ and the other variables, and let $F_1$ and $F_2$ be expressed as functions of $\phi, y, z$, &c. We may now express the general form of the function by any formula which is sensibly equal to $F_2$ when $\phi$ is positive, and to $F_1$ when $\phi$ is negative. Such a formula is the following—

$$F = \frac{F_1 + e^{n\phi} F_2}{1 + e^{n\phi}}.$$

As long as $n$ is a finite quantity, however great, $F$ will be a continuous function, but if we make $n$ infinite $F$ will be equal to $F_2$ when $\phi$ is positive, and equal to $F_1$ when $\phi$ is negative.

*Discontinuity of the Derivatives of a Continuous Function.*

The first derivatives of a continuous function may be discontinuous. Let the values of the variables for which the discontinuity of the derivatives occurs be connected by the equation

$$\phi = \phi(x, y, z \ldots) = 0,$$

and let $F_1$ and $F_2$ be expressed in terms of $\phi$ and $n-1$ other variables, say $(y, z \ldots)$.

Then, when $\phi$ is negative, $F_1$ is to be taken, and when $\phi$ is positive $F_2$ is to be taken, and, since $F$ is itself continuous, when $\phi$ is zero, $F_1 = F_2$.

Hence, when $\phi$ is zero, the derivatives $\frac{dF_1}{d\phi}$ and $\frac{dF_2}{d\phi}$ may be different, but the derivatives with respect to any of the other variables, such as $\frac{dF_1}{dy}$ and $\frac{dF_2}{dy}$, must be the same. The discontinuity is therefore confined to the derivative with respect to $\phi$, all the other derivatives being continuous.

*Periodic and Multiple Functions.*

9.] If $u$ is a function of $x$ such that its value is the same for $x$, $x+a$, $x+na$, and all values of $x$ differing by $a$, $u$ is called a periodic function of $x$, and $a$ is called its period.

If $x$ is considered as a function of $u$, then, for a given value of $u$, there must be an infinite series of values of $x$ differing by multiples of $a$. In this case $x$ is called a multiple function of $u$, and $a$ is called its cyclic constant.

The differential coefficient $\frac{dx}{du}$ has only a finite number of values corresponding to a given value of $u$.

*On the Relation of Physical Quantities to Directions in Space.*

10.] In distinguishing the kinds of physical quantities, it is of great importance to know how they are related to the directions of those coordinate axes which we usually employ in defining the positions of things. The introduction of coordinate axes into geometry by Des Cartes was one of the greatest steps in mathematical progress, for it reduced the methods of geometry to calculations performed on numerical quantities. The position of a point is made to depend on the lengths of three lines which are always drawn in determinate directions, and the line joining two points is in like manner considered as the resultant of three lines.

But for many purposes of physical reasoning, as distinguished from calculation, it is desirable to avoid explicitly introducing the Cartesian coordinates, and to fix the mind at once on a point of space instead of its three coordinates, and on the magnitude and direction of a force instead of its three components. This mode of contemplating geometrical and physical quantities is more primitive and more natural than the other, although the ideas connected with it did not receive their full development till Hamilton made the next great step in dealing with space, by the invention of his Calculus of Quaternions *.

As the methods of Des Cartes are still the most familiar to students of science, and as they are really the most useful for purposes of calculation, we shall express all our results in the Cartesian form. I am convinced, however, that the introduction of the ideas, as distinguished from the operations and methods of Quaternions, will be of great use to us in the study of all parts of our subject, and especially in electrodynamics, where we have to deal with a number of physical quantities, the relations of which to each other can be expressed far more simply by a few expressions of Hamilton's, than by the ordinary equations.

* {For an elementary account of Quaternions, the reader may be referred to Kelland and Tait's 'Introduction to Quaternions,' Tait's 'Elementary Treatise on Quaternions,' and Hamilton's 'Elements of Quaternions.'}

11.] One of the most important features of Hamilton's method is the division of quantities into Scalars and Vectors.

A Scalar quantity is capable of being completely defined by a single numerical specification. Its numerical value does not in any way depend on the directions we assume for the coordinate axes.

A Vector, or Directed quantity, requires for its definition three numerical specifications, and these may most simply be understood as having reference to the directions of the coordinate axes.

Scalar quantities do not involve direction. The volume of a geometrical figure, the mass and the energy of a material body, the hydrostatical pressure at a point in a fluid, and the potential at a point in space, are examples of scalar quantities.

A vector quantity has direction as well as magnitude, and is such that a reversal of its direction reverses its sign. The displacement of a point, represented by a straight line drawn from its original to its final position, may be taken as the typical vector quantity, from which indeed the name of Vector is derived.

The velocity of a body, its momentum, the force acting on it, an electric current, the magnetization of a particle of iron, are instances of vector quantities.

There are physical quantities of another kind which are related to directions in space, but which are not vectors. Stresses and strains in solid bodies are examples of these, and so are some of the properties of bodies considered in the theory of elasticity and in the theory of double refraction. Quantities of this class require for their definition *nine* numerical specifications. They are expressed in the language of quaternions by linear and vector functions of a vector.

The addition of one vector quantity to another of the same kind is performed according to the rule given in Statics for the composition of forces. In fact, the proof which Poisson gives of the 'parallelogram of forces' is applicable to the composition of any quantities such that turning them end for end is equivalent to a reversal of their sign.

When we wish to denote a vector quantity by a single symbol, and to call attention to the fact that it is a vector, so that we must consider its direction as well as its magnitude, we shall denote it by a German capital letter, as $\mathfrak{A}$, $\mathfrak{B}$, &c.

In the calculus of quaternions, the position of a point in space is defined by the vector drawn from a fixed point, called the origin, to that point. If we have to consider any physical quantity whose value depends on the position of the point, that quantity is treated as a function of the vector drawn from the origin. The function may be itself either scalar or vector. The density of a body, its temperature, its hydrostatical pressure, the potential at a point, are examples of scalar functions. The resultant force at a point, the velocity of a fluid at a point, the velocity of rotation of an element of the fluid, and the couple producing rotation, are examples of vector functions.

12.] Physical vector quantities may be divided into two classes, in one of which the quantity is defined with reference to a line, while in the other the quantity is defined with reference to an area.

For instance, the resultant of an attractive force in any direction may be measured by finding the work which it would do on a body if the body were moved a short distance in that direction and dividing it by that short distance. Here the attractive force is defined with reference to a line.

On the other hand, the flux of heat in any direction at any point of a solid body may be defined as the quantity of heat which crosses a small area drawn perpendicular to that direction divided by that area and by the time. Here the flux is defined with reference to an area.

There are certain cases in which a quantity may be measured with reference to a line as well as with reference to an area.

Thus, in treating of the displacements of elastic solids, we may direct our attention either to the original and the actual positions of a particle, in which case the displacement of the particle is measured by the line drawn from the first position to the second, or we may consider a small area fixed in space, and determine what quantity of the solid passes across that area during the displacement.

In the same way the velocity of a fluid may be investigated either with respect to the actual velocity of the individual particles, or with respect to the quantity of the fluid which flows through any fixed area.

But in these cases we require to know separately the density of the body as well as the displacement or velocity, in order to apply the first method, and whenever we attempt to form a molecular theory we have to use the second method.

In the case of the flow of electricity we do not know anything of its density or its velocity in the conductor, we only know the value of what, on the fluid theory, would correspond to the product of the density and the velocity. Hence in all such cases we must apply the more general method of measurement of the flux across an area.

In electrical science, electromotive and magnetic intensity belong to the first class, being defined with reference to lines. When we wish to indicate this fact, we may refer to them as Intensities.

On the other hand, electric and magnetic induction, and electric currents, belong to the second class, being defined with reference to areas. When we wish to indicate this fact, we shall refer to them as Fluxes.

Each of these intensities may be considered as producing, or tending to produce, its corresponding flux. Thus, electromotive intensity produces electric currents in conductors, and tends to produce them in dielectrics. It produces electric induction in dielectrics, and probably in conductors also. In the same sense, magnetic intensity produces magnetic induction.

13.] In some cases the flux is simply proportional to the intensity and in the same direction, but in other cases we can only affirm that the direction and magnitude of the flux are functions of the direction and magnitude of the intensity.

The case in which the components of the flux are *linear* functions of those of the intensity is discussed in the chapter on the Equations of Conduction, Art. 297. There are in general nine coefficients which determine the relation between the intensity and the flux. In certain cases we have reason to believe that six of these coefficients form three pairs of equal quantities. In such cases the relation between the line of direction of the intensity and the normal plane of the flux is of the same kind as that between a semi-diameter of an ellipsoid and its conjugate diametral plane. In Quaternion language, the one vector is said to be a linear and vector function of the other, and when there are three pairs of equal coefficients the function is said to be self-conjugate.

In the case of magnetic induction in iron, the flux (the magnetization of the iron) is not a linear function of the magnetizing intensity. In all cases, however, the product of the intensity and the flux resolved in its direction, gives a result of scientific importance, and this is always a scalar quantity.

14.] There are two mathematical operations of frequent occurrence which are appropriate to these two classes of vectors, or directed quantities.

In the case of intensity, we have to take the integral along a line of the product of an element of the line, and the resolved part of the intensity along that element. The result of this operation is called the Line-integral of the intensity. It represents the work done on a body carried along the line. In certain cases in which the line-integral does not depend on the form of the line, but only on the positions of its extremities, the line-integral is called the Potential.

In the case of fluxes, we have to take the integral, over a surface, of the flux through every element of the surface. The result of this operation is called the Surface-integral of the flux. It represents the quantity which passes through the surface.

There are certain surfaces across which there is no flux. If two of these surfaces intersect, their line of intersection is a line of flux. In those cases in which the flux is in the same direction as the force, lines of this kind are often called Lines of Force. It would be more correct, however, to speak of them in electrostatics and magnetics as Lines of Induction, and in electrokinematics as Lines of Flow.

15.] There is another distinction between different kinds of directed quantities, which, though very important from a physical point of view, is not so necessary to be observed for the sake of the mathematical methods. This is the distinction between longitudinal and rotational properties.

The direction and magnitude of a quantity may depend upon some action or effect which takes place entirely along a certain line, or it may depend upon something of the nature of rotation about that line as an axis. The laws of combination of directed quantities are the same whether they are longitudinal or rotational, so that there is no difference in the mathematical treatment of the two classes, but there may be physical circumstances which indicate to which class we must refer a particular phenomenon. Thus, electrolysis consists of the transfer of certain substances along a line in one direction, and of certain other substances in the opposite direction, which is evidently a longitudinal phenomenon, and there is no evidence of any rotational effect about the direction of the force. Hence we infer that the electric current which causes or accompanies electrolysis is a longitudinal, and not a rotational phenomenon.

On the other hand, the north and south poles of a magnet do not differ as oxygen and hydrogen do, which appear at opposite places during electrolysis, so that we have no evidence that magnetism is a longitudinal phenomenon, while the effect of magnetism in rotating the plane of polarization of plane polarized light distinctly shews that magnetism is a rotational phenomenon*.

### *On Line-integrals.*

16.] The operation of integration of the resolved part of a vector quantity along a line is important in physical science generally, and should be clearly understood.

Let $x$, $y$, $z$ be the coordinates of a point $P$ on a line whose length, measured from a certain point $A$, is $s$. These coordinates will be functions of a single variable $s$.

Let $R$ be the numerical value of the vector quantity at $P$, and let the tangent to the curve at $P$ make with the direction of $R$ the angle $\epsilon$, then $R\cos\epsilon$ is the resolved part of $R$ along the line, and the integral

$$L = \int_0^s R\cos\epsilon\, ds$$

is called the line-integral of $R$ along the line $s$.

We may write this expression

$$L = \int_0^s \left( X\frac{dx}{ds} + Y\frac{dy}{ds} + Z\frac{dz}{ds}\right) ds,$$

where $X$, $Y$, $Z$ are the components of $R$ parallel to $x$, $y$, $z$ respectively.

This quantity is, in general, different for different lines drawn

* {This must not be taken to imply that in any theory in which electric and magnetic phenomena are supposed to be due to the motion of a medium, the electric current must *necessarily* be due to a motion of translation and magnetic force to one of rotation. There are rotatory effects connected with a current, for example, a magnetic pole is turned round it, and it is probable that if the medium in which electrostatic phenomena have their seat has an electric displacement through it whose components are $f$, $g$, $h$, and is moving with the velocity $u$, $v$, $w$, it will be the seat of a magnetic force whose components are $4\pi(wg-vh)$, $4\pi(uh-wf)$, $4\pi(vf-ug)$ respectively: thus, in this case, a motion of translation could produce a magnetic field. *Phil. Mag.* July, 1889.}

between $A$ and $P$. When, however, within a certain region, the quantity
$$X\,dx + Y\,dy + Z\,dz = -D\Psi,$$
that is, when it is an exact differential within that region, the value of $L$ becomes
$$L = \Psi_A - \Psi_P,$$
and is the same for any two forms of the path between $A$ and $P$, provided the one form can be changed into the other by continuous motion without passing out of this region.

*On Potentials.*

The quantity $\Psi$ is a scalar function of the position of the point, and is therefore independent of the directions of reference. It is called the Potential Function, and the vector quantity whose components are $X$, $Y$, $Z$ is said to have a potential $\Psi$, if
$$X = -\left(\frac{d\Psi}{dx}\right), \qquad Y = -\left(\frac{d\Psi}{dy}\right), \qquad Z = -\left(\frac{d\Psi}{dz}\right).$$

When a potential function exists, surfaces for which the potential is constant are called Equipotential surfaces. The direction of $R$ at any point of such a surface coincides with the normal to the surface, and if $n$ be a normal at the point $P$, then $R = -\dfrac{d\Psi}{dn}$.

The method of considering the components of a vector as the first derivatives of a certain function of the coordinates with respect to these coordinates was invented by Laplace * in his treatment of the theory of attractions. The name of Potential was first given to this function by Green †, who made it the basis of his treatment of electricity. Green's essay was neglected by mathematicians till 1846, and before that time most of its important theorems had been rediscovered by Gauss, Chasles, Sturm, and Thomson ‡.

In the theory of gravitation the potential is taken with the opposite sign to that which is here used, and the resultant force in any direction is then measured by the rate of *increase* of the potential function in that direction. In electrical and magnetic

* *Méc. Céleste*, liv. iii.

† Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism, Nottingham, 1828. Reprinted in *Crelle's Journal*, and in Mr. Ferrers' edition of Green's Works.

‡ Thomson and Tait, *Natural Philosophy*, § 483.

investigations the potential is defined so that the resultant force in any direction is measured by the *decrease* of the potential in that direction. This method of using the expression makes it correspond in sign with potential energy, which always decreases when the bodies are moved in the direction of the forces acting on them.

17.] The geometrical nature of the relation between the potential and the vector thus derived from it receives great light from Hamilton's discovery of the form of the operator by which the vector is derived from the potential.

The resolved part of the vector in any direction is, as we have seen, the first derivative of the potential with respect to a coordinate drawn in that direction, the sign being reversed.

Now if $i$, $j$, $k$ are three unit vectors at right angles to each other, and if $X$, $Y$, $Z$ are the components of the vector $\mathfrak{F}$ resolved parallel to these vectors, then

$$\mathfrak{F} = iX + jY + kZ; \tag{1}$$

and by what we have said above, if $\Psi$ is the potential,

$$\mathfrak{F} = -\left(i\frac{d\Psi}{dx} + j\frac{d\Psi}{dy} + k\frac{d\Psi}{dz}\right). \tag{2}$$

If we now write $\nabla$ for the operator,

$$i\frac{d}{dx} + j\frac{d}{dy} + k\frac{d}{dz}, \tag{3}$$

$$\mathfrak{F} = -\nabla\Psi. \tag{4}$$

The symbol of operation $\nabla$ may be interpreted as directing us to measure, in each of three rectangular directions, the rate of increase of $\Psi$, and then, considering the quantities thus found as vectors, to compound them into one. This is what we are directed to do by the expression (3). But we may also consider it as directing us first to find out in what direction $\Psi$ increases fastest, and then to lay off in that direction a vector representing this rate of increase.

M. Lamé, in his *Traité des Fonctions Inverses*, uses the term Differential Parameter to express the magnitude of this greatest rate of increase, but neither the term itself, nor the mode in which Lamé uses it, indicates that the quantity referred to has direction as well as magnitude. On those rare occasions in which I shall have to refer to this relation as a purely geometrical one, I shall call the vector $\mathfrak{F}$ the space-variation of the scalar function $\Psi$, using the phrase to indicate the direction, as well as the magnitude, of the most rapid decrease of $\Psi$.

18.] There are cases, however, in which the conditions

$$\frac{dZ}{dy}-\frac{dY}{dz}=0,\quad \frac{dX}{dz}-\frac{dZ}{dx}=0,\ \text{and}\ \frac{dY}{dx}-\frac{dX}{dy}=0,$$

which are those of $Xdx+Ydy+Zdz$ being a complete differential, are satisfied throughout a certain region of space, and yet the line-integral from $A$ to $P$ may be different for two lines, each of which lies wholly within that region. This may be the case if the region is in the form of a ring, and if the two lines from $A$ to $P$ pass through opposite segments of the ring. In this case, the one path cannot be transformed into the other by continuous motion without passing out of the region.

We are here led to considerations belonging to the Geometry of Position, a subject which, though its importance was pointed out by Leibnitz and illustrated by Gauss, has been little studied. The most complete treatment of this subject has been given by J. B. Listing*.

Let there be $p$ points in space, and let $l$ lines of any form be drawn joining these points so that no two lines intersect each other, and no point is left isolated. We shall call a figure composed of lines in this way a Diagram. Of these lines, $p-1$ are sufficient to join the $p$ points so as to form a connected system. Every new line completes a loop or closed path, or, as we shall call it, a Cycle. The number of independent cycles in the diagram is therefore $\kappa = l-p+1$.

Any closed path drawn along the lines of the diagram is composed of these independent cycles, each being taken any number of times and in either direction.

The existence of cycles is called Cyclosis, and the number of cycles in a diagram is called its Cyclomatic number.

### *Cyclosis in Surfaces and Regions.*

Surfaces are either complete or bounded. Complete surfaces are either infinite or closed. Bounded surfaces are limited by one or more closed lines, which may in the limiting cases become double finite lines or points.

* *Der Census Raümlicher Complexe*, Gött. Abh., Bd. x. S. 97 (1861). {For an elementary account of those properties of multiply connected space which are necessary for physical purposes see Lamb's *Treatise on the Motion of Fluids*, p. 47.}

A finite region of space is bounded by one or more closed surfaces. Of these one is the external surface, the others are included in it and exclude each other, and are called internal surfaces.

If the region has one bounding surface, we may suppose that surface to contract inwards without breaking its continuity or cutting itself. If the region is one of simple continuity, such as a sphere, this process may be continued till it is reduced to a point; but if the region is like a ring, the résult will be a closed curve; and if the region has multiple connections, the result will be a diagram of lines, and the cyclomatic number of the diagram will be that of the region. The space outside the region has the same cyclomatic number as the region itself. Hence, if the region is bounded by internal as well as external surfaces, its cyclomatic number is the sum of those due to all the surfaces.

When a region encloses within itself other regions, it is called a Periphractic region.

The number of internal bounding surfaces of a region is called its periphractic number. A closed surface is also periphractic, its periphractic number being unity.

The cyclomatic number of a closed surface is twice that of either of the regions which it bounds. To find the cyclomatic number of a bounded surface, suppose all the boundaries to contract inwards, without breaking continuity, till they meet. The surface will then be reduced to a point in the case of an acyclic surface, or to a linear diagram in the case of cyclic surfaces. The cyclomatic number of the diagram is that of the surface.

19.] THEOREM I. *If throughout any acyclic region*

$$X\,dx + Y\,dy + Z\,dz = -D\Psi,$$

*the value of the line-integral from a point A to a point P taken along any path within the region will be the same.*

We shall first shew that the line-integral taken round any closed path within the region is zero.

Suppose the equipotential surfaces drawn. They are all either closed surfaces or are bounded entirely by the surface of the region, so that a closed line within the region, if it cuts any of the surfaces at one part of its path, must cut the same surface in the opposite direction at some other part of its path, and the corresponding portions of the line-integral being equal and opposite, the total value is zero.

Hence if $AQP$ and $AQ'P$ are two paths from $A$ to $P$, the line-integral for $AQ'P$ is the sum of that for $AQP$ and the closed path $AQ'PQA$. But the line-integral of the closed path is zero, therefore those of the two paths are equal.

Hence if the potential is given at any one point of such a region, that at any other point is determinate.

20.] THEOREM II. *In a cyclic region in which the equation*

$$X\,dx + Y\,dy + Z\,dz = -D\Psi$$

*is everywhere satisfied, the line-integral from $A$ to $P$ along a line drawn within the region, will not in general be determinate unless the channel of communication between $A$ and $P$ be specified.*

Let $N$ be the cyclomatic number of the region, then $N$ sections of the region may be made by surfaces which we may call Diaphragms, so as to close up $N$ of the channels of communication, and reduce the region to an acyclic condition without destroying its continuity.

The line-integral from $A$ to any point $P$ taken along a line which does not cut any of these diaphragms will be, by the last theorem, determinate in value.

Now let $A$ and $P$ be taken indefinitely near to each other, but on opposite sides of a diaphragm, and let $K$ be the line-integral from $A$ to $P$.

Let $A'$ and $P'$ be two other points on opposite sides of the same diaphragm and indefinitely near to each other, and let $K'$ be the line-integral from $A'$ to $P'$. Then $K' = K$.

For if we draw $AA'$ and $PP'$, nearly coincident, but on opposite sides of the diaphragm, the line-integrals along these lines will be equal*. Suppose each equal to $L$, then $K'$, the line-integral of $A'P'$, is equal to that of $A'A + AP + PP' = -L + K + L = K =$ that of $AP$.

Hence the line-integral round a closed curve which passes through one diaphragm of the system in a given direction is a constant quantity $K$. This quantity is called the Cyclic constant corresponding to the given cycle.

Let any closed curve be drawn within the region, and let it cut the diaphragm of the first cycle $p$ times in the positive direction

* {Since $X$, $Y$, $Z$, are continuous.}

and $p'$ times in the negative direction, and let $p-p' = n_1$. Then the line-integral of the closed curve will be $n_1 K_1$.

Similarly the line-integral of any closed curve will be

$$n_1 K_1 + n_2 K_2 + \dots + n_s K_s;$$

where $n_s$ represents the excess of the number of positive passages of the curve through the diaphragm of the cycle $S$ over the number of negative passages.

If two curves are such that one of them may be transformed into the other by continuous motion without at any time passing through any part of space for which the condition of having a potential is not fulfilled, these two curves are called Reconcileable curves. Curves for which this transformation cannot be effected are called Irreconcileable curves *.

The condition that $X\,dx + Y\,dy + Z\,dz$ is a complete differential of some function $\Psi$ for all points within a certain region, occurs in several physical investigations in which the directed quantity and the potential have different physical interpretations.

In pure kinematics we may suppose $X, Y, Z$ to be the components of the displacement of a point of a continuous body whose original coordinates are $x, y, z$; the condition then expresses that these displacements constitute a *non-rotational strain* †.

If $X, Y, Z$ represent the components of the velocity of a fluid at the point $x, y, z$, then the condition expresses that the motion of the fluid is irrotational.

If $X, Y, Z$ represent the components of the force at the point $x, y, z$, then the condition expresses that the work done on a particle passing from one point to another is the difference of the potentials at these points, and the value of this difference is the same for all reconcileable paths between the two points.

### *On Surface-Integrals.*

21.] Let $dS$ be the element of a surface, and $\epsilon$ the angle which a normal to the surface drawn towards the positive side of the surface makes with the direction of the vector quantity $R$, then $\iint R \cos\epsilon \, dS$ is called the *surface-integral of $R$ over the surface $S$* ‡.

* See Sir W. Thomson 'On Vortex Motion,' *Trans. R. S. Edin.*, 1867–8.

† See Thomson and Tait's *Natural Philosophy*, § 190 (*i*).

‡ {In the following investigations the positive direction of the normal is outwards from the surface.}

THEOREM III. *The surface-integral of the flux inwards through a closed surface may be expressed as the volume-integral of its convergence taken within the surface.* (See Art. 25.)

Let $X$, $Y$, $Z$ be the components of $R$, and let $l$, $m$, $n$ be the direction-cosines of the normal to $S$ measured outwards. Then the surface-integral of $R$ over $S$ is

$$\iint R\cos\epsilon\, dS = \iint Xl\,dS + \iint Ym\,dS + \iint Zn\,dS, \qquad (1)$$

the values of $X$, $Y$, $Z$ being those at a point in the surface, and the integrations being extended over the whole surface.

If the surface is a closed one, then, when $y$ and $z$ are given, the coordinate $x$ must have an even number of values, since a line parallel to $x$ must enter and leave the enclosed space an equal number of times provided it meets the surface at all.

At each entrance

$$l\,dS = -\,dy\,dz,$$

and at each exit

$$l\,dS = \quad dy\,dz.$$

Let a point travelling from $x = -\infty$ to $x = +\infty$ first enter the space when $x = x_1$, then leave it when $x = x_2$, and so on; and let the values of $X$ at these points be $X_1$, $X_2$, &c., then

$$\iint Xl\,dS = -\iint \{(X_1 - X_2) + (X_3 - X_4) + \text{\&c.} + (X_{2n-1} - X_{2n})\}\,dy\,dz. \qquad (2)$$

If $X$ is a quantity which is continuous, and has no infinite values between $x_1$ and $x_2$, then

$$X_2 - X_1 = \int_{x_1}^{x_2} \frac{dX}{dx}\,dx; \qquad (3)$$

where the integration is extended from the first to the second intersection, that is, along the first segment of $x$ which is within the closed surface. Taking into account all the segments which lie within the closed surface, we find

$$\iint Xl\,dS = \iiint \frac{dX}{dx}\,dx\,dy\,dz, \qquad (4)$$

the double integration being confined to the closed surface, but the triple integration being extended to the whole enclosed space. Hence, if $X$, $Y$, $Z$ are continuous and finite within a closed surface $S$, the total surface-integral of $R$ over that surface will be

$$\iint R\cos\epsilon\, dS = \iiint \left(\frac{dX}{dx} + \frac{dY}{dy} + \frac{dZ}{dz}\right) dx\,dy\,dz, \qquad (5)$$

the triple integration being extended over the whole space within $S$.

Let us next suppose that $X$, $Y$, $Z$ are not continuous within the closed surface, but that at a certain surface $F(x, y, z) = 0$ the values of $X$, $Y$, $Z$ alter abruptly from $X$, $Y$, $Z$ on the negative side of the surface to $X'$, $Y'$, $Z'$ on the positive side.

If this discontinuity occurs, say, between $x_1$ and $x_2$, the value of $X_2 - X_1$ will be

$$\int_{x_1}^{x_2} \frac{dX}{dx}\, dx + (X' - X), \tag{6}$$

where in the expression under the integral sign only the finite values of the derivative of $X$ are to be considered.

In this case therefore the total surface-integral of $R$ over the closed surface will be expressed by

$$\iint R \cos \epsilon\, dS = \iiint \left(\frac{dX}{dx} + \frac{dY}{dy} + \frac{dZ}{dz}\right) dx\, dy\, dz + \iint (X' - X)\, dy\, dz$$
$$+ \iint (Y' - Y)\, dz\, dx + \iint (Z' - Z)\, dx\, dy\,; \tag{7}$$

or, if $l'$, $m'$, $n'$ are the direction-cosines of the normal to the surface of discontinuity, and $dS'$ an element of that surface,

$$\iint R \cos \epsilon\, dS = \iiint \left(\frac{dX}{dx} + \frac{dY}{dy} + \frac{dZ}{dz}\right) dx\, dy\, dz$$
$$+ \iint \{l'(X' - X) + m'(Y' - Y) + n'(Z' - Z)\} dS', \tag{8}$$

where the integration of the last term is to be extended over the surface of discontinuity.

If at every point where $X$, $Y$, $Z$ are continuous

$$\frac{dX}{dx} + \frac{dY}{dy} + \frac{dZ}{dz} = 0, \tag{9}$$

and at every surface where they are discontinuous

$$l'X' + m'Y' + n'Z' = l'X + m'Y + n'Z, \tag{10}$$

then the surface-integral over every closed surface is zero, and the distribution of the vector quantity is said to be Solenoidal.

We shall refer to equation (9) as the General solenoidal condition, and to equation (10) as the Superficial solenoidal condition.

22.] Let us now consider the case in which at every point within the surface $S$ the equation

$$\frac{dX}{dx} + \frac{dY}{dy} + \frac{dZ}{dz} = 0 \tag{11}$$

is satisfied. We have as a consequence of this the surface-integral over the closed surface equal to zero.

Now let the closed surface $S$ consist of three parts $S_1$, $S_0$, and $S_2$. Let $S_1$ be a surface of any form bounded by a closed line $L_1$. Let $S_0$ be formed by drawing lines from every point of $L_1$ always coinciding with the direction of $R$. If $l$, $m$, $n$ are the direction-cosines of the normal at any point of the surface $S_0$, we have

$$R \cos \epsilon = Xl + Ym + Zn = 0. \tag{12}$$

Hence this part of the surface contributes nothing towards the value of the surface-integral.

Let $S_2$ be another surface of any form bounded by the closed curve $L_2$ in which it meets the surface $S_0$.

Let $Q_1$, $Q_0$, $Q_2$ be the surface-integrals of the surfaces $S_1$, $S_0$, $S_2$, and let $Q$ be the surface-integral of the closed surface $S$. Then

$$Q = Q_1 + Q_0 + Q_2 = 0; \tag{13}$$

and we know that

$$Q_0 = 0; \tag{14}$$

therefore

$$Q_2 = -Q_1; \tag{15}$$

or, in other words, the surface-integral over the surface $S_2$ is equal and opposite to that over $S_1$ whatever be the form and position of $S_2$, provided that the intermediate surface $S_0$ is one for which $R$ is always tangential.

If we suppose $L_1$ a closed curve of small area, $S_0$ will be a tubular surface having the property that the surface-integral over every complete section of the tube is the same.

Since the whole space can be divided into tubes of this kind provided

$$\frac{dX}{dx} + \frac{dY}{dy} + \frac{dZ}{dz} = 0, \tag{16}$$

a distribution of a vector quantity consistent with this equation is called a Solenoidal Distribution.

### *On Tubes and Lines of Flow.*

If the space is so divided into tubes that the surface-integral for every tube is unity, the tubes are called Unit tubes, and the surface-integral over any finite surface $S$ bounded by a closed curve $L$ is equal to the *number* of such tubes which pass through $S$ in the positive direction, or, what is the same thing, the number which pass through the closed curve $L$.

Hence the surface-integral of $S$ depends only on the form of its boundary $L$, and not on the form of the surface within its boundary.

*On Periphractic Regions.*

If, throughout the whole region bounded externally by the single closed surface $S$, the solenoidal condition

$$\frac{dX}{dx} + \frac{dY}{dy} + \frac{dZ}{dz} = 0$$

is satisfied, then the surface-integral taken over any closed surface drawn within this region will be zero, and the surface-integral taken over a bounded surface within the region will depend only on the form of the closed curve which forms its boundary.

It is not, however, generally true that the same results follow if the region within which the solenoidal condition is satisfied is bounded otherwise than by a single surface.

For if it is bounded by more than one continuous surface, one of these is the external surface and the others are internal surfaces, and the region $S$ is a periphractic region, having within it other regions which it completely encloses.

If within one of these enclosed regions, say, that bounded by the closed surface $S_1$, the solenoidal condition is not satisfied, let

$$Q_1 = \iint R \cos \epsilon \, dS_1$$

be the surface-integral for the surface enclosing this region, and let $Q_2$, $Q_3$, &c. be the corresponding quantities for the other enclosed regions $S_2$, $S_3$, &c.

Then, if a closed surface $S'$ is drawn within the region $S$, the value of its surface-integral will be zero only when this surface $S'$ does not include any of the enclosed regions $S_1$, $S_2$, &c. If it includes any of these, the surface-integral is the sum of the surface-integrals of the different enclosed regions which lie within it.

For the same reason, the surface-integral taken over a surface bounded by a closed curve is the same for such surfaces only, bounded by the closed curve, as are reconcileable with the given surface by continuous motion of the surface within the region $S$.

When we have to deal with a periphractic region, the first thing to be done is to reduce it to an aperiphractic region by drawing lines $L_1$, $L_2$, &c. joining the internal surfaces $S_1$, $S_2$, &c. to the external surface $S$. Each of these lines, provided it joins surfaces which were not already in continuous connexion, reduces the periphractic number by unity, so that the whole number of lines to be drawn to remove the periphraxy is equal to the periphractic number, or the number of internal surfaces. In drawing these lines we must remember that any line joining surfaces which are already connected does not diminish the periphraxy, but introduces cyclosis. When these lines have been drawn we may assert that if the solenoidal condition is satisfied in the region $S$, any closed surface drawn entirely within $S$, and not cutting any of the lines, has its surface-integral zero. If it cuts any line, say $L_1$, once or any odd number of times, it encloses the surface $S_1$ and the surface-integral is $Q_1$.

The most familiar example of a periphractic region within which the solenoidal condition is satisfied is the region surrounding a mass attracting or repelling inversely as the square of the distance.

In the latter case we have

$$X = m\frac{x}{r^3}, \quad Y = m\frac{y}{r^3}, \quad Z = m\frac{z}{r^3};$$

where $m$ is the mass, supposed to be at the origin of coordinates.

At any point where $r$ is finite

$$\frac{dX}{dx} + \frac{dY}{dy} + \frac{dZ}{dz} = 0,$$

but at the origin these quantities become infinite. For any closed surface not including the origin, the surface-integral is zero. If a closed surface includes the origin, its surface-integral is $4\pi m$.

If, for any reason, we wish to treat the region round $m$ as if it were not periphractic, we must draw a line from $m$ to an infinite distance, and in taking surface-integrals we must remember to add $4\pi m$ whenever this line crosses from the negative to the positive side of the surface.

*On Right-handed and Left-handed Relations in Space.*

23.] In this treatise the motions of translation along any axis and of rotation about that axis will be assumed to be of the same sign when their directions correspond to those of the translation and rotation of an ordinary or right-handed screw *.

* The combined action of the muscles of the arm when we turn the upper side of the right-hand outwards, and at the same time thrust the hand forwards, will impress the right-handed screw motion on the memory more firmly than any verbal definition. A common corkscrew may be used as a material symbol of the same relation.

Professor W. H. Miller has suggested to me that as the tendrils of the vine are right-handed screws and those of the hop left-handed, the two systems of relations in space might be called those of the vine and the hop respectively.

The system of the vine, which we adopt, is that of Linnæus, and of screw-makers in all civilized countries except Japan. De Candolle was the first who called the hop-tendril right-handed, and in this he is followed by Listing, and by most writers on the circular polarization of light. Screws like the hop-tendril are made for the couplings of railway-carriages, and for the fittings of wheels on the left side of ordinary carriages, but they are always called left-handed screws by those who use them.

For instance, if the actual rotation of the earth from west to east is taken positive, the direction of the earth's axis from south to north will be taken positive, and if a man walks forward in the positive direction, the positive rotation is in the order, head, right-hand, feet, left-hand.

If we place ourselves on the positive side of a surface, the positive direction along its bounding curve will be opposite to the motion of the hands of a watch with its face towards us.

This is the right-handed system which is adopted in Thomson and Tait's *Natural Philosophy*, and in Tait's *Quaternions*. The opposite, or left-handed system, is adopted in Hamilton's *Quaternions* (*Lectures*, p. 76, and *Elements*, p. 108, and p. 117 note). The operation of passing from the one system to the other is called by Listing, *Perversion*.

The reflexion of an object in a mirror is a perverted image of the object.

When we use the Cartesian axes of $x$, $y$, $z$, we shall draw them so that the ordinary conventions about the cyclic order of the symbols lead to a right-handed system of directions in space. Thus, if $x$ is drawn eastward and $y$ northward, $z$ must be drawn upward *.

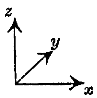
The areas of surfaces will be taken positive when the order of integration coincides with the cyclic order of the symbols. Thus, the area of a closed curve in the plane of $xy$ may be written either

$$\int x\,dy \quad \text{or} \quad -\int y\,dx;$$

the order of integration being $x$, $y$ in the first expression, and $y$, $x$ in the second.

This relation between the two products $dx\,dy$ and $dy\,dx$ may be compared with the rule for the product of two perpendicular vectors in the method of Quaternions, the sign of which depends on the order of multiplication; and with the reversal of the sign of a determinant when the adjoining rows or columns are exchanged.

For similar reasons a volume-integral is to be taken positive when the order of integration is in the cyclic order of the variables $x$, $y$, $z$, and negative when the cyclic order is reversed.

* {As in the diagram 

.}

We now proceed to prove a theorem which is useful as establishing a connection between the surface-integral taken over a finite surface and a line-integral taken round its boundary.

24.] THEOREM IV. *A line-integral taken round a closed curve may be expressed in terms of a surface-integral taken over a surface bounded by the curve.*

Let $X$, $Y$, $Z$ be the components of a vector quantity $\mathfrak{A}$ whose line-integral is to be taken round a closed curve $s$.

Let $S$ be any continuous finite surface bounded entirely by the closed curve $s$, and let $\xi$, $\eta$, $\zeta$ be the components of another vector quantity $\mathfrak{B}$, related to $X$, $Y$, $Z$ by the equations

$$\xi = \frac{dZ}{dy} - \frac{dY}{dz}, \quad \eta = \frac{dX}{dz} - \frac{dZ}{dx}, \quad \zeta = \frac{dY}{dx} - \frac{dX}{dy}. \tag{1}$$

Then the surface-integral of $\mathfrak{B}$ taken over the surface $S$ is equal to the line-integral of $\mathfrak{A}$ taken round the curve $s$. It is manifest that $\xi$, $\eta$, $\zeta$ satisfy of themselves the solenoidal condition.

$$\frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz} = 0.$$

Let $l$, $m$, $n$ be the direction-cosines of the normal to an element of the surface $dS$, reckoned in the positive direction. Then the value of the surface-integral of $\mathfrak{B}$ may be written

$$\iint (l\xi + m\eta + n\zeta)\, dS. \tag{2}$$

In order to form a definite idea of the meaning of the element $dS$, we shall suppose that the values of the coordinates $x, y, z$ for every point of the surface are given as functions of two independent variables $\alpha$ and $\beta$. If $\beta$ is constant and $\alpha$ varies, the point $(x, y, z)$ will describe a curve on the surface, and if a series of values is given to $\beta$, a series of such curves will be traced, all lying on the surface $S$. In the same way, by giving a series of constant values to $\alpha$, a second series of curves may be traced, cutting the first series, and dividing the whole surface into elementary portions, any one of which may be taken as the element $dS$.

The projection of this element on the plane of $yz$ is, by the ordinary formula,

$$l\,dS = \left(\frac{dy}{d\alpha}\frac{dz}{d\beta} - \frac{dy}{d\beta}\frac{dz}{d\alpha}\right) d\beta\, d\alpha. \tag{3}$$

The expressions for $m\,dS$ and $n\,dS$ are obtained from this by substituting $x$, $y$, $z$ in cyclic order.

The surface-integral which we have to find is

$$\iint (l\xi + m\eta + n\zeta)\, dS; \tag{4}$$

or, substituting the values of $\xi$ $\eta$, $\zeta$ in terms of $X$, $Y$, $Z$,

$$\iint \left(m\frac{dX}{dz} - n\frac{dX}{dy} + n\frac{dY}{dx} - l\frac{dY}{dz} + l\frac{dZ}{dy} - m\frac{dZ}{dx}\right) dS. \tag{5}$$

The part of this which depends on $X$ may be written

$$\iint \left\{\frac{dX}{dz}\left(\frac{dz}{da}\frac{dx}{d\beta} - \frac{dz}{d\beta}\frac{dx}{da}\right) - \frac{dX}{dy}\left(\frac{dx}{da}\frac{dy}{d\beta} - \frac{dx}{d\beta}\frac{dy}{da}\right)\right\} d\beta\, da; \tag{6}$$

adding and subtracting $\frac{dX}{dx}\frac{dx}{da}\frac{dx}{d\beta}$, this becomes

$$\iint \left\{\frac{dx}{d\beta}\left(\frac{dX}{dx}\frac{dx}{da} + \frac{dX}{dy}\frac{dy}{da} + \frac{dX}{dz}\frac{dz}{da}\right)\right.$$
$$\left. - \frac{dx}{da}\left(\frac{dX}{dx}\frac{dx}{d\beta} + \frac{dX}{dy}\frac{dy}{d\beta} + \frac{dX}{dz}\frac{dz}{d\beta}\right)\right\} d\beta\, da; \tag{7}$$

$$= \iint \left(\frac{dX}{da}\frac{dx}{d\beta} - \frac{dX}{d\beta}\frac{dx}{da}\right) d\beta\, da. \tag{8}$$

Let us now suppose that the curves for which $a$ is constant form a series of closed curves surrounding a point on the surface for which $a$ has its minimum value, $a_0$, and let the last curve of the series, for which $a = a_1$, coincide with the closed curve $s$.

Let us also suppose that the curves for which $\beta$ is constant form a series of lines drawn from the point at which $a = a_0$ to the closed curve $s$, the first, $\beta_0$, and the last, $\beta_1$, being identical.

Integrating (8) by parts, the first term with respect to $a$ and the second with respect to $\beta$, the double integrals destroy each other and the expression becomes

$$\int_{\beta_0}^{\beta_1} \left(X\frac{dx}{d\beta}\right)_{a=a_1} d\beta - \int_{\beta_0}^{\beta_1} \left(X\frac{dx}{d\beta}\right)_{a=a_0} d\beta - \int_{a_0}^{a_1} \left(X\frac{dx}{da}\right)_{\beta=\beta_1} da$$
$$+ \int_{a_0}^{a_1} \left(X\frac{dx}{da}\right)_{\beta=\beta_0} da. \tag{9}$$

Since the point $(a, \beta_1)$ is identical with the point $(a, \beta_0)$, the third and fourth terms destroy each other; and since there is but one value of $x$ at the point where $a = a_0$, the second term is zero, and the expression is reduced to the first term:

Since the curve $a = a_1$ is identical with the closed curve $s$, we may write the expression in the form

$$\int X \frac{dx}{ds} ds, \tag{10}$$

where the integration is to be performed round the curve $s$. We may treat in the same way the parts of the surface-integral which depend upon $Y$ and $Z$, so that we get finally,

$$\iint (l\xi + m\eta + n\zeta)\, dS = \int \left(X\frac{dx}{ds} + Y\frac{dy}{ds} + Z\frac{dz}{ds}\right) ds; \tag{11}$$

where the first integral is extended over the surface $S$, and the second round the bounding curve $s$ *.

*On the effect of the operator $\nabla$ on a vector function.*

25.] We have seen that the operation denoted by $\nabla$ is that by which a vector quantity is deduced from its potential. The same operation, however, when applied to a vector function, produces results which enter into the two theorems we have just proved (III and IV). The extension of this operator to vector displacements, and most of its further development, are due to Professor Tait †.

Let $\sigma$ be a vector function of $\rho$, the vector of a variable point. Let us suppose, as usual, that

$$\rho = ix + jy + kz,$$

and $$\sigma = iX + jY + kZ;$$

where $X$, $Y$, $Z$ are the components of $\sigma$ in the directions of the axes.

We have to perform on $\sigma$ the operation

$$\nabla = i\frac{d}{dx} + j\frac{d}{dy} + k\frac{d}{dz}.$$

Performing this operation, and remembering the rules for the multiplication of $i$, $j$, $k$, we find that $\nabla\sigma$ consists of two parts, one scalar and the other vector.

* This theorem was given by Professor Stokes, *Smith's Prize Examination*, 1854, question 8. It is proved in Thomson and Tait's *Natural Philosophy*, § 190 ($j$).

† See *Proc. R. S. Edin.*, April 28, 1862. 'On Green's and other allied Theorems,' *Trans. R. S. Edin.*, 1869–70, a very valuable paper; and 'On some Quaternion Integrals,' *Proc. R. S. Edin.*, 1870–71.

The scalar part is

$$S\nabla\sigma = -\left(\frac{dX}{dx} + \frac{dY}{dy} + \frac{dZ}{dz}\right), \text{ see Theorem III,}$$

and the vector part is

$$V\nabla\sigma = i\left(\frac{dZ}{dy} - \frac{dY}{dz}\right) + j\left(\frac{dX}{dz} - \frac{dZ}{dx}\right) + k\left(\frac{dY}{dx} - \frac{dX}{dy}\right).$$

If the relation between $X$, $Y$, $Z$ and $\xi$, $\eta$, $\zeta$ is that given by equation (1) of the last theorem, we may write

$$V\nabla\sigma = i\xi + j\eta + k\zeta. \quad \text{See Theorem IV.}$$

It appears therefore that the functions of $X$, $Y$, $Z$ which occur in the two theorems are both obtained by the operation $\nabla$ on the vector whose components are $X$, $Y$, $Z$. The theorems themselves may be written

$$\iiint S\nabla\sigma\, d\varsigma = \iint S\,.\,\sigma U\nu\, ds, \quad \text{(III)}$$

$$\text{and} \quad \int S\sigma\, d\rho = -\iint S\,.\,\nabla\sigma U\nu\, ds; \quad \text{(IV)}$$

where $d\varsigma$ is an element of a volume, $ds$ of a surface, $d\rho$ of a curve, and $U\nu$ a unit-vector in the direction of the normal.

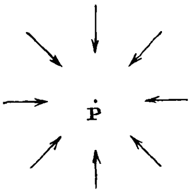


Fig. 1.

To understand the meaning of these functions of a vector, let us suppose that $\sigma_0$ is the value of $\sigma$ at a point $P$, and let us examine the value of $\sigma - \sigma_0$ in the neighbourhood of $P$. If we draw a closed surface round $P$, then, if the surface-integral of $\sigma$ over this surface is directed inwards, $S\nabla\sigma$ will be positive, and the vector $\sigma - \sigma_0$ near the point $P$ will be on the whole directed towards $P$, as in the figure (1).

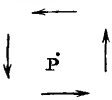


Fig. 2.

I propose therefore to call the scalar part of $\nabla\sigma$ the *convergence* of $\sigma$ at the point $P$.

To interpret the vector part of $\nabla\sigma$, let the direction of the vector whose components are $\xi$, $\eta$, $\zeta$ be upwards from the paper and at right angles to it, and let us examine the vector $\sigma - \sigma_0$ near the point $P$. It will appear as in the figure (2), this vector being arranged on the whole tangentially in the direction opposite to the hands of a watch.

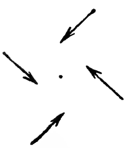


Fig. 3.

I propose (with great diffidence) to call the vector part of $\nabla\sigma$ the *rotation* of $\sigma$ at the point $P$.

In Fig. 3 we have an illustration of rotation combined with convergence.

Let us now consider the meaning of the equation

$$V\nabla\sigma = 0.$$

This implies that $\nabla\sigma$ is a scalar, or that the vector $\sigma$ is the space-variation of some scalar function $\Psi$.

26.] One of the most remarkable properties of the operator $\nabla$ is that when repeated it becomes

$$\nabla^2 = -\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}\right),$$

an operator occurring in all parts of Physics, which we may refer to as Laplace's Operator.

This operator is itself essentially scalar. When it acts on a scalar function the result is scalar, when it acts on a vector function the result is a vector.

If, with any point $P$ as centre, we draw a small sphere whose radius is $r$, then if $q_0$ is the value of $q$ at the centre, and $\bar{q}$ the mean value of $q$ for all points within the sphere,

$$q_0 - \bar{q} = \tfrac{1}{10} r^2 \nabla^2 q;$$

so that the value at the centre exceeds or falls short of the mean value according as $\nabla^2 q$ is positive or negative.

I propose therefore to call $\nabla^2 q$ the *concentration* of $q$ at the point $P$, because it indicates the excess of the value of $q$ at that point over its mean value in the neighbourhood of the point.

If $q$ is a scalar function, the method of finding its mean value is well known. If it is a vector function, we must find its mean value by the rules for integrating vector functions. The result of course is a vector.

# PART I.

## ELECTROSTATICS.

## CHAPTER I.

### DESCRIPTION OF PHENOMENA.

*Electrification by Friction.*

27.] EXPERIMENT I*. Let a piece of glass and a piece of resin, neither of which exhibits any electrical properties, be rubbed together and left with the rubbed surfaces in contact. They will still exhibit no electrical properties. Let them be separated. They will now attract each other.

If a second piece of glass be rubbed with a second piece of resin, and if the pieces be then separated and suspended in the neighbourhood of the former pieces of glass and resin, it may be observed—

(1) That the two pieces of glass repel each other.

(2) That each piece of glass attracts each piece of resin.

(3) That the two pieces of resin repel each other.

These phenomena of attraction and repulsion are called Electrical phenomena, and the bodies which exhibit them are said to be *electrified*, or to be *charged with electricity*.

Bodies may be electrified in many other ways, as well as by friction.

The electrical properties of the two pieces of glass are similar to each other but opposite to those of the two pieces of resin: the glass attracts what the resin repels and repels what the resin attracts.

* See Sir W. Thomson 'On the Mathematical Theory of Electricity in Equilibrium,' *Cambridge and Dublin Mathematical Journal*, March, 1848.

If a body electrified in any manner whatever behaves as the glass does, that is, if it repels the glass and attracts the resin, the body is said to be *vitreously* electrified, and if it attracts the glass and repels the resin it is said to be *resinously* electrified. All electrified bodies are found to be either vitreously or resinously electrified.

It is the established practice of men of science to call the vitreous electrification positive, and the resinous electrification negative. The exactly opposite properties of the two kinds of electrification justify us in indicating them by opposite signs, but the application of the positive sign to one rather than to the other kind must be considered as a matter of arbitrary convention, just as it is a matter of convention in mathematical diagrams to reckon positive distances towards the right hand.

No force, either of attraction or of repulsion, can be observed between an electrified body and a body not electrified. When, in any case, bodies not previously electrified are observed to be acted on by an electrified body, it is because they have become *electrified by induction.*

*Electrification by Induction.*

28.] EXPERIMENT II *. Let a hollow vessel of metal be hung up by white silk threads, and let a similar thread be attached to the lid of the vessel so that the vessel may be opened or closed without touching it.

Let the pieces of glass and resin be similarly suspended and electrified as before.

Let the vessel be originally unelectrified, then if an electrified piece of glass is hung up within it by its thread without touching the vessel, and the lid closed, the outside of the vessel will be found to be vitreously electrified, and it may be shewn that the electrification outside of the vessel is exactly the same in whatever part of the interior space the glass is suspended †.

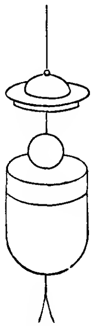


Fig. 4.

If the glass is now taken out of the vessel without touching it, the electrification of the glass will be the same as before it was put in, and that of the vessel will have disappeared.

* This, and several experiments which follow, are due to Faraday, 'On Static Electrical Inductive Action,' *Phil. Mag.*, 1843, or *Exp. Res.*, vol. ii. p. 279.

† {This is an illustration of Art. 100 *c.*}

This electrification of the vessel, which depends on the glass being within it, and which vanishes when the glass is removed, is called electrification by Induction.

Similar effects would be produced if the glass were suspended near the vessel on the outside, but in that case we should find an electrification, vitreous in one part of the outside of the vessel and resinous in another. When the glass is inside the vessel the whole of the outside is vitreously and the whole of the inside resinously electrified.

### *Electrification by Conduction.*

29.] EXPERIMENT III. Let the metal vessel be electrified by induction, as in the last experiment, let a second metallic body be suspended by white silk threads near it, and let a metal wire, similarly suspended, be brought so as to touch simultaneously the electrified vessel and the second body.

The second body will now be found to be vitreously electrified, and the vitreous electrification of the vessel will have diminished.

The electrical condition has been transferred from the vessel to the second body by means of the wire. The wire is called a *conductor* of electricity, and the second body is said to be *electrified by conduction.*

### *Conductors and Insulators.*

EXPERIMENT IV. If a glass rod, a stick of resin or gutta-percha, or a white silk thread, had been used instead of the metal wire, no transfer of electricity would have taken place. Hence these latter substances are called Non-conductors of electricity. Non-conductors are used in electrical experiments to support electrified bodies without carrying off their electricity. They are then called Insulators.

The metals are good conductors; air, glass, resins, gutta-percha, vulcanite, paraffin, &c. are good insulators; but, as we shall see afterwards, all substances resist the passage of electricity, and all substances allow it to pass, though in exceedingly different degrees. This subject will be considered when we come to treat of the motion of electricity. For the present we shall consider only two classes of bodies, good conductors, and good insulators.

In Experiment II an electrified body produced electrification in the metal vessel while separated from it by air, a non-conducting medium. Such a medium, considered as transmitting these electrical effects without conduction, has been called by Faraday a Dielectric medium, and the action which takes place through it is called Induction.

In Experiment III the electrified vessel produced electrification in the second metallic body through the medium of the wire. Let us suppose the wire removed, and the electrified piece of glass taken out of the vessel without touching it, and removed to a sufficient distance. The second body will still exhibit vitreous electrification, but the vessel, when the glass is removed, will have resinous electrification. If we now bring the wire into contact with both bodies, conduction will take place along the wire, and all electrification will disappear from both bodies, shewing that the electrification of the two bodies was equal and opposite.

30.] EXPERIMENT V. In Experiment II it was shewn that if a piece of glass, electrified by rubbing it with resin, is hung up in an insulated metal vessel, the electrification observed outside does not depend on the position of the glass. If we now introduce the piece of resin with which the glass was rubbed into the same vessel, without touching it or the vessel, it will be found that there is no electrification outside the vessel. From this we conclude that the electrification of the resin is exactly equal and opposite to that of the glass. By putting in any number of bodies, electrified in any way, it may be shewn that the electrification of the outside of the vessel is that due to the algebraic sum of all the electrifications, those being reckoned negative which are resinous. We have thus a practical method of adding the electrical effects of several bodies without altering their electrification.

31.] EXPERIMENT VI. Let a second insulated metallic vessel, *B*, be provided, and let the electrified piece of glass be put into the first vessel *A*, and the electrified piece of resin into the second vessel *B*. Let the two vessels be then put in communication by the metal wire, as in Experiment III. All signs of electrification will disappear.

Next, let the wire be removed, and let the pieces of glass and of resin be taken out of the vessels without touching them. It will be found that *A* is electrified resinously and *B* vitreously.

If now the glass and the vessel *A* be introduced together into a larger insulated metal vessel *C*, it will be found that there is no electrification outside $C$. This shews that the electrification of $A$ is exactly equal and opposite to that of the piece of glass, and that of $B$ may be shewn in the same way to be equal and opposite to that of the piece of resin.

We have thus obtained a method of charging a vessel with a quantity of electricity exactly equal and opposite to that of an electrified body without altering the electrification of the latter, and we may in this way charge any number of vessels with exactly equal quantities of electricity of either kind, which we may take for provisional units.

32.] EXPERIMENT VII. Let the vessel $B$, charged with a quantity of positive electricity, which we shall call, for the present, unity, be introduced into the larger insulated vessel $C$ without touching it. It will produce a positive electrification on the outside of $C$. Now let $B$ be made to touch the inside of $C$. No change of the external electrification will be observed. If $B$ is now taken out of $C$ without touching it, and removed to a sufficient distance, it will be found that $B$ is completely discharged, and that $C$ has become charged with a unit of positive electricity.

We have thus a method of transferring the charge of $B$ to $C$.

Let $B$ be now recharged with a unit of electricity, introduced into $C$ already charged, made to touch the inside of $C$, and removed. It will be found that $B$ is again completely discharged, so that the charge of $C$ is doubled.

If this process is repeated, it will be found that however highly $C$ is previously charged, and in whatever way $B$ is charged, when $B$ is first entirely enclosed in $C$, then made to touch $C$, and finally removed without touching $C$, the charge of $B$ is completely transferred to $C$, and $B$ is entirely free from electrification.

This experiment indicates a method of charging a body with any number of units of electricity. We shall find, when we come to the mathematical theory of electricity, that the result of this experiment affords an accurate test of the truth of the theory *.

* {The difficulties which would have to be overcome to make several of the preceding experiments conclusive are so great as to be almost insurmountable. Their description however serves to illustrate the properties of Electricity in a very striking way. In Experiment V no method is given by which the electrification of the outer vessel can be measured.}

33.] Before we proceed to the investigation of the law of electrical force, let us enumerate the facts we have already established.

By placing any electrified system inside an insulated hollow conducting vessel, and examining the resultant effect on the outside of the vessel, we ascertain the character of the total electrification of the system placed inside, without any communication of electricity between the different bodies of the system.

The electrification of the outside of the vessel may be tested with great delicacy by putting it in communication with an electroscope.

We may suppose the electroscope to consist of a strip of gold leaf hanging between two bodies charged, one positively, and the other negatively. If the gold leaf becomes electrified it will incline towards the body whose electrification is opposite to its own. By increasing the electrification of the two bodies and the delicacy of the suspension, an exceedingly small electrification of the gold leaf may be detected.

When we come to describe electrometers and multipliers we shall find that there are still more delicate methods of detecting electrification and of testing the accuracy of our theories, but at present we shall suppose the testing to be made by connecting the hollow vessel with a gold leaf electroscope.

This method was used by Faraday in his very admirable demonstration of the laws of electrical phenomena *.

34.] I. The total electrification of a body, or system of bodies, remains always the same, except in so far as it receives electrification from or gives electrification to other bodies.

In all electrical experiments the electrification of bodies is found to change, but it is always found that this change is due to want of perfect insulation, and that as the means of insulation are improved, the loss of electrification becomes less. We may therefore assert that the electrification of a body placed in a perfectly insulating medium would remain perfectly constant.

II. When one body electrifies another by conduction, the total electrification of the two bodies remains the same, that is, the one loses as much positive or gains as much negative

* 'On Static Electrical Inductive Action,' *Phil. Mag.*, 1843 or *Exp. Res.*, vol. ii. p. 279.

electrification as the other gains of positive or loses of negative electrification.

For if the two bodies are enclosed in the hollow vessel, no change of the total electrification is observed.

III. When electrification is produced by friction, or by any other known method, equal quantities of positive and negative electrification are produced.

For the electrification of the whole system may be tested in the hollow vessel, or the process of electrification may be carried on within the vessel itself, and however intense the electrification of the parts of the system may be, the electrification of the whole, as indicated by the gold leaf electroscope, is invariably zero.

The electrification of a body is therefore a physical quantity capable of measurement, and two or more electrifications can be combined experimentally with a result of the same kind as when two quantities are added algebraically. We therefore are entitled to use language fitted to deal with electrification as a quantity as well as a quality, and to speak of any electrified body as 'charged with a certain quantity of positive or negative electricity.'

35.] While admitting electricity, as we have now done, to the rank of a physical quantity, we must not too hastily assume that it is, or is not, a substance, or that it is, or is not, a form of energy, or that it belongs to any known category of physical quantities. All that we have hitherto proved is that it cannot be created or annihilated, so that if the total quantity of electricity within a closed surface is increased or diminished, the increase or diminution must have passed in or out through the closed surface.

This is true of matter, and is expressed by the equation known as the Equation of Continuity in Hydrodynamics.

It is not true of heat, for heat may be increased or diminished within a closed surface, without passing in or out through the surface, by the transformation of some other form of energy into heat, or of heat into some other form of energy.

It is not true even of energy in general if we admit the immediate action of bodies at a distance. For a body outside the closed surface may make an exchange of energy with a body within the surface. But if all apparent action at a distance is the result of the action between the parts of an intervening medium, it is conceivable that in all cases of the increase or diminution of the energy within a closed surface we may be able, when the nature of this action of the parts of the medium is clearly understood, to trace the passage of the energy in or out through that surface.

There is, however, another reason which warrants us in asserting that electricity, as a physical quantity, synonymous with the total electrification of a body, is not, like heat, a form of energy. An electrified system has a certain amount of energy, and this energy can be calculated by multiplying the quantity of electricity in each of its parts by another physical quantity, called the Potential of that part, and taking half the sum of the products. The quantities 'Electricity' and 'Potential,' when multiplied together, produce the quantity 'Energy.' It is impossible, therefore, that electricity and energy should be quantities of the same category, for electricity is only one of the factors of energy, the other factor being 'Potential.' *

Energy, which is the product of these factors, may also be considered as the product of several other pairs of factors, such as

| | |
|---|---|
| A Force | × A distance through which the force is to act. |
| A Mass | × Gravitation acting through a certain height. |
| A Mass | × Half the square of its velocity. |
| A Pressure | × A volume of fluid introduced into a vessel at that pressure. |
| A Chemical Affinity | × A chemical change, measured by the number of electro-chemical equivalents which enter into combination. |

If we ever should obtain distinct mechanical ideas of the nature of electric potential, we may combine these with the idea of energy to determine the physical category in which 'Electricity' is to be placed.

36.] In most theories on the subject, Electricity is treated as a substance, but inasmuch as there are two kinds of electrification which, being combined, annul each other, and since we cannot conceive of two substances annulling each other, a distinction has been drawn between Free Electricity and Combined Electricity.

* {It is shown afterwards that 'Potential' is not of zero dimensions.}

*Theory of Two Fluids.*

In what is called the Theory of Two Fluids, all bodies, in their unelectrified state, are supposed to be charged with equal quantities of positive and negative electricity. These quantities are supposed to be so great that no process of electrification has ever yet deprived a body of all the electricity of either kind. The process of electrification, according to this theory, consists in taking a certain quantity $P$ of positive electricity from the body $A$ and communicating it to $B$, or in taking a quantity $N$ of negative electricity from $B$ and communicating it to $A$, or in some combination of these processes.

The result will be that $A$ will have $P+N$ units of negative electricity over and above its remaining positive electricity, which is supposed to be in a state of combination with an equal quantity of negative electricity. This quantity $P+N$ is called the Free electricity, the rest is called the Combined, Latent, or Fixed electricity.

In most expositions of this theory the two electricities are called 'Fluids,' because they are capable of being transferred from one body to another, and are, within conducting bodies, extremely mobile. The other properties of fluids, such as their inertia, weight, and elasticity, are not attributed to them by those who have used the theory for merely mathematical purposes; but the use of the word Fluid has been apt to mislead the vulgar, including many men of science who are not natural philosophers, and who have seized on the word Fluid as the only term in the statement of the theory which seemed intelligible to them.

We shall see that the mathematical treatment of the subject has been greatly developed by writers who express themselves in terms of the 'Two Fluids' theory. Their results, however, have been deduced entirely from data which can be proved by experiment, and which must therefore be true, whether we adopt the theory of two fluids or not. The experimental verification of the mathematical results therefore is no evidence for or against the peculiar doctrines of this theory.

The introduction of two fluids permits us to consider the negative electrification of $A$ and the positive electrification of $B$ as the effect of *any one* of three different processes which would lead to the same result. We have already supposed it produced by the transfer of $P$ units of positive electricity from $A$ to $B$, together with the transfer of $N$ units of negative electricity from $B$ to $A$. But if $P+N$ units of positive electricity had been transferred from $A$ to $B$, or if $P+N$ units of negative electricity had been transferred from $B$ to $A$, the resulting 'free electricity' on $A$ and on $B$ would have been the same as before, but the quantity of 'combined electricity' in $A$ would have been less in the second case and greater in the third than it was in the first.

It would appear therefore, according to this theory, that it is possible to alter not only the amount of free electricity in a body, but the amount of combined electricity. But no phenomena have ever been observed in electrified bodies which can be traced to the varying amount of their combined electricities. Hence either the combined electricities have no observable properties, or the amount of the combined electricities is incapable of variation. The first of these alternatives presents no difficulty to the mere mathematician, who attributes no properties to the fluids except those of attraction and repulsion, for he conceives the two fluids simply to annul one another, like $+e$ and $-e$, and their combination to be a true mathematical zero. But to those who cannot use the word Fluid without thinking of a substance it is difficult to conceive how the combination of the two fluids can have no properties at all, so that the addition of more or less of the combination to a body shall not in any way affect it, either by increasing its mass or its weight, or altering some of its other properties. Hence it has been supposed by some, that in every process of electrification exactly equal quantities of the two fluids are transferred in opposite directions, so that the total quantity of the two fluids in any body taken together remains always the same. By this new law they 'contrive to save appearances,' forgetting that there would have been no need of the law except to reconcile the 'Two Fluids' theory with facts, and to prevent it from predicting non-existent phenomena.

### *Theory of One Fluid.*

37.] In the theory of One Fluid everything is the same as in the theory of Two Fluids except that, instead of supposing the two substances equal and opposite in all respects, one of them, generally the negative one, has been endowed with the properties and name of Ordinary Matter, while the other retains the name of The Electric Fluid. The particles of the fluid are supposed to repel one another according to the law of the inverse square of the distance, and to attract those of matter according to the same law. Those of matter are supposed to repel each other and attract those of electricity.

If the quantity of the electric fluid in a body is such that a particle of the electric fluid outside the body is as much repelled by the electric fluid in the body as it is attracted by the matter of the body, the body is said to be Saturated. If the quantity of fluid in the body is greater than that required for saturation, the excess is called the Redundant fluid, and the body is said to be Overcharged. If it is less, the body is said to be Undercharged, and the quantity of fluid which would be required to saturate it is sometimes called the Deficient fluid. The number of units of electricity required to saturate one gramme of ordinary matter must be very great, because a gramme of gold may be beaten out to an area of a square metre, and when in this form may have a negative charge of at least 60,000 units of electricity. In order to saturate the gold leaf when so charged, this quantity of electric fluid must be communicated to it, so that the whole quantity required to saturate it must be greater than this. The attraction between the matter and the fluid in two saturated bodies is supposed to be a very little greater than the repulsion between the two portions of matter and that between the two portions of fluid. This residual force is supposed to account for the attraction of gravitation.

This theory does not, like the Two Fluid theory, explain too much. It requires us, however, to suppose the mass of the electric fluid so small that no attainable positive or negative electrification has yet perceptibly increased or diminished either the mass or the weight of a body *, and it has not yet been able to assign sufficient reasons why the vitreous rather than the resinous electrification should be supposed due to an *excess* of electricity.

One objection has sometimes been urged against this theory by men who ought to have reasoned better. It has been said that the doctrine that the particles of matter uncombined with

* {The apparent mass of a body is increased by a charge of electricity whether vitreous or resinous (see *Phil. Mag.* 1861, v. xi. p. 229).}

electricity *repel* one another, is in direct antagonism with the well-established fact that every particle of matter *attracts* every other particle throughout the universe. If the theory of One Fluid were true we should have the heavenly bodies repelling one another.

It is manifest however that the heavenly bodies, according to this theory, if they consisted of matter uncombined with electricity, would be in the highest state of negative electrification, and would repel each other. We have no reason to believe that they are in such a highly electrified state, or could be maintained in that state. The earth and all the bodies whose attraction has been observed are rather in an unelectrified state, that is, they contain the normal charge of electricity, and the only action between them is the residual force lately mentioned. The artificial manner, however, in which this residual force is introduced is a much more valid objection to the theory.

In the present treatise I propose, at different stages of the investigation, to test the different theories in the light of additional classes of phenomena. For my own part, I look for additional light on the nature of electricity from a study of what takes place in the space intervening between the electrified bodies. Such is the essential character of the mode of investigation pursued by Faraday in his *Experimental Researches*, and as we go on I intend to exhibit the results, as developed by Faraday, W. Thomson, &c., in a connected and mathematical form, so that we may perceive what phenomena are explained equally well by all the theories, and what phenomena indicate the peculiar difficulties of each theory.

### *Measurement of the Force between Electrified Bodies.*

38.] Forces may be measured in various ways. For instance, one of the bodies may be suspended from one arm of a delicate balance, and weights suspended from the other arm, till the body, when unelectrified, is in equilibrium. The other body may then be placed at a known distance beneath the first, so that the attraction or repulsion of the bodies when electrified may increase or diminish the apparent weight of the first. The weight which must be added to or taken from the other arm, when expressed in dynamical measure, will measure the force between the bodies. This arrangement was used by Sir W. Snow Harris, and is that adopted in Sir W. Thomson's absolute electrometers. See Art. 217.

It is sometimes more convenient to use a torsion-balance, in which a horizontal arm is suspended by a fine wire or fibre, so as to be capable of vibrating about the vertical wire as an axis, and the body is attached to one end of the arm and acted on by the force in the tangential direction, so as to turn the arm round the vertical axis, and so twist the suspension wire through a certain angle. The torsional rigidity of the wire is found by observing the time of oscillation of the arm, the moment of inertia of the arm being otherwise known, and from the angle of torsion and the torsional rigidity the force of attraction or repulsion can be deduced. The torsion-balance was devised by Michell for the determination of the force of gravitation between small bodies, and was used by Cavendish for this purpose. Coulomb, working independently of these philosophers, reinvented it, thoroughly studied its action, and successfully applied it to discover the laws of electric and magnetic forces; and the torsion-balance has ever since been used in researches where small forces have to be measured. See Art. 215.

39.] Let us suppose that by either of these methods we can measure the force between two electrified bodies. We shall suppose the dimensions of the bodies small compared with the distance between them, so that the result may not be much altered by any inequality of distribution of the electrification on either body, and we shall suppose that both bodies are so suspended in air as to be at a considerable distance from other bodies on which they might induce electrification.

It is then found that if the bodies are placed at a fixed distance and charged respectively with $e$ and $e'$ of our provisional units of electricity, they will repel each other with a force proportional to the product of $e$ and $e'$. If either $e$ or $e'$ is negative, that is, if one of the charges is vitreous and the other resinous, the force will be attractive, but if both $e$ and $e'$ are negative the force is again repulsive.

We may suppose the first body, $A$, charged with $m$ units of positive and $n$ units of negative electricity, which may be conceived separately placed within the body, as in Experiment V.

Let the second body, $B$, be charged with $m'$ units of positive and $n'$ units of negative electricity.

Then each of the $m$ positive units in $A$ will repel each of the $m'$ positive units in $B$ with a certain force, say $f$, making a total effect equal to $mm'f$.

Since the effect of negative electricity is exactly equal and opposite to that of positive electricity, each of the $m$ positive units in $A$ will attract each of the $n'$ negative units in $B$ with the same force $f$, making a total effect equal to $mn'f$.

Similarly the $n$ negative units in $A$ will attract the $m'$ positive units in $B$ with a force $nm'f$, and will repel the $n'$ negative units in $B$ with a force $nn'f$.

The total repulsion will therefore be $(mm'+nn')f$; and the total attraction will be $(mn'+m'n)f$.

The resultant repulsion will be

$$(mm'+nn'-mn'-nm')f \quad \text{or} \quad (m-n)(m'-n')f.$$

Now $m-n=e$ is the algebraical value of the charge on $A$, and $m'-n'=e'$ is that of the charge on $B$, so that the resultant repulsion may be written $ee'f$, the quantities $e$ and $e'$ being always understood to be taken with their proper signs.

### *Variation of the Force with the Distance.*

40.] Having established the law of force at a fixed distance, we may measure the force between bodies charged in a constant manner and placed at different distances. It is found by direct measurement that the force, whether of attraction or repulsion, varies inversely as the square of the distance, so that if $f$ is the repulsion between two units at unit distance, the repulsion at distance $r$ will be $fr^{-2}$, and the general expression for the repulsion between $e$ units and $e'$ units at distance $r$ will be

$$fee'r^{-2}.$$

### *Definition of the Electrostatic Unit of Electricity.*

41.] We have hitherto used a wholly arbitrary standard for our unit of electricity, namely, the electrification of a certain piece of glass as it happened to be electrified at the commencement of our experiments. We are now able to select a unit on a definite principle, and in order that this unit may belong to a general system we define it so that $f$ may be unity, or in other words—

*The electrostatic unit of electricity is that quantity of positive* *electricity which, when placed at unit of distance from an equal quantity, repels it with unit of force**.

This unit is called the Electrostatic unit to distinguish it from the Electromagnetic unit, to be afterwards defined.

We may now write the general law of electrical action in the simple form $$F = ee' r^{-2}; \quad \text{or,}$$

*The repulsion between two small bodies charged respectively with e and e′ units of electricity is numerically equal to the product of the charges divided by the square of the distance.*

*Dimensions of the Electrostatic Unit of Quantity.*

42.] If $[Q]$ is the concrete electrostatic unit of quantity itself, and $e$, $e'$ the numerical values of particular quantities; if $[L]$ is the unit of length, and $r$ the numerical value of the distance; and if $[F]$ is the unit of force, and $F$ the numerical value of the force, then the equation becomes

$$F[F] = ee'r^{-2}[Q^2][L^{-2}];$$

whence

$$[Q] = [LF^{\frac{1}{2}}]$$
$$= [L^{\frac{3}{2}}T^{-1}M^{\frac{1}{2}}].$$

This unit is called the Electrostatic Unit of electricity. Other units may be employed for practical purposes, and in other departments of electrical science, but in the equations of electrostatics quantities of electricity are understood to be estimated in electrostatic units, just as in physical astronomy we employ a unit of mass which is founded on the phenomena of gravitation, and which differs from the units of mass in common use.

*Proof of the Law of Electrical Force.*

43.] The experiments of Coulomb with the torsion-balance may be considered to have established the law of force with a certain approximation to accuracy. Experiments of this kind, however, are rendered difficult, and in some degree uncertain, by several disturbing causes, which must be carefully traced and corrected for.

In the first place, the two electrified bodies must be of sensible dimensions relative to the distance between them, in order to be

* {In this definition and in the enunciation of the law of electrical action the medium surrounding the electrified bodies is supposed to be air. See Art. 94.}

capable of carrying charges sufficient to produce measurable forces. The action of each body will then produce an effect on the distribution of electricity on the other, so that the charge cannot be considered as evenly distributed over the surface, or collected at the centre of gravity; but its effect must be calculated by an intricate investigation. This, however, has been done as regards two spheres by Poisson in an extremely able manner, and the investigation has been greatly simplified by Sir W. Thomson in his *Theory of Electrical Images*. See Arts. 172–175.

Another difficulty arises from the action of the electricity induced on the sides of the case containing the instrument. By making the inner surface of the instrument of metal, this effect can be rendered definite and measurable.

An independent difficulty arises from the imperfect insulation of the bodies, on account of which the charge continually decreases. Coulomb investigated the law of dissipation, and made corrections for it in his experiments.

The methods of insulating charged conductors, and of measuring electrical effects, have been greatly improved since the time of Coulomb, particularly by Sir W. Thomson; but the perfect accuracy of Coulomb's law of force is established, not by any direct experiments and measurements (which may be used as illustrations of the law), but by a mathematical consideration of the phenomenon described as Experiment VII, namely, that an electrified conductor $B$, if made to touch the inside of a hollow closed conductor $C$ and then withdrawn without touching $C$, is perfectly discharged, in whatever manner the outside of $C$ may be electrified. By means of delicate electroscopes it is easy to shew that no electricity remains on $B$ after the operation, and by the mathematical theory given at Arts. 74 $c$, 74 $d$, this can only be the case if the force varies inversely as the square of the distance, for if the law were of any different form $B$ would be electrified.

## *The Electric Field.*

44.] The Electric Field is the portion of space in the neighbourhood of electrified bodies, considered with reference to electric phenomena. It may be occupied by air or other bodies, or it may be a so-called vacuum, from which we have withdrawn every substance which we can act upon with the means at our disposal.

If an electrified body be placed at any part of the electric field it will, in general, produce a sensible disturbance in the electrification of the other bodies.

But if the body is very small, and its charge also very small, the electrification of the other bodies will not be sensibly disturbed, and we may consider the position of the body as determined by its centre of mass. The force acting on the body will then be proportional to its charge, and will be reversed when the charge is reversed.

Let $e$ be the charge of the body, and $F$ the force acting on the body in a certain direction, then when $e$ is very small $F$ is proportional to $e$, or

$$F = Re,$$

where $R$ depends on the distribution of electricity on the other bodies in the field. If the charge $e$ could be made equal to unity without disturbing the electrification of other bodies we should have $F = R$.

We shall call $R$ the Resultant Electromotive Intensity at the given point of the field. When we wish to express the fact that this quantity is a vector we shall denote it by the German letter $\mathfrak{E}$.

*Total Electromotive Force and Potential.*

45.] If the small body carrying the small charge $e$ be moved from one given point, $A$, to another $B$, along a given path, it will experience at each point of its course a force $Re$, where $R$ varies from point to point of the course. Let the whole work done on the body by the electrical force be $Ee$, then $E$ is called the Total Electromotive Force along the path $AB$. If the path forms a complete circuit, and if the total electromotive force round the circuit does not vanish, the electricity cannot be in equilibrium but a current will be produced. Hence in Electrostatics the total electromotive force round any closed circuit must be zero, so that if $A$ and $B$ are two points on the circuit, the total electromotive force from $A$ to $B$ is the same along either of the two paths into which the circuit is broken, and since either of these can be altered independently of the other, the total electromotive force from $A$ to $B$ is the same for all paths from $A$ to $B$.

If $B$ is taken as a point of reference for all other points, then the total electromotive force from $A$ to $B$ is called the Potential of $A$.

It depends only on the position of $A$. In mathematical investigations, $B$ is generally taken at an infinite distance from the electrified bodies.

A body charged positively tends to move from places of greater positive potential to places of smaller positive, or of negative, potential, and a body charged negatively tends to move in the opposite direction.

In a conductor the electrification is free to move relatively to the conductor. If therefore two parts of a conductor have different potentials, positive electricity will move from the part having greater potential to the part having less potential as long as that difference continues. A conductor therefore cannot be in electrical equilibrium unless every point in it has the same potential. This potential is called the Potential of the Conductor.

### *Equipotential Surfaces.*

46.] If a surface described or supposed to be described in the electric field is such that the electric potential is the same at every point of the surface it is called an Equipotential surface.

An electrified particle constrained to rest upon such a surface will have no tendency to move from one part of the surface to another, because the potential is the same at every point. An equipotential surface is therefore a surface of equilibrium or a level surface.

The resultant force at any point of the surface is in the direction of the normal to the surface, and the magnitude of the force is such that the work done on an electrical unit in passing from the surface $V$ to the surface $V'$ is $V-V'$.

No two equipotential surfaces having different potentials can meet one another, because the same point cannot have more than one potential, but one equipotential surface may meet itself, and this takes place at all points and along all lines of equilibrium.

The surface of a conductor in electrical equilibrium is necessarily an equipotential surface. If the electrification of the conductor is positive over the whole surface, then the potential will diminish as we move away from the surface on every side, and the conductor will be surrounded by a series of surfaces of lower potential.

But if (owing to the action of external electrified bodies) some regions of the conductor are charged positively and others negatively, the complete equipotential surface will consist of the surface of the conductor itself together with a system of other surfaces, meeting the surface of the conductor in the lines which divide the positive from the negative regions *. These lines will be lines of equilibrium, and an electrified particle placed on one of these lines will experience no force in any direction.

When the surface of a conductor is charged positively in some parts and negatively in others, there must be some other electrified body in the field besides itself. For if we allow a positively electrified particle, starting from a positively charged part of the surface, to move always in the direction of the resultant force upon it, the potential at the particle will continually diminish till the particle reaches either a negatively charged surface at a potential less than that of the first conductor, or moves off to an infinite distance. Since the potential at an infinite distance is zero, the latter case can only occur when the potential of the conductor is positive.

In the same way a negatively electrified particle, moving off from a negatively charged part of the surface, must either reach a positively charged surface, or pass off to infinity, and the latter case can only happen when the potential of the conductor is negative.

Therefore, if both positive and negative charges exist on a conductor, there must be some other body in the field whose potential has the same sign as that of the conductor but a greater numerical value, and if a conductor of any form is alone in the field the charge of every part is of the same sign as the potential of the conductor.

The interior surface of a hollow conducting vessel containing no charged bodies is entirely free from charge. For if any part of the surface were charged positively, a positively electrified particle moving in the direction of the force upon it, must reach a negatively charged surface at a lower potential. But the whole interior surface has the same potential. Hence it can have no charge †.

* {See Arts. 80, 114.}

† {To make the proof rigid it is necessary to state that by Art. 80 the force cannot vanish where the surface is charged, and that by Art. 112 the potential cannot have a maximum or minimum value at a point where there is no electrification.}

A conductor placed inside the vessel and communicating with it, may be considered as bounded by the interior surface. Hence such a conductor has no charge.

### *Lines of Force.*

47.] The line described by a point moving always in the direction of the resultant intensity is called a Line of Force. It cuts the equipotential surfaces at right angles. The properties of lines of force will be more fully explained afterwards, because Faraday has expressed many of the laws of electrical action in terms of his conception of lines of force drawn in the electric field, and indicating both the direction and the intensity at every point.

### *Electric Tension.*

48.] Since the surface of a conductor is an equipotential surface, the resultant intensity is normal to the surface, and it will be shewn in Art. 80 that it is proportional to the superficial density of the electrification. Hence the electricity on any small area of the surface will be acted on by a force tending *from* the conductor and proportional to the product of the resultant intensity and the density, that is, proportional to the square of the resultant intensity.

This force, which acts outwards as a tension on every part of the conductor, will be called electric Tension. It is measured like ordinary mechanical tension, by the force exerted on unit of area.

The word Tension has been used by electricians in several vague senses, and it has been attempted to adopt it in mathematical language as a synonym for Potential; but on examining the cases in which the word has been used, I think it will be more consistent with usage and with mechanical analogy to understand by tension a pulling force of so many pounds weight per square inch exerted on the surface of a conductor or elsewhere. We shall find that the conception of Faraday, that this electric tension exists not only at the electrified surface but all along the lines of force, leads to a theory of electric action as a phenomenon of stress in a medium.

### *Electromotive Force.*

49.] When two conductors at different potentials are connected by a thin conducting wire, the tendency of electricity to flow along the wire is measured by the difference of the potentials of the two bodies. The difference of potentials between two conductors or two points is therefore called the Electromotive force between them.

Electromotive force cannot in all cases be expressed in the form of a difference of potentials. These cases, however, are not treated of in Electrostatics. We shall consider them when we come to heterogeneous circuits, chemical actions, motions of magnets, inequalities of temperature, &c.

### *Capacity of a Conductor.*

50.] If one conductor is insulated while all the surrounding conductors are kept at the zero potential by being put in communication with the earth, and if the conductor, when charged with a quantity $E$ of electricity, has a potential $V$, the ratio of $E$ to $V$ is called the Capacity of the conductor. If the conductor is completely enclosed within a conducting vessel without touching it, then the charge on the inner conductor will be equal and opposite to the charge on the inner surface of the outer conductor, and will be equal to the capacity of the inner conductor multiplied by the difference of the potentials of the two conductors.

### *Electric Accumulators.*

A system consisting of two conductors whose opposed surfaces are separated from each other by a thin stratum of an insulating medium is called an electric Accumulator. The two conductors are called the Electrodes and the insulating medium is called the Dielectric. The capacity of the accumulator is directly proportional to the area of the opposed surfaces and inversely proportional to the thickness of the stratum between them. A Leyden jar is an accumulator in which glass is the insulating medium. Accumulators are sometimes called Condensers, but I prefer to restrict the term 'condenser' to an instrument which is used not to hold electricity but to increase its superficial density.

## PROPERTIES OF BODIES IN RELATION TO STATICAL ELECTRICITY.

### *Resistance to the Passage of Electricity through a Body.*

51.] When a charge of electricity is communicated to any part of a mass of metal the electricity is rapidly transferred from places of high to places of low potential till the potential of the whole mass becomes the same. In the case of pieces of metal used in ordinary experiments this process is completed in a time too short to be observed, but in the case of very long and thin wires, such as those used in telegraphs, the potential does not become uniform till after a sensible time, on account of the resistance of the wire to the passage of electricity through it.

The resistance to the passage of electricity is exceedingly different in different substances, as may be seen from the tables at Arts. 362, 364, and 367, which will be explained in treating of Electric Currents.

All the metals are good conductors, though the resistance of lead is 12 times that of copper or silver, that of iron 6 times, and that of mercury 60 times that of copper. The resistance of all metals increases as their temperature rises.

Many liquids conduct electricity by electrolysis. This mode of conduction will be considered in Part II. For the present, we may regard all liquids containing water and all damp bodies as conductors, far inferior to the metals but incapable of insulating a charge of electricity for a sufficient time to be observed. The resistance of electrolytes diminishes as the temperature rises.

On the other hand, the gases at the atmospheric pressure, whether dry or moist, are insulators so nearly perfect when the electric tension is small that we have as yet obtained no evidence of electricity passing through them by ordinary conduction. The gradual loss of charge by electrified bodies may in every case be traced to imperfect insulation in the supports, the electricity either passing through the substance of the support or creeping over its surface. Hence, when two charged bodies are hung up near each other, they will preserve their charges longer if they are electrified in opposite ways, than if they are electrified in the same way. For though the electromotive force tending to make the electricity pass through the air between them is much greater when they are oppositely electrified, no perceptible loss occurs in this way. The actual loss takes place through the supports, and the electromotive force through the supports is greatest when the bodies are electrified in the same way. The result appears anomalous only when we expect the loss to occur by the passage of electricity through the air between the bodies. The passage of electricity through gases takes place, in general, by disruptive discharge, and does not begin till the electromotive intensity has reached a certain value. The value of the electromotive intensity which can exist in a dielectric without a discharge taking place is called the Electric Strength of the dielectric. The electric strength of air diminishes as the pressure is reduced from the atmospheric pressure to that of about three millimetres of mercury *. When the pressure is still further reduced, the electric strength rapidly increases; and when the exhaustion is carried to the highest degree hitherto attained, the electromotive intensity required to produce a spark of a quarter of an inch is greater than that which will give a spark of eight inches in air at the ordinary pressure.

A vacuum, that is to say, that which remains in a vessel after we have removed everything which we can remove from it, is therefore an insulator of very great electric strength.

The electric strength of hydrogen is much less than that of air at the same pressure.

Certain kinds of glass when cold are marvellously perfect insulators, and Sir W. Thomson has preserved charges of electricity for years in bulbs hermetically sealed. The same glass, however, becomes a conductor at a temperature below that of boiling water.

Gutta-percha, caoutchouc, vulcanite, paraffin, and resins are good insulators, the resistance of gutta-percha at 75° F. being about $6 \times 10^{19}$ times that of copper.

Ice, crystals, and solidified electrolytes, are also insulators.

Certain liquids, such as naphtha, turpentine, and some oils, are insulators, but inferior to the best solid insulators.

## DIELECTRICS.

### *Specific Inductive Capacity.*

52.] All bodies whose insulating power is such that when they are placed between two conductors at different potentials the electromotive force acting on them does not immediately distribute their electricity so as to reduce the potential to a constant value, are called by Faraday Dielectrics.

It appears from the hitherto unpublished researches of Cavendish † that he had, before 1773, measured the capacity of plates of glass, resin, bees-wax, and shellac, and had determined

* {The pressure at which the electric strength is a minimum depends on the shape and size of the vessel in which the gas is contained.}

† {See *Electrical Researches of the Honourable Henry Cavendish.*}

the ratios in which their capacities exceeded that of plates of air of the same dimensions.

Faraday, to whom these researches were unknown, discovered that the capacity of an accumulator depends on the nature of the insulating medium between the two conductors, as well as on the dimensions and relative position of the conductors themselves. By substituting other insulating media for air as the dielectric of the accumulator, without altering it in any other respect, he found that when air and other gases were employed as the insulating medium the capacity of the accumulator remained sensibly the same, but that when shellac, sulphur, glass, &c. were substituted for air, the capacity was increased in a ratio which was different for each substance.

By a more delicate method of measurement Boltzmann succeeded in observing the variation of the inductive capacities of gases at different pressures.

This property of dielectrics, which Faraday called Specific Inductive Capacity, is also called the Dielectric Constant of the substance. It is defined as the ratio of the capacity of an accumulator when its dielectric is the given substance, to its capacity when the dielectric is a vacuum.

If the dielectric is not a good insulator, it is difficult to measure its inductive capacity, because the accumulator will not hold a charge for a sufficient time to allow it to be measured; but it is certain that inductive capacity is a property not confined to good insulators, and it is probable that it exists in all bodies *.

## *Absorption of Electricity.*

53.] It is found that when an accumulator is formed of certain dielectrics, the following phenomena occur.

When the accumulator has been for some time electrified and is then suddenly discharged and again insulated, it becomes recharged in the same sense as at first, but to a smaller degree, so that it may be discharged again several times in succession, these discharges always diminishing. This phenomenon is called that of the Residual Discharge.

* {Cohn and Arons (*Wiedemann's Annalen*, v. 33, p. 13) have investigated the specific inductive capacities of some non-insulating fluids such as water and alcohol: they find that these are very large; thus, that of distilled water is about 76 and that of ethyl alcohol about 26 times that of air.}

The instantaneous discharge appears always to be proportional to the difference of potentials at the instant of discharge, and the ratio of these quantities is the true capacity of the accumulator; but if the contact of the discharger is prolonged so as to include some of the residual discharge, the apparent capacity of the accumulator, calculated from such a discharge, will be too great.

The accumulator if charged and left insulated appears to lose its charge by conduction, but it is found that the proportionate rate of loss is much greater at first than it is afterwards, so that the measure of conductivity, if deduced from what takes place at first, would be too great. Thus, when the insulation of a submarine cable is tested, the insulation appears to improve as the electrification continues.

Thermal phenomena of a kind at first sight analogous take place in the case of the conduction of heat when the opposite sides of a body are kept at different temperatures. In the case of heat we know that they depend on the heat taken in and given out by the body itself. Hence, in the case of the electrical phenomena, it has been supposed that electricity is absorbed and emitted by the parts of the body. We shall see, however, in Art. 329, that the phenomena can be explained without the hypothesis of absorption of electricity, by supposing the dielectric in some degree heterogeneous.

That the phenomena called Electric Absorption are not an actual absorption of electricity by the substance may be shewn by charging the substance in any manner with electricity while it is surrounded by a closed metallic insulated vessel. If, when the substance is charged and insulated, the vessel be instantaneously discharged and then left insulated, no charge is ever communicated to the vessel by the gradual dissipation of the electrification of the charged substance within it *.

54.] This fact is expressed by the statement of Faraday that it is impossible to charge matter with an absolute and independent charge of one kind of electricity †.

In fact it appears from the result of every experiment which has been tried that in whatever way electrical actions may take

* {For a detailed account of the phenomena of Electric absorption, see *Wiedemann's Elektricität*, v. 2, p. 83.}

† *Exp. Res.*, vol. i. series xi. ¶ ii. 'On the Absolute Charge of Matter,' and § 1244.

place among a system of bodies surrounded by a metallic vessel, the charge on the outside of that vessel is not altered.

Now if any portion of electricity could be forced into a body so as to be absorbed in it, or to become latent, or in any way to exist in it, without being connected with an equal portion of the opposite electricity by lines of induction, or if, after having been absorbed, it could gradually emerge and return to its ordinary mode of action, we should find some change of electrification in the surrounding vessel.

As this is never found to be the case, Faraday concluded that it is impossible to communicate an absolute charge to matter, and that no portion of matter can by any change of state evolve or render latent one kind of electricity or the other. He therefore regarded induction as 'the essential function both in the first development and the consequent phenomena of electricity.' His 'induction' is (1298) a polarized state of the particles of the dielectric, each particle being positive on one side and negative on the other, the positive and the negative electrification of each particle being always exactly equal.

## *Disruptive Discharge.**

55.] If the electromotive intensity at any point of a dielectric is gradually increased, a limit is at length reached at which there is a sudden electrical discharge through the dielectric, generally accompanied with light and sound, and with a temporary or permanent rupture of the dielectric.

The electromotive intensity when this takes place is a measure of what we may call the electric strength of the dielectric. It depends on the nature of the dielectric, and is greater in dense air than in rare air, and greater in glass than in air, but in every case, if the electromotive force be made great enough, the dielectric gives way and its insulating power is destroyed, so that a current of electricity takes place through it. It is for this reason that distributions of electricity for which the electromotive intensity becomes anywhere infinite cannot exist.

* See Faraday, *Exp. Res.*, vol. i., series xii. and xiii.
{So many investigations have been made on the passage of electricity through gases since the first edition of this book was published that the mere enumeration of them would stretch beyond the limits of a foot-note. A summary of the results obtained by these researches will be given in the Supplementary Volume.}

*The Electric Glow.*

Thus, when a conductor having a sharp point is electrified, the theory, based on the hypothesis that it retains its charge, leads to the conclusion that as we approach the point the superficial density of the electricity increases without limit, so that at the point itself the surface-density, and therefore the resultant electromotive intensity, would be infinite. If the air, or other surrounding dielectric, had an invincible insulating power, this result would actually occur; but the fact is, that as soon as the resultant intensity in the neighbourhood of the point has reached a certain limit, the insulating power of the air gives way, so that the air close to the point becomes a conductor. At a certain distance from the point the resultant intensity is not sufficient to break through the insulation of the air, so that the electric current is checked, and the electricity accumulates in the air round the point.

The point is thus surrounded by particles of air * charged with electricity of the same kind as its own. The effect of this charged air round the point is to relieve the air at the point itself from part of the enormous electromotive intensity which it would have experienced if the conductor alone had been electrified. In fact the surface of the electrified body is no longer pointed, because the point is enveloped by a rounded mass of charged air, the surface of which, rather than that of the solid conductor, may be regarded as the outer electrified surface.

If this portion of charged air could be kept still, the electrified body would retain its charge, if not on itself at least in its neighbourhood, but the charged particles of air being free to move under the action of electrical force, tend to move away from the electrified body because it is charged with the same kind of electricity. The charged particles of air therefore tend to move off in the direction of the lines of force and to approach those surrounding bodies which are oppositely electrified. When they are gone, other uncharged particles take their place round the point, and since these cannot shield those next the point itself from the excessive electric tension, a new discharge takes place, after which the newly charged particles move off, and so on as long as the body remains electrified.

* {Or dust? It is doubtful whether air free from dust and aqueous vapour can be electrified except at very high temperatures; see Supplementary Volume.}

In this way the following phenomena are produced:—At and close to the point there is a steady glow, arising from the constant discharges which are taking place between the point and the air very near it.

The charged particles of air tend to move off in the same general direction, and thus produce a current of air from the point, consisting of the charged particles, and probably of others carried along by them. By artificially aiding this current we may increase the glow, and by checking the formation of the current we may prevent the continuance of the glow *.

The electric wind in the neighbourhood of the point is sometimes very rapid, but it soon loses its velocity, and the air with its charged particles is carried about with the general motions of the atmosphere, and constitutes an invisible electric cloud. When the charged particles come near to any conducting surface, such as a wall, they induce on that surface a charge opposite to their own, and are then attracted towards the wall, but since the electromotive force is small they may remain for a long time near the wall without being drawn up to the surface and discharged. They thus form an electrified atmosphere clinging to conductors, the presence of which may sometimes be detected by the electrometer. The electrical forces, however, acting between large masses of charged air and other bodies are exceedingly feeble compared with the ordinary forces which produce winds, and which depend on inequalities of density due to differences of temperature, so that it is very improbable that any observable part of the motion of ordinary thunder clouds arises from electrical causes.

The passage of electricity from one place to another by the motion of charged particles is called Electrical Convection or Convective Discharge.

The electrical glow is therefore produced by the constant passage of electricity through a small portion of air in which the tension is very high, so as to charge the surrounding particles of air which are continually swept off by the electric wind, which is an essential part of the phenomenon.

The glow is more easily formed in rare air than in dense air, and more easily when the point is positive than when it is negative.

* See Priestley's *History of Electricity*, pp. 117 and 591; and Cavendish's 'Electrical Researches,' *Phil. Trans.*, 1771, § 4, or Art. 125 of *Electrical Researches of the Honourable Henry Cavendish.*

This and many other differences between positive and negative electrification must be studied by those who desire to discover something about the nature of electricity. They have not, however, been satisfactorily brought to bear upon any existing theory.

### *The Electric Brush.*

56.] The electric brush is a phenomenon which may be produced by electrifying a blunt point or small ball so as to produce an electric field in which the tension diminishes as the distance increases, but in a less rapid manner than when a sharp point is used. It consists of a succession of discharges, ramifying as they diverge from the ball into the air, and terminating either by charging portions of air or by reaching some other conductor. It is accompanied by a sound, the pitch of which depends on the interval between the successive discharges, and there is no current of air as in the case of the glow.

### *The Electric Spark.*

57.] When the tension in the space between two conductors is considerable all the way between them, as in the case of two balls whose distance is not great compared with their radii, the discharge, when it occurs, usually takes the form of a spark, by which nearly the whole electrification is discharged at once.

In this case, when any part of the dielectric has given way, the parts on either side of it in the direction of the electric force are put into a state of greater tension so that they also give way, and so the discharge proceeds right through the dielectric, just as when a little rent is made in the edge of a piece of paper a tension applied to the paper in the direction of the edge causes the paper to be torn through, beginning at the rent, but diverging occasionally where there are weak places in the paper. The electric spark in the same way begins at the point where the electric tension first overcomes the insulation of the dielectric, and proceeds from that point, in an apparently irregular path, so as to take in other weak points, such as particles of dust floating in air.

All these phenomena differ considerably in different gases, and in the same gas at different densities. Some of the forms of electrical discharge through rare gases are exceedingly remarkable. In some cases there is a regular alternation of luminous and dark strata, so that if the electricity, for example, is passing along a tube containing a very small quantity of gas, a number of luminous disks will be seen arranged transversely at nearly equal intervals along the axis of the tube and separated by dark strata. If the strength of the current be increased a new disk will start into existence, and it and the old disks will arrange themselves in closer order. In a tube described by Mr. Gassiot * the light of each of the disks is bluish on the negative and reddish on the positive side, and bright red in the central stratum.

These, and many other phenomena of electrical discharge, are exceedingly important, and when they are better understood they will probably throw great light on the nature of electricity as well as on the nature of gases and of the medium pervading space. At present, however, they must be considered as outside the domain of the mathematical theory of electricity.

### *Electric Phenomena of Tourmaline* †.

58.] Certain crystals of tourmaline, and of other minerals, possess what may be called Electric Polarity. Suppose a crystal of tourmaline to be at a uniform temperature, and apparently free from electrification on its surface. Let its temperature be now raised, the crystal remaining insulated. One end will be found positively and the other end negatively electrified. Let the surface be deprived of this apparent electrification by means of a flame or otherwise, then if the crystal be made still hotter, electrification of the same kind as before will appear, but if the crystal be cooled the end which was positive when the crystal was heated will become negative.

These electrifications are observed at the extremities of the crystallographic axis. Some crystals are terminated by a six-sided pyramid at one end and by a three-sided pyramid at the other. In these the end having the six-sided pyramid becomes positive when the crystal is heated.

Sir W. Thomson supposes every portion of these and other hemihedral crystals to have a definite electric polarity, the intensity of which depends on the temperature. When the surface is passed through a flame, every part of the surface becomes electrified to such an extent as to exactly neutralize,

* *Intellectual Observer*, March 1866.

† {For a fuller account of this property and the electrification of crystals by radiant light and heat, see *Wiedemann's Elektricität*, v. 2, p. 316.}

for all external points, the effect of the internal polarity. The crystal then has no external electrical action, nor any tendency to change its mode of electrification. But if it be heated or cooled the interior polarization of each particle of the crystal is altered, and can no longer be balanced by the superficial electrification, so that there is a resultant external action.

### *Plan of this Treatise.*

59.] In the following treatise I propose first to explain the ordinary theory of electrical action, which considers it as depending only on the electrified bodies and on their relative position, without taking account of any phenomena which may take place in the intervening media. In this way we shall establish the law of the inverse square, the theory of the potential, and the equations of Laplace and Poisson. We shall next consider the charges and potentials of a system of electrified conductors as connected by a system of equations, the coefficients of which may be supposed to be determined by experiment in those cases in which our present mathematical methods are not applicable, and from these we shall determine the mechanical forces acting between the different electrified bodies.

We shall then investigate certain general theorems by which Green, Gauss, and Thomson have indicated the conditions of solution of problems in the distribution of electricity. One result of these theorems is, that if Poisson's equation is satisfied by any function, and if at the surface of every conductor the function has the value of the potential of that conductor, then the function expresses the actual potential of the system at every point. We also deduce a method of finding problems capable of exact solution.

In Thomson's theorem, the total energy of the system is expressed in the form of the integral of a certain quantity extended over the whole space between the electrified bodies, and also in the form of an integral extended over the electrified surfaces only. The equality of these two expressions may be thus interpreted physically. We may conceive the physical relation between the electrified bodies, either as the result of the state of the intervening medium, or as the result of a direct action between the electrified bodies at a distance. If we adopt the latter conception, we may determine the law of the action, but we can go no further in speculating on its cause. If, on the other hand, we adopt the conception of action through a medium, we are led to enquire into the nature of that action in each part of the medium.

It appears from the theorem, that if we are to look for the seat of the electric energy in the different parts of the dielectric medium, the amount of energy in any small part must depend on the square of the resultant electromotive intensity at that place multiplied by a coefficient called the specific inductive capacity of the medium.

It is better, however, in considering the theory of dielectrics from the most general point of view, to distinguish between the electromotive intensity at any point and the electric polarization of the medium at that point, since these directed quantities, though related to one another, are not, in some solid substances, in the same direction. The most general expression for the electric energy of the medium per unit of volume is half the product of the electromotive intensity and the electric polarization multiplied by the cosine of the angle between their directions. In all fluid dielectrics the electromotive intensity and the electric polarization are in the same direction and in a constant ratio.

If we calculate on this hypothesis the total energy residing in the medium, we shall find it equal to the energy due to the electrification of the conductors on the hypothesis of direct action at a distance. Hence the two hypotheses are mathematically equivalent.

If we now proceed to investigate the mechanical state of the medium on the hypothesis that the mechanical action observed between electrified bodies is exerted through and by means of the medium, as in the familiar instances of the action of one body on another by means of the tension of a rope or the pressure of a rod, we find that the medium must be in a state of mechanical stress.

The nature of this stress is, as Faraday pointed out *, a tension along the lines of force combined with an equal pressure in all directions at right angles to these lines. The magnitude of these stresses is proportional to the energy of the electrification per unit of volume, or, in other words, to the square of the resultant electromotive intensity multiplied by the specific inductive capacity of the medium.

* *Exp. Res.*, series xi. 1297.

This distribution of stress is the only one consistent * with the observed mechanical action on the electrified bodies, and also with the observed equilibrium of the fluid dielectric which surrounds them. I have therefore thought it a warrantable step in scientific procedure to assume the actual existence of this state of stress, and to follow the assumption into its consequences. Finding the phrase *electric tension* used in several vague senses, I have attempted to confine it to what I conceive to have been in the minds of some of those who have used it, namely, the state of stress in the dielectric medium which causes motion of the electrified bodies, and leads, when continually augmented, to disruptive discharge. Electric tension, in this sense, is a tension of exactly the same kind, and measured in the same way, as the tension of a rope, and the dielectric medium, which can support a certain tension and no more, may be said to have a certain strength in exactly the same sense as the rope is said to have a certain strength. Thus, for example, Thomson has found that air at the ordinary pressure and temperature can support an electric tension of 9600 grains weight per square foot before a spark passes.

60.] From the hypothesis that electric action is not a direct action between bodies at a distance, but is exerted by means of the medium between the bodies, we have deduced that this medium must be in a state of stress. We have also ascertained the character of the stress, and compared it with the stresses which may occur in solid bodies. Along the lines of force there is tension, and perpendicular to them there is pressure, the numerical magnitude of these forces being equal, and each proportional to the square of the resultant intensity at the point. Having established these results, we are prepared to take another step, and to form an idea of the nature of the electric polarization of the dielectric medium.

An elementary portion of a body may be said to be polarized when it acquires equal and opposite properties on two opposite sides. The idea of internal polarity may be studied to the greatest advantage as exemplified in permanent magnets, and it will be explained at greater length when we come to treat of magnetism.

* {This statement requires modification: the distribution of stress referred to is only one among many such distributions which will all produce the required effect.}

The electric polarization of an elementary portion of a dielectric is a forced state into which the medium is thrown by the action of electromotive force, and which disappears when that force is removed. We may conceive it to consist in what we may call an electric displacement, produced by the electromotive intensity. When the electromotive force acts on a conducting medium it produces a current through it, but if the medium is a non-conductor or dielectric, the current cannot {continue to} flow through the medium, but the electricity is displaced within the medium in the direction of the electromotive intensity, the extent of this displacement depending on the magnitude of the electromotive intensity, so that if the electromotive intensity increases or diminishes, the electric displacement increases or diminishes in the same ratio.

The amount of the displacement is measured by the quantity of electricity which crosses unit of area, while the displacement increases from zero to its actual amount. This, therefore, is the measure of the electric polarization.

The analogy between the action of electromotive intensity in producing electric displacement and of ordinary mechanical force in producing the displacement of an elastic body is so obvious that I have ventured to call the ratio of the electromotive intensity to the corresponding electric displacement the *coefficient of electric elasticity* of the medium. This coefficient is different in different media, and varies inversely as the specific inductive capacity of each medium.

The variations of electric displacement evidently constitute electric currents *. These currents, however, can only exist during the variation of the displacement, and therefore, since the displacement cannot exceed a certain value without causing disruptive discharge, they cannot be continued indefinitely in the same direction, like the currents through conductors.

In tourmaline, and other pyro-electric crystals, it is probable that a state of electric polarization exists, which depends upon temperature, and does not require an external electromotive force to produce it. If the interior of a body were in a state of permanent electric polarization, the outside would gradually become charged in such a manner as to neutralize the action of the internal polarization for all points outside the body. This

* {If we assume the views enunciated in the preceding paragraph.}

external superficial charge could not be detected by any of the ordinary tests, and could not be removed by any of the ordinary methods for discharging superficial electrification. The internal polarization of the substance would therefore never be discovered unless by some means, such as change of temperature, the amount of the internal polarization could be increased or diminished. The external electrification would then be no longer capable of neutralizing the external effect of the internal polarization, and an apparent electrification would be observed, as in the case of tourmaline.

If a charge $e$ is uniformly distributed over the surface of a sphere, the resultant intensity at any point of the medium surrounding the sphere is proportional to the charge $e$ divided by the square of the distance from the centre of the sphere. This resultant intensity, according to our theory, is accompanied by a displacement of electricity in a direction outwards from the sphere.

If we now draw a concentric spherical surface of radius $r$, the whole displacement, $E$, through this surface will be proportional to the resultant intensity multiplied by the area of the spherical surface. But the resultant intensity is directly as the charge $e$ and inversely as the square of the radius, while the area of the surface is directly as the square of the radius.

Hence the whole displacement, $E$, is proportional to the charge $e$, and is independent of the radius.

To determine the ratio between the charge $e$, and the quantity of electricity, $E$, displaced outwards through any one of the spherical surfaces, let us consider the work done upon the medium in the region between two concentric spherical surfaces, while the displacement is increased from $E$ to $E+\delta E$. If $V_1$ and $V_2$ denote the potentials at the inner and the outer of these surfaces respectively, the electromotive force by which the additional displacement is produced is $V_1 - V_2$, so that the work spent in augmenting the displacement is $(V_1 - V_2)\,\delta E$.

If we now make the inner surface coincide with that of the electrified sphere, and make the radius of the outer infinite, $V_1$ becomes $V$, the potential of the sphere, and $V_2$ becomes zero, so that the whole work done in the surrounding medium is $V\delta E$.

But by the ordinary theory, the work done in augmenting the charge is $V\delta e$, and if this is spent, as we suppose, in augmenting the displacement, $\delta E = \delta e$, and since $E$ and $e$ vanish together, $E = e$, or—

*The displacement outwards through any spherical surface concentric with the sphere is equal to the charge on the sphere.*

To fix our ideas of electric displacement, let us consider an accumulator formed of two conducting plates $A$ and $B$, separated by a stratum of a dielectric $C$. Let $W$ be a conducting wire joining $A$ and $B$, and let us suppose that by the action of an electromotive force a quantity $Q$ of positive electricity is transferred along the wire from $B$ to $A$. The positive electrification of $A$ and the negative electrification of $B$ will produce a certain electromotive force acting from $A$ towards $B$ in the dielectric stratum, and this will produce an electric displacement from $A$ towards $B$ within the dielectric. The amount of this displacement, as measured by the quantity of electricity forced across an imaginary section of the dielectric dividing it into two strata, will be, according to our theory, exactly $Q$. See Arts. 75, 76, 111.

It appears, therefore, that at the same time that a quantity $Q$ of electricity is being transferred along the wire by the electromotive force from $B$ towards $A$, so as to cross every section of the wire, the same quantity of electricity crosses every section of the dielectric from $A$ towards $B$ by reason of the electric displacement.

The displacements of electricity during the discharge of the accumulator will be the reverse of these. In the wire the discharge will be $Q$ from $A$ to $B$, and in the dielectric the displacement will subside, and a quantity of electricity $Q$ will cross every section from $B$ towards $A$.

Every case of charge or discharge may therefore be considered as a motion in a closed circuit, such that at every section of the circuit the same quantity of electricity crosses in the same time, and this is the case, not only in the voltaic circuit where it has always been recognized, but in those cases in which electricity has been generally supposed to be accumulated in certain places.

61.] We are thus led to a very remarkable consequence of the theory which we are examining, namely, that the motions of electricity are like those of an *incompressible* fluid, so that the total quantity within an imaginary fixed closed surface remains always the same. This result appears at first sight in direct contradiction to the fact that we can charge a conductor and then introduce it into the closed space, and so alter the quantity of electricity within that space. But we must remember that the ordinary theory takes no account of the electric displacement in the substance of dielectrics which we have been investigating, but confines its attention to the electrification at the bounding surfaces of the conductors and dielectrics. In the case of the charged conductor let us suppose the charge to be positive, then if the surrounding dielectric extends on all sides beyond the closed surface there will be electric polarization, accompanied with displacement from within outwards all over the closed surface, and the surface-integral of the displacement taken over the surface will be equal to the charge on the conductor within.

Thus when the charged conductor is introduced into the closed space there is immediately a displacement of a quantity of electricity equal to the charge through the surface from within outwards, and the whole quantity within the surface remains the same.

The theory of electric polarization will be discussed at greater length in Chapter V, and a mechanical illustration of it will be given in Art. 334, but its importance cannot be fully understood till we arrive at the study of electromagnetic phenomena.

62.] The peculiar features of the theory are :—

That the energy of electrification resides in the dielectric medium, whether that medium be solid, liquid, or gaseous, dense or rare, or even what is called a vacuum, provided it be still capable of transmitting electrical action.

That the energy in any part of the medium is stored up in the form of a state of constraint called electric polarization, the amount of which depends on the resultant electromotive intensity at the place.

That electromotive force acting on a dielectric produces what we have called electric displacement, the relation between the intensity and the displacement being in the most general case of a kind to be afterwards investigated in treating of conduction, but in the most important cases the displacement is in the same direction as the intensity, and is numerically equal to the intensity multiplied by $\frac{1}{4\pi}K$, where $K$ is the specific inductive capacity of the dielectric.

That the energy per unit of volume of the dielectric arising from the electric polarization is half the product of the electromotive intensity and the electric displacement, multiplied, if necessary, by the cosine of the angle between their directions.

That in fluid dielectrics the electric polarization is accompanied by a tension in the direction of the lines of induction, combined with an equal pressure in all directions at right angles to the lines of induction, the tension or pressure per unit of area being numerically equal to the energy per unit of volume at the same place.

That the surface of any elementary portion into which we may conceive the volume of the dielectric divided must be conceived to be charged so that the surface-density at any point of the surface is equal in magnitude to the displacement through that point of the surface *reckoned inwards*. If the displacement is in the positive direction, the surface of the element will be charged negatively on the positive side of the element, and positively on the negative side. These superficial charges will in general destroy one another when consecutive elements are considered, except where the dielectric has an internal charge, or at the surface of the dielectric.

That whatever electricity may be, and whatever we may understand by the movement of electricity, the phenomenon which we have called electric displacement is a movement of electricity in the same sense as the transference of a definite quantity of electricity through a wire is a movement of electricity, the only difference being that in the dielectric there is a force which we have called electric elasticity which acts against the electric displacement, and forces the electricity back when the electromotive force is removed; whereas in the conducting wire the electric elasticity is continually giving way, so that a current of true conduction is set up, and the resistance depends not on the total quantity of electricity displaced from its position of equilibrium, but on the quantity which crosses a section of the conductor in a given time.

That in every case the motion of electricity is subject to the same condition as that of an incompressible fluid, namely, that at every instant as much must flow out of any given closed surface as flows into it.

It follows from this that every electric current must form a closed circuit. The importance of this result will be seen when we investigate the laws of electro-magnetism.

Since, as we have seen, the theory of direct action at a distance is mathematically identical with that of action by means of a medium, the actual phenomena may be explained by the one theory as well as by the other, provided suitable hypotheses be introduced when any difficulty occurs. Thus, Mossotti has deduced the mathematical theory of dielectrics from the ordinary theory of attraction merely by giving an electric instead of a magnetic interpretation to the symbols in the investigation by which Poisson has deduced the theory of magnetic induction from the theory of magnetic fluids. He assumes the existence within the dielectric of small conducting elements, capable of having their opposite surfaces oppositely electrified by induction, but not capable of losing or gaining electricity on the whole, owing to their being insulated from each other by a non-conducting medium. This theory of dielectrics is consistent with the laws of electricity, and may be actually true. If it is true, the specific inductive capacity of a dielectric may be greater, but cannot be less, than that of a vacuum. No instance has yet been found of a dielectric having an inductive capacity less than that of a vacuum, but if such should be discovered, Mossotti's physical theory must be abandoned, although his formulae would all remain exact, and would only require us to alter the sign of a coefficient.

In many parts of physical science, equations of the same form are found applicable to phenomena which are certainly of quite different natures, as, for instance, electric induction through dielectrics, conduction through conductors, and magnetic induction. In all these cases the relation between the intensity and the effect produced is expressed by a set of equations of the same kind, so that when a problem in one of these subjects is solved, the problem and its solution may be translated into the language of the other subjects and the results in their new form will still be true.

# CHAPTER II.

## ELEMENTARY MATHEMATICAL THEORY OF STATICAL ELECTRICITY.

### *Definition of Electricity as a Mathematical Quantity.*

63.] WE have seen that the properties of charged bodies are such that the charge of one body may be equal to that of another, or to the sum of the charges of two bodies, and that when two bodies are equally and oppositely charged they have no electrical effect on external bodies when placed together within a closed insulated conducting vessel. We may express all these results in a concise and consistent manner by describing an electrified body as *charged* with a certain *quantity of electricity*, which we may denote by $e$. When the charge is positive, that is, according to the usual convention, vitreous, $e$ will be a positive quantity. When the charge is negative or resinous, $e$ will be negative, and the quantity $-e$ may be interpreted either as a negative quantity of vitreous electricity or as a positive quantity of resinous electricity.

The effect of adding together two equal and opposite charges of electricity, $+e$ and $-e$, is to produce a state of no charge expressed by zero. We may therefore regard a body not charged as virtually charged with equal and opposite charges of indefinite magnitude, and a charged body as virtually charged with unequal quantities of positive and negative electricity, the algebraic sum of these charges constituting the observed electrification. It is manifest, however, that this way of regarding an electrified body is entirely artificial, and may be compared to the conception of the velocity of a body as compounded of two or more different velocities, no one of which is the actual velocity of the body.

## ON ELECTRIC DENSITY.

### *Distribution in Three Dimensions.*

64.] *Definition.* The electric volume-density at a given point in space is the limiting ratio of the quantity of electricity within a sphere whose centre is the given point to the volume of the sphere, when its radius is diminished without limit.

We shall denote this ratio by the symbol $\rho$, which may be positive or negative.

### *Distribution over a Surface.*

It is a result alike of theory and of experiment, that, in certain cases, the charge of a body is entirely on the surface. The density at a point on the surface, if defined according to the method given above, would be infinite. We therefore adopt a different method for the measurement of surface-density.

*Definition.* The electric density at a given point on a surface is the limiting ratio of the quantity of electricity within a sphere whose centre is the given point to the area of the surface contained within the sphere, when its radius is diminished without limit.

We shall denote the surface-density by the symbol $\sigma$.

Those writers who supposed electricity to be a material fluid or a collection of particles, were obliged in this case to suppose the electricity distributed on the surface in the form of a stratum of a certain thickness $\theta$, its density being $\rho_0$, or that value of $\rho$ which would result from the particles having the closest contact of which they are capable. It is manifest that on this theory

$$\rho_0 \theta = \sigma.$$

When $\sigma$ is negative, according to this theory, a certain stratum of thickness $\theta$ is left entirely devoid of positive electricity, and filled entirely with negative electricity, or, on the theory of one fluid, with matter.

There is, however, no experimental evidence either of the electric stratum having any thickness, or of electricity being a fluid or a collection of particles. We therefore prefer to do without the symbol for the thickness of the stratum, and to use a special symbol for surface-density.

*Distribution on a Line.*

It is sometimes convenient to suppose electricity distributed on a line, that is, a long narrow body of which we neglect the thickness. In this case we may define the line-density at any point to be the limiting ratio of the charge on an element of the line to the length of that element when the element is diminished without limit.

If $\lambda$ denotes the line-density, then the whole quantity of electricity on a curve is $e = \int \lambda ds$, where $ds$ is the element of the curve. Similarly, if $\sigma$ is the surface-density, the whole quantity of electricity on the surface is

$$e = \iint \sigma dS,$$

where $dS$ is the element of surface.

If $\rho$ is the volume-density at any point of space, then the whole electricity with a certain volume is

$$e = \iiint \rho \, dx \, dy \, dz,$$

where $dx\,dy\,dz$ is the element of volume. The limits of integration in each case are those of the curve, the surface, or the portion of space considered.

It is manifest that $e$, $\lambda$, $\sigma$ and $\rho$ are quantities differing in kind, each being one dimension in space lower than the preceding, so that if $l$ be a line, the quantities $e$, $l\lambda$, $l^2 \sigma$, and $l^3 \rho$ will be all of the same kind, and if $[L]$ be the unit of length, and $[\lambda]$, $[\sigma]$, $[\rho]$ the units of the different kinds of density, $[e]$, $[L\lambda]$, $[L^2 \sigma]$, and $[L^3 \rho]$ will each denote one unit of electricity.

*Definition of the Unit of Electricity.*

65.] Let $A$ and $B$ be two points the distance between which is the unit of length. Let two bodies, whose dimensions are small compared with the distance $AB$, be charged with equal quantities of positive electricity and placed at $A$ and $B$ respectively, and let the charges be such that the force with which they repel each other is the unit of force, measured as in Art. 6. Then the charge of either body is said to be the unit of electricity *.

If the charge of the body at $B$ were a unit of negative

* {In this definition the dielectric separating the charged bodies is supposed to be air.}

electricity, then, since the action between the bodies would be reversed, we should have an attraction equal to the unit of force. If the charge of $A$ were also negative, and equal to unity, the force would be repulsive, and equal to unity.

Since the action between any two portions of electricity is not affected by the presence of other portions, the repulsion between $e$ units of electricity at $A$ and $e'$ units at $B$ is $ee'$, the distance $AB$ being unity. See Art. 39.

*Law of Force between Charged Bodies.*

66.] Coulomb shewed by experiment that the force between charged bodies whose dimensions are small compared with the distance between them, varies inversely as the square of the distance. Hence the repulsion between two such bodies charged with quantities $e$ and $e'$ and placed at a distance $r$ is

$$\frac{ee'}{r^2}.$$

We shall prove in Arts. 74 $c$, 74 $d$, 74 $e$ that this law is the only one consistent with the observed fact that a conductor, placed in the inside of a closed hollow conductor and in contact with it, is deprived of all electrical charge. Our conviction of the accuracy of the law of the inverse square of the distance may be considered to rest on experiments of this kind, rather than on the direct measurements of Coulomb.

*Resultant Force between Two Bodies.*

67.] In order to calculate the resultant force between two bodies we might divide each of them into its elements of volume, and consider the repulsion between the electricity in each of the elements of the first body and the electricity in each of the elements of the second body. We should thus get a system of forces equal in number to the product of the numbers of the elements into which we have divided each body, and we should have to combine the effects of these forces by the rules of Statics. Thus, to find the component in the direction of $x$ we should have to find the value of the sextuple integral

$$\iiiiiint \frac{\rho\rho'(x-x')\,dx\,dy\,dz\,dx'\,dy'\,dz'}{\{(x-x')^2+(y-y')^2+(z-z')^2\}^{\frac{3}{2}}},$$

where $x$, $y$, $z$ are the coordinates of a point in the first body at which the electrical density is $\rho$, and $x'$, $y'$, $z'$, and $\rho'$ are the corresponding quantities for the second body, and the integration is extended first over the one body and then over the other.

### *Resultant Intensity at a Point.*

68.] In order to simplify the mathematical process, it is convenient to consider the action of an electrified body, not on another body of any form, but on an indefinitely small body, charged with an indefinitely small amount of electricity, and placed at any point of the space to which the electrical action extends. By making the charge of this body indefinitely small we render insensible its disturbing action on the charge of the first body.

Let $e$ be the charge of the small body, and let the force acting on it when placed at the point $(x, y, z)$ be $Re$, and let the direction-cosines of the force be $l, m, n$, then we may call $R$ the resultant electric intensity at the point $(x, y, z)$.

If $X, Y, Z$ denote the components of $R$, then

$$X = Rl, \qquad Y = Rm, \qquad Z = Rn.$$

In speaking of the resultant electric intensity at a point, we do not necessarily imply that any force is actually exerted there, but only that if an electrified body were placed there it would be acted on by a force $Re$, where $e$ is the charge of the body *.

*Definition.* The resultant electric intensity at any point is the force which would be exerted on a small body charged with the unit of positive electricity, if it were placed there without disturbing the actual distribution of electricity.

This force not only tends to move a body charged with electricity, but to move the electricity within the body, so that the positive electricity tends to move in the direction of $R$ and the negative electricity in the opposite direction. Hence the quantity $R$ is also called the Electromotive Intensity at the point $(x, y, z)$.

When we wish to express the fact that the resultant intensity is a vector, we shall denote it by the German letter $\mathfrak{E}$. If the body is a dielectric, then, according to the theory adopted in this treatise, the electricity is displaced within it, so that the

* The Electric and Magnetic Intensities correspond, in electricity and magnetism, to the intensity of gravity, commonly denoted by $g$, in the theory of heavy bodies.

quantity of electricity which is forced in the direction of $\mathfrak{E}$ across unit of area fixed perpendicular to $\mathfrak{E}$ is

$$\mathfrak{D} = \frac{1}{4\pi} K \mathfrak{E};$$

where $\mathfrak{D}$ is the displacement, $\mathfrak{E}$ the resultant intensity, and $K$ the specific inductive capacity of the dielectric.

If the body is a conductor, the state of constraint is continually giving way, so that a current of conduction is produced and maintained as long as $\mathfrak{E}$ acts on the medium.

### *Line-Integral of Electric Intensity, or Electromotive Force along an Arc of a Curve.*

69.] The Electromotive force along a given arc $AP$ of a curve is numerically measured by the work which would be done by the electric intensity on a unit of positive electricity carried along the curve from $A$, the beginning, to $P$, the end of the arc.

If $s$ is the length of the arc, measured from $A$, and if the resultant intensity $R$ at any point of the curve makes an angle $\epsilon$ with the tangent drawn in the positive direction, then the work done on unit of electricity in moving along the element of the curve $ds$ will be

$$R \cos \epsilon \, ds,$$

and the total electromotive force $E$ will be

$$E = \int_0^s R \cos \epsilon \, ds,$$

the integration being extended from the beginning to the end of the arc.

If we make use of the components of the intensity, the expression becomes

$$E = \int_0^s \left( X \frac{dx}{ds} + Y \frac{dy}{ds} + Z \frac{dz}{ds} \right) ds.$$

If $X$, $Y$, and $Z$ are such that $X dx + Y dy + Z dz$ is the complete differential of $-V$, a function of $x$, $y$, $z$, then

$$E = \int_A^P (X dx + Y dy + Z dz) = -\int_A^P dV = V_A - V_P;$$

where the integration is performed in any way from the point $A$ to the point $P$, whether along the given curve or along any other line between $A$ and $P$.

In this case $V$ is a scalar function of the position of a point in space, that is, when we know the coordinates of the point, the value of $V$ is determinate, and this value is independent of the position and direction of the axes of reference. See Art. 16.

*On Functions of the Position of a Point.*

In what follows, when we describe a quantity as a function of the position of a point, we mean that for every position of the point the function has a determinate value. We do not imply that this value can always be expressed by the same formula for all points of space, for it may be expressed by one formula on one side of a given surface and by another formula on the other side.

*On Potential Functions.*

70.] The quantity $X\,dx + Y\,dy + Z\,dz$ is an exact differential whenever the force arises from attractions or repulsions whose intensity is a function of the distances from any number of points. For if $r_1$ be the distance of one of the points from the point $(x, y, z)$, and if $R_1$ be the repulsion, then

$$X_1 = R_1\frac{x-x_1}{r_1} = R_1\frac{dr_1}{dx},$$

with similar expressions for $Y_1$ and $Z_1$, so that

$$X_1dx + Y_1dy + Z_1dz = R_1dr_1;$$

and since $R_1$ is a function of $r_1$ only, $R_1dr_1$ is an exact differential of some function of $r_1$, say $-V_1$.

Similarly for any other force $R_2$, acting from a centre at distance $r_2$,

$$X_2dx + Y_2dy + Z_2dz = R_2dr_2 = -dV_2.$$

But $X = X_1 + X_2 + \&c.$, and $Y$ and $Z$ are compounded in the same way, therefore

$$X\,dx + Y\,dy + Z\,dz = -dV_1 - dV_2 - \&c. = -dV.$$

The integral of this quantity, under the condition that it vanishes at an infinite distance, is called the Potential Function.

The use of this function in the theory of attractions was introduced by Laplace in the calculation of the attraction of the earth. Green, in his essay 'On the Application of Mathematical Analysis to Electricity,' gave it the name of the Potential Function. Gauss, working independently of Green, also used the word Potential. Clausius and others have applied the term Potential to the work which would be done if two bodies or systems were removed to an infinite distance from one another. We shall follow the use of the word in recent English works, and avoid ambiguity by adopting the following definition due to Sir W. Thomson.

*Definition of Potential.* The Potential at a Point is the work which would be done on a unit of positive electricity by the electric forces if it were placed at that point without disturbing the electric distribution, and carried from that point to an infinite distance: or, what comes to the same thing, the work which must be done by an external agent in order to bring the unit of positive electricity from an infinite distance (or from any place where the potential is zero) to the given point.

### 71.] *Expressions for the Resultant Intensity and its components in terms of the Potential.*

Since the total electromotive force along any arc $AB$ is

$$E_{AB} = V_A - V_B,$$

if we put $ds$ for the arc $AB$ we shall have for the intensity resolved in the direction of $ds$,

$$R\cos\epsilon = -\frac{dV}{ds};$$

whence, by assuming $ds$ parallel to each of the axes in succession, we get

$$X = -\frac{dV}{dx}, \qquad Y = -\frac{dV}{dy}, \qquad Z = -\frac{dV}{dz};$$

$$R = \left\{\overline{\frac{dV}{dx}}\Big|^2 + \overline{\frac{dV}{dy}}\Big|^2 + \overline{\frac{dV}{dz}}\Big|^2\right\}^{\frac{1}{2}}.$$

We shall denote the intensity itself, whose magnitude, or tensor, is $R$ and whose components are $X$, $Y$, $Z$, by the German letter $\mathfrak{E}$, as in Art. 68.

### *The Potential at all Points within a Conductor is the same.*

72.] A conductor is a body which allows the electricity within it to move from one part of the body to any other when acted on by electromotive force. When the electricity is in equilibrium there can be no electromotive intensity acting within the conductor. Hence $R = 0$ throughout the whole space occupied by the conductor. From this it follows that

$$\frac{dV}{dx} = 0, \qquad \frac{dV}{dy} = 0, \qquad \frac{dV}{dz} = 0;$$

and therefore for every point of the conductor

$$V = C,$$

where $C$ is a constant quantity.

Since the potential at all points within the substance of the conductor is $C$, the quantity $C$ is called the Potential of the conductor. $C$ may be defined as the work which must be done by external agency in order to bring a unit of electricity from an infinite distance to the conductor, the distribution of electricity being supposed not to be disturbed by the presence of the unit *.

It will be shewn at Art. 246 that in general when two bodies of different kinds are in contact, an electromotive force acts from one to the other through the surface of contact, so that when they are in equilibrium the potential of the latter is higher than that of the former. For the present, therefore, we shall suppose all our conductors made of the same metal, and at the same temperature.

If the potentials of the conductors $A$ and $B$ be $V_A$ and $V_B$ respectively, then the electromotive force along a wire joining $A$ and $B$ will be

$$V_A - V_B$$

in the direction $AB$, that is, positive electricity will tend to pass from the conductor of higher potential to the other.

Potential, in electrical science, has the same relation to Electricity that Pressure, in Hydrostatics, has to Fluid, or that Temperature, in Thermodynamics, has to Heat. Electricity, Fluids, and Heat all tend to pass from one place to another, if the Potential, Pressure, or Temperature is greater in the first place than in the second. A fluid is certainly a substance, heat is as certainly not a substance, so that though we may find assistance from analogies of this kind in forming clear ideas of formal relations of electrical quantities, we must be careful not to let the one or the other analogy suggest to us that electricity is either a substance like water, or a state of agitation like heat.

* {If there is any discontinuity in the potential as we pass from the dielectric to the conductor it is necessary to state whether the electrified point is brought inside the conductor or merely to the surface.}

## *Potential due to any Electrical System.*

73.] Let there be a single electrified point charged with a quantity $e$ of electricity, and let $r$ be the distance of the point $x', y', z'$ from it, then

$$V = \int_r^\infty R\,dr = \int_r^\infty \frac{e}{r^2}\,dr = \frac{e}{r}.$$

Let there be any number of electrified points whose coordinates are $(x_1, y_1, z_1)$, $(x_2, y_2, z_2)$, &c. and their charges $e_1$, $e_2$, &c., and let their distances from the point $(x', y', z')$ be $r_1$, $r_2$, &c., then the potential of the system at $(x', y', z')$ will be

$$V = \Sigma\left(\frac{e}{r}\right).$$

Let the electric density at any point $(x, y, z)$ within an electrified body be $\rho$, then the potential due to the body is

$$V = \iiint \frac{\rho}{r}\,dx\,dy\,dz;$$

where $$r = \{(x-x')^2 + (y-y')^2 + (z-z')^2\}^{\frac{1}{2}},$$

the integration being extended throughout the body.

## *On the Proof of the Law of the Inverse Square.*

74*a*.] The fact that the force between electrified bodies is inversely as the square of the distance may be considered to be established by Coulomb's direct experiments with the torsion-balance. The results, however, which we derive from such experiments must be regarded as affected by an error depending on the probable error of each experiment, and unless the skill of the operator be very great, the probable error of an experiment with the torsion-balance is considerable.

A far more accurate verification of the law of force may be deduced from an experiment similar to that described at Art. 32 (Exp. VII).

Cavendish, in his hitherto unpublished work on electricity, makes the evidence of the law of force depend on an experiment of this kind.

He fixed a globe on an insulating support, and fastened two hemispheres by glass rods to two wooden frames hinged to an axis so that the hemispheres, when the frames were brought together, formed an insulated spherical shell concentric with the globe.

The globe could then be made to communicate with the hemispheres by means of a short wire, to which a silk string was fastened so that the wire could be removed without discharging the apparatus.

The globe being in communication with the hemispheres, he charged the hemispheres by means of a Leyden jar, the potential of which had been previously measured by an electrometer, and immediately drew out the communicating wire by means of the silk string, removed and discharged the hemispheres, and tested the electrical condition of the globe by means of a pith ball electrometer.

No indication of any charge of the globe could be detected by the pith ball electrometer, which at that time (1773) was considered the most delicate electroscope.

Cavendish next communicated to the globe a known fraction of the charge formerly communicated to the hemispheres, and tested the globe again with his electrometer.

He thus found that the charge of the globe in the original experiment must have been less than $\frac{1}{60}$ of the charge of the whole apparatus, for if it had been greater it would have been detected by the electrometer.

He then calculated the ratio of the charge of the globe to that of the hemispheres on the hypothesis that the repulsion is inversely as a power of the distance differing slightly from 2, and found that if this difference was $\frac{1}{50}$ there would have been a charge on the globe equal to $\frac{1}{57}$ of that of the whole apparatus, and therefore capable of being detected by the electrometer.

74 *b.*] The experiment has recently been repeated at the Cavendish Laboratory in a somewhat different manner.

The hemispheres were fixed on an insulating stand, and the globe fixed in its proper position within them by means of an ebonite ring. By this arrangement the insulating support of the globe was never exposed to the action of any sensible electric force, and therefore never became charged, so that the disturbing effect of electricity creeping along the surface of the insulators was entirely removed.

Instead of removing the hemispheres before testing the potential of the globe, they were left in their position, but discharged to earth. The effect of a given charge of the globe on the electrometer was not so great as if the hemispheres had been removed, but this disadvantage was more than compensated by the perfect security afforded by the conducting vessel against all external electric disturbances.

The short wire which made the connexion between the shell and the globe was fastened to a small metal disk which acted as a lid to a small hole in the shell, so that when the wire and the lid were lifted up by a silk string, the electrode of the electrometer could be made to dip into the hole and rest on the globe within.

The electrometer was Thomson's Quadrant Electrometer described in Art. 219. The case of the electrometer and one of the electrodes were always connected to earth, and the testing electrode was connected to earth till the electricity of the shell had been discharged.

To estimate the original charge of the shell, a small brass ball was placed on an insulating support at a considerable distance from the shell.

The operations were conducted as follows:—

The shell was charged by communication with a Leyden jar.

The small ball was connected to earth so as to give it a negative charge by induction, and was then left insulated.

The communicating wire between the globe and the shell was removed by a silk string.

The shell was then discharged, and kept connected to earth.

The testing electrode was disconnected from earth, and made to touch the globe, passing through the hole in the shell.

Not the slightest effect on the electrometer could be observed.

To test the sensitiveness of the apparatus the shell was disconnected from earth and the small ball was discharged to earth. The electrometer {the testing electrode remaining in contact with the globe} then shewed a positive deflection, $D$.

The negative charge of the brass ball was about $\frac{1}{54}$ of the original charge of the shell, and the positive charge induced by the ball when the shell was put to earth was about $\frac{1}{9}$ of that of the ball. Hence when the ball was put to earth the potential of the shell, as indicated by the electrometer, was about $\frac{1}{486}$ of its original potential.

But if the repulsion had been as $r^{q-2}$, the potential of the globe would have been $-0{\cdot}1478\,q$ of that of the shell by equation (22), p. 85.

Hence if $\pm d$ be the greatest deflection of the electrometer which could escape observation, and $D$ the deflection observed in the second part of the experiment, {since $\cdot 1478\, qV/\frac{1}{486}V$ must be less than $d/D$,} $q$ cannot exceed

$$\pm \frac{1}{72}\frac{d}{D}.$$

Now even in a rough experiment $D$ was more than $300\,d$, so that $q$ cannot exceed

$$\pm \frac{1}{21600}.$$

### *Theory of the Experiment.*

74 *c*.] To find the potential at any point due to a uniform spherical shell, the repulsion between two units of matter being any given function of the distance.

Let $\phi(r)$ be the repulsion between two units at distance $r$, and let $f(r)$ be such that

$$\frac{df(r)}{dr}\left(=f'(r)\right) = r\int_r^\infty \phi(r)\,dr. \tag{1}$$

Let the radius of the shell be $a$, and its surface density $\sigma$, then, if $\alpha$ denotes the whole charge of the shell,

$$\alpha = 4\pi a^2 \sigma. \tag{2}$$

Let $b$ denote the distance of the given point from the centre of the shell, and let $r$ denote its distance from any given point of the shell.

If we refer the point on the shell to spherical coordinates, the pole being the centre of the shell, and the axis the line drawn to the given point, then

$$r^2 = a^2 + b^2 - 2ab\cos\theta. \tag{3}$$

The mass of the element of the shell is

$$\sigma a^2 \sin\theta\, d\phi\, d\theta, \tag{4}$$

and the potential due to this element at the given point is

$$\sigma a^2 \sin\theta \frac{f'(r)}{r}\, d\theta\, d\phi; \tag{5}$$

and this has to be integrated with respect to $\phi$ from $\phi = 0$ to $\phi = 2\pi$, which gives

$$2\pi\sigma a^2 \sin\theta \frac{f'(r)}{r} d\theta, \tag{6}$$

which has to be integrated from $\theta = 0$ to $\theta = \pi$.

Differentiating (3) we find

$$r dr = ab \sin\theta d\theta. \tag{7}$$

Substituting the value of $d\theta$ in (6) we obtain

$$2\pi\sigma\frac{a}{b}f'(r)\,dr, \tag{8}$$

the integral of which is

$$V = 2\pi\sigma\frac{a}{b}\{f(r_1)-f(r_2)\}, \tag{9}$$

where $r_1$ is the greatest value of $r$, which is always $a+b$, and $r_2$ is the least value of $r$, which is $b-a$ when the given point is outside the shell and $a-b$ when it is within the shell.

If we write $\alpha$ for the whole charge of the shell, and $V$ for its potential at the given point, then for a point outside the shell

$$V = \frac{\alpha}{2ab}\{f(b+a)-f(b-a)\}. \tag{10}$$

For a point on the shell itself

$$V = \frac{\alpha}{2a^2}f(2a),^* \tag{11}$$

and for a point inside the shell

$$V = \frac{\alpha}{2ab}\{f(a+b)-f(a-b)\}. \tag{12}$$

We have next to determine the potentials of two concentric spherical shells, the radii of the outer and inner shells being $a$ and $b$, and their charges $\alpha$ and $\beta$.

Calling the potential of the outer shell $A$, and that of the inner $B$, we have by what precedes

$$A = \frac{\alpha}{2a^2}f(2a) + \frac{\beta}{2ab}\{f(a+b)-f(a-b)\}, \tag{13}$$

$$B = \frac{\beta}{2b^2}f(2b) + \frac{\alpha}{2ab}\{f(a+b)-f(a-b)\}. \tag{14}$$

In the first part of the experiment the shells communicate by the short wire and are both raised to the same potential, say $V$.

* {Strictly $f(2a)-f(0)$, but the conclusions arrived at in Art. 74 *d* are not altered if we write $f(2a)-f(0)$ for $f(2a)$ and $f(2b)-f(0)$ for $f(2b)$ all through.}

By putting $A = B = V$, and solving the equations (13) and (14) for $\beta$, we find for the charge of the inner shell

$$\beta = 2Vb\frac{bf(2a)-a\,[f(a+b)-f(a-b)]}{f(2a)f(2b)-[f(a+b)-f(a-b)]^2}. \tag{15}$$

In the experiment of Cavendish, the hemispheres forming the outer shell were removed to a distance which we may suppose infinite, and discharged. The potential of the inner shell (or globe) would then become

$$B_1 = \frac{\beta}{2b^2}f(2b). \tag{16}$$

In the form of the experiment as repeated at the Cavendish Laboratory the outer shell was left in its place, but connected to earth, so that $A = 0$. In this case we find for the potential of the inner globe in terms of $V$

$$B_2 = V\left\{1 - \frac{a}{b}\frac{f(a+b)-f(a-b)}{f(2a)}\right\}. \tag{17}$$

74 *d.*] Let us now assume, with Cavendish, that the law of force is some inverse power of the distance, not differing much from the inverse square, and let us put

$$\phi(r) = r^{q-2}; \tag{18}$$

then

$$f(r) = \frac{1}{1-q^2}r^{q+1}\,*. \tag{19}$$

If we suppose $q$ to be small, we may expand this by the exponential theorem in the form

$$f(r) = \frac{1}{1-q^2}r\left\{1+q\log r+\frac{1}{1\,.\,2}(q\log r)^2+\&c.\right\}; \tag{20}$$

and if we neglect terms involving $q^2$, equations (16) and (17) become

$$B_1 = \tfrac{1}{2}\frac{a}{a-b}Vq\left[\log\frac{4a^2}{a^2-b^2}-\frac{a}{b}\log\frac{a+b}{a-b}\right], \tag{21}$$

$$B_2 = \tfrac{1}{2}Vq\left[\log\frac{4a^2}{a^2-b^2}-\frac{a}{b}\log\frac{a+b}{a-b}\right], \tag{22}$$

from which we may determine $q$ in terms of the results of the experiment.

74 *e.*] Laplace gave the first demonstration that no function of the distance except the inverse square satisfies the condition that a uniform spherical shell exerts no force on a particle within it †.

* {Strictly $f(r)-f(0) = \frac{1}{1-q^2}r^{q+1}$ if $q^2$ be less than unity.}

† *Mec. Cel.*, I. 2.

If we suppose that $\beta$ in equation (15) is always zero, we may apply the method of Laplace to determine the form of $f(r)$. We have by (15),

$$bf(2a) - af(a+b) + af(a-b) = 0.$$

Differentiating twice with respect to $b$, and dividing by $a$, we find

$$f''(a+b) = f''(a-b).$$

If this equation is generally true

$$f''(r) = C_0, \text{ a constant.}$$

Hence,

$$f'(r) = C_0 r + C_1;$$

and by (1)

$$\int_r^\infty \phi(r)\,dr = \frac{f'(r)}{r} = C_0 + \frac{C_1}{r},$$

$$\phi(r) = \frac{C_1}{r^2}.$$

We may observe, however, that though the assumption of Cavendish, that the force varies as some power of the distance, may appear less general than that of Laplace, who supposes it to be any function of the distance, it is the only one consistent with the fact that similar surfaces can be electrified so as to have similar electrical properties, {so that the lines of force are similar}.

For if the force were any function of the distance except a power of the distance, the ratio of the forces at two different distances would not be a function of the ratio of the distances, but would depend on the absolute value of the distances, and would therefore involve the ratios of these distances to an absolutely fixed length.

Indeed Cavendish himself points out * that on his own hypothesis as to the constitution of the electric fluid, it is impossible for the distribution of electricity to be accurately similar in two conductors geometrically similar, unless the charges are proportional to the volumes. For he supposes the particles of the electric fluid to be closely pressed together near the surface of the body, and this is equivalent to supposing that the law of repulsion is no longer the inverse square †, but that as soon as the particles come very close together, their repulsion begins to increase at a much greater rate with any further diminution of their distance.

* {*Electrical Researches of the Hon. H. Cavendish*, pp. 27, 28.}
† {Idem, Note 2, p. 370.}

*Surface-Integral of Electric Induction, and Electric Displacement through a surface.*

75.] Let $R$ be the resultant intensity at any point of the surface, and $\epsilon$ the angle which $R$ makes with the normal drawn towards the positive side of the surface, then $R\cos\epsilon$ is the component of the intensity normal to the surface, and if $dS$ is the element of the surface, the electric displacement through $dS$ will be, by Art. 68,

$$\frac{1}{4\pi}KR\cos\epsilon\,dS.$$

Since we do not at present consider any dielectric except air, $K=1$.

We may, however, avoid introducing at this stage the theory of electric displacement, by calling $R\cos\epsilon\,dS$ the Induction through the element $dS$. This quantity is well known in mathematical physics, but the name of induction is borrowed from Faraday. The surface-integral of induction is

$$\iint R\cos\epsilon\,dS,$$

and it appears by Art. 21, that if $X$, $Y$, $Z$ are the components of $R$, and if these quantities are continuous within a region bounded by a closed surface $S$, the induction reckoned from within outwards is

$$\iint R\cos\epsilon\,dS = \iiint\left(\frac{dX}{dx}+\frac{dY}{dy}+\frac{dZ}{dz}\right)dx\,dy\,dz,$$

the integration being extended through the whole space within the surface.

*Induction through a Closed Surface due to a single Centre of Force.*

76.] Let a quantity $e$ of electricity be supposed to be placed at a point $O$, and let $r$ be the distance of any point $P$ from $O$, the intensity at that point is $R = er^{-2}$ in the direction $OP$.

Let a line be drawn from $O$ in any direction to an infinite distance. If $O$ is without the closed surface this line will either not cut the surface at all, or it will issue from the surface as many times as it enters. If $O$ is within the surface the line must first issue from the surface, and then it may enter and issue any number of times alternately, ending by issuing from it.

Let $\epsilon$ be the angle between $OP$ and the normal to the surface drawn outwards where $OP$ cuts it, then where the line issues from the surface, $\cos \epsilon$ will be positive, and where it enters, $\cos \epsilon$ will be negative.

Now let a sphere be described with centre $O$ and radius unity, and let the line $OP$ describe a conical surface of small angular aperture about $O$ as vertex.

This cone will cut off a small element $d\omega$ from the surface of the sphere, and small elements $dS_1$, $dS_2$, &c. from the closed surface at the different places where the line $OP$ intersects it.

Then, since any one of these elements $dS$ intersects the cone at a distance $r$ from the vertex and at an obliquity $\epsilon$,

$$dS = \pm r^2 \sec \epsilon \, d\omega ;$$

and, since $R = er^{-2}$, we shall have

$$R \cos \epsilon \, dS = \pm e d\omega ;$$

the positive sign being taken when $r$ issues from the surface, and the negative when it enters it.

If the point $O$ is without the closed surface, the positive values are equal in number to the negative ones, so that for any direction of $r$,

$$\Sigma R \cos \epsilon \, dS = 0,$$

and therefore

$$\iint R \cos \epsilon \, dS = 0,$$

the integration being extended over the whole closed surface.

If the point $O$ is within the closed surface the radius vector $OP$ first issues from the closed surface, giving a positive value of $e\,d\omega$, and then has an equal number of entrances and issues, so that in this case

$$\Sigma R \cos \epsilon \, dS = e d\omega.$$

Extending the integration over the whole closed surface, we shall include the whole of the spherical surface, the area of which is $4\pi$, so that

$$\iint R \cos \epsilon \, dS = e \iint d\omega = 4\pi e.$$

Hence we conclude that the total induction outwards through a closed surface due to a centre of force $e$ placed at a point $O$ is zero when $O$ is without the surface, and $4\pi e$ when $O$ is within the surface.

Since in air the displacement is equal to the induction divided by $4\pi$, the displacement through a closed surface, reckoned outwards, is equal to the electricity within the surface.

*Corollary.* It also follows that if the surface is not closed but is bounded by a given closed curve, the total induction through it is $\omega e$, where $\omega$ is the solid angle subtended by the closed curve at $O$. This quantity, therefore, depends only on the closed curve, and the form of the surface of which it is the boundary may be changed in any way, provided it does not pass from one side to the other of the centre of force.

*On the Equations of Laplace and Poisson.*

77.] Since the value of the total induction of a single centre of force through a closed surface depends only on whether the centre is within the surface or not, and does not depend on its position in any other way, if there are a number of such centres $e_1$, $e_2$, &c. within the surface, and $e_1'$, $e_2'$, &c. without the surface, we shall have

$$\iint R \cos\epsilon \, dS = 4\pi e;$$

where $e$ denotes the algebraical sum of the quantities of electricity at all the centres of force within the closed surface, that is, the total electricity within the surface, resinous electricity being reckoned negative.

If the electricity is so distributed within the surface that the density is nowhere infinite, we shall have by Art. 64,

$$4\pi e = 4\pi \iiint \rho \, dx\, dy\, dz,$$

and by Art. 75,

$$\iint R \cos\epsilon \, dS = \iiint \left(\frac{dX}{dx} + \frac{dY}{dy} + \frac{dZ}{dz}\right) dx\, dy\, dz.$$

If we take as the closed surface that of the element of volume $dx\, dy\, dz$, we shall have, by equating these expressions,

$$\frac{dX}{dx} + \frac{dY}{dy} + \frac{dZ}{dz} = 4\pi\rho;$$

and if a potential $V$ exists, we find by Art. 71,

$$\frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} + \frac{d^2V}{dz^2} + 4\pi\rho = 0.$$

This equation, in the case in which the density is zero, is called Laplace's Equation. In its more general form it was first given by Poisson. It enables us, when we know the potential at every point, to determine the distribution of electricity.

We shall denote, as in Art. 26, the quantity

$$\frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} + \frac{d^2V}{dz^2} \text{ by } -\nabla^2 V,$$

and we may express Poisson's equation in words by saying that the electric density multiplied by $4\pi$ is the concentration of the potential. Where there is no electrification, the potential has no concentration, and this is the interpretation of Laplace's equation.

By Art. 72, $V$ is constant within a conductor. Hence within a conductor the volume-density is zero, and the whole charge must be on the surface.

If we suppose that in the superficial and linear distributions of electricity the volume-density $\rho$ remains finite, and that the electricity exists in the form of a thin stratum or a narrow fibre, then, by increasing $\rho$ and diminishing the depth of the stratum or the section of the fibre, we may approach the limit of true superficial or linear distribution, and the equation being true throughout the process will remain true at the limit, if interpreted in accordance with the actual circumstances.

*Variation of the Potential at a Charged Surface.*

78 *a*.] The potential function, $V$, must be physically continuous in the sense defined in Art. 7, except at the bounding surface of two different media, in which case, as we shall see in Art. 246, there may be a difference of potential between the substances, so that when the electricity is in equilibrium, the potential at a point in one substance is higher than the potential at the contiguous point in the other substance by a constant quantity, $C$, depending on the natures of the two substances and on their temperatures.

But the first derivatives of $V$ with respect to $x, y$, or $z$ may be discontinuous, and, by Art. 8, the points at which this discontinuity occurs must lie on a surface, the equation of which may be expressed in the form

$$\phi = \phi(x, y, z) = 0. \tag{1}$$

This surface separates the region in which $\phi$ is negative from the region in which $\phi$ is positive.

Let $V_1$ denote the potential at any given point in the negative region, and $V_2$ that at any given point in the positive region, then at any point in the surface at which $\phi = 0$, and which may be said to belong to both regions,

$$V_1 + C = V_2, \tag{2}$$

where $C$ is the constant excess of potential, if any, in the substance on the positive side of the surface.

Let $l, m, n$ be the direction-cosines of the normal $\nu_2$ drawn from a given point of the surface into the positive region. Those of the normal $\nu_1$ drawn from the same point into the negative region will be $-l$, $-m$, and $-n$.

The rates of variation of $V$ along the normals are

$$\frac{dV_1}{d\nu_1} = -l\frac{dV_1}{dx} - m\frac{dV_1}{dy} - n\frac{dV_1}{dz}, \tag{3}$$

$$\frac{dV_2}{d\nu_2} = \quad l\frac{dV_2}{dx} + m\frac{dV_2}{dy} + n\frac{dV_2}{dz}. \tag{4}$$

Let any line be drawn on the surface, and let its length, measured from a fixed point in it, be $s$, then at every point of the surface, and therefore at every point of this line, $V_2 - V_1 = C$. Differentiating this equation with respect to $s$, we get

$$\left(\frac{dV_2}{dx} - \frac{dV_1}{dx}\right)\frac{dx}{ds} + \left(\frac{dV_2}{dy} - \frac{dV_1}{dy}\right)\frac{dy}{ds} + \left(\frac{dV_2}{dz} - \frac{dV_1}{dz}\right)\frac{dz}{ds} = 0; \tag{5}$$

and since the normal is perpendicular to this line

$$l\frac{dx}{ds} + m\frac{dy}{ds} + n\frac{dz}{ds} = 0. \tag{6}$$

From (3), (4), (5), (6) we find

$$\frac{dV_2}{dx} - \frac{dV_1}{dx} = l\left(\frac{dV_1}{d\nu_1} + \frac{dV_2}{d\nu_2}\right), \tag{7}$$

$$\frac{dV_2}{dy} - \frac{dV_1}{dy} = m\left(\frac{dV_1}{d\nu_1} + \frac{dV_2}{d\nu_2}\right), \tag{8}$$

$$\frac{dV_2}{dz} - \frac{dV_1}{dz} = n\left(\frac{dV_1}{d\nu_1} + \frac{dV_2}{d\nu_2}\right) *. \tag{9}$$

If we consider the variation of the electromotive intensity at a point in passing through the surface, that component of the intensity which is normal to the surface may change abruptly at the surface, but the other two components parallel to the tangent plane remain continuous in passing through the surface.

78 *b*.] To determine the charge of the surface, let us consider a closed surface which is partly in the positive region and partly in the negative region, and which therefore encloses a portion of the surface of discontinuity.

* {Since (5) and (6) are true for an infinite number of values of $\frac{dx}{ds} : \frac{dy}{ds} : \frac{dz}{ds}$, we have

$$\frac{\frac{dV_2}{dx} - \frac{dV_1}{dx}}{l} = \frac{\frac{dV_2}{dy} - \frac{dV_1}{dy}}{m} = \frac{\frac{dV_2}{dz} - \frac{dV_1}{dz}}{n} = l\left(\frac{dV_2}{dx} - \frac{dV_1}{dx}\right) + m\left(\frac{dV_2}{dy} - \frac{dV_1}{dy}\right) + n\left(\frac{dV_2}{dz} - \frac{dV_1}{dz}\right);$$

and therefore by equations (3) and (4) each of these ratios $= \frac{dV_1}{d\nu_1} + \frac{dV_2}{d\nu_2}$}.

The surface integral,
$$\iint R\cos\epsilon\, dS,$$
extended over this surface, is equal to $4\pi e$, where $e$ is the quantity of electricity within the closed surface.

Proceeding as in Art. 21, we find
$$\iint R\cos\epsilon\, dS = \iiint\left(\frac{dX}{dx}+\frac{dY}{dy}+\frac{dZ}{dz}\right)dx\,dy\,dz$$
$$+\iint\{l(X_2-X_1)+m(Y_2-Y_1)+n(Z_2-Z_1)\}\,dS, \quad (10)$$
where the triple integral is extended throughout the closed surface, and the double integral over the surface of discontinuity.

Substituting for the terms of this equation their values from (7), (8), (9),
$$4\pi e = \iiint 4\pi\rho\, dx\,dy\,dz - \iint\left(\frac{dV_1}{dv_1}+\frac{dV_2}{dv_2}\right)dS. \quad (11)$$

But by the definition of the volume-density, $\rho$, and the surface-density, $\sigma$,
$$4\pi e = 4\pi\iiint\rho\, dx\,dy\,dz + 4\pi\iint\sigma\, dS. \quad (12)$$

Hence, comparing the last terms of these two equations,
$$\frac{dV_1}{dv_1}+\frac{dV_2}{dv_2}+4\pi\sigma=0. \quad (13)$$
This equation is called the characteristic equation of $V$ at an electrified surface of which the surface-density is $\sigma$.

78 $c$.] If $V$ is a function of $x$, $y$, $z$ which, throughout a given continuous region of space, satisfies Laplace's equation
$$\frac{d^2V}{dx^2}+\frac{d^2V}{dy^2}+\frac{d^2V}{dz^2}=0,$$
and if throughout a finite portion of this region $V$ is constant and equal to $C$, then $V$ must be constant and equal to $C$ throughout the whole region in which Laplace's equation is satisfied *.

If $V$ is not equal to $C$ throughout the whole region, let $S$ be the surface which bounds the finite portion within which $V=C$.

At the surface $S$, $V=C$.

Let $\nu$ be a normal drawn outwards from the surface $S$. Since $S$ is the boundary of the continuous region for which $V=C$, the value of $V$ as we travel from the surface along the normal begins

* {It would perhaps be clearer to say that the potential is equal to $C$ at any point which can be reached from the region of constant potential without passing through electricity.}

to differ from $C$. Hence $\frac{dV}{d\nu}$ just outside the surface may be positive or negative, but cannot be zero except for normals drawn from the boundary line between a positive and a negative area.

But if $\nu'$ is the normal drawn inwards from the surface $S$, $V' = C$ and $\frac{dV'}{d\nu'} = 0$.

Hence, at every point of the surface except certain boundary lines,

$$\frac{dV}{d\nu} + \frac{dV'}{d\nu'} \;(= -4\pi\sigma)$$

is a finite quantity, positive or negative, and therefore the surface $S$ has a continuous distribution of electricity over all parts of it except certain boundary lines which separate positively from negatively charged areas.

Laplace's equation is not satisfied at the surface $S$ except at points lying on certain lines on the surface. The surface $S$ therefore, within which $V = C$, includes the whole of the continuous region within which Laplace's equation is satisfied.

### *Force Acting on a Charged Surface.*

79.] The general expressions for the components of the force acting on a charged body parallel to the three axes are of the form

$$A = \iiint \rho X \, dx \, dy \, dz, \tag{14}$$

with similar expressions for $B$ and $C$, the components parallel to $y$ and $z$.

But at a charged surface $\rho$ is infinite, and $X$ may be discontinuous, so that we cannot calculate the force directly from expressions of this form.

We have proved, however, that the discontinuity affects only that component of the intensity which is normal to the charged surface, the other two components being continuous.

Let us therefore assume the axis of $x$ normal to the surface at the given point, and let us also assume, at least in the first part of our investigation, that $X$ is not really discontinuous, but that it changes continuously from $X_1$ to $X_2$ while $x$ changes from $x_1$ to $x_2$. If the result of our calculation gives a definite limiting value for the force when $x_2 - x_1$ is diminished without limit, we may consider it correct when $x_2 = x_1$, and the charged surface has no thickness.

Substituting for $\rho$ its value as found in Art. 77,

$$A = \frac{1}{4\pi}\iiint\left(\frac{dX}{dx} + \frac{dY}{dy} + \frac{dZ}{dz}\right) X\,dx\,dy\,dz. \tag{15}$$

Integrating this expression with respect to $x$ from $x = x_1$ to $x = x_2$ it becomes

$$A = \frac{1}{4\pi}\iint\left[\tfrac{1}{2}(X_2^2 - X_1^2) + \int_{x_1}^{x_2}\left(\frac{dY}{dy} + \frac{dZ}{dz}\right) X\,dx\right] dy\,dz. \tag{16}$$

This is the value of $A$ for a stratum parallel to $yz$ of which the thickness is $x_2 - x_1$.

Since $Y$ and $Z$ are continuous, $\frac{dY}{dy} + \frac{dZ}{dz}$ is finite, and since $X$ is also finite,

$$\int_{x_1}^{x_2}\left(\frac{dY}{dy} + \frac{dZ}{dz}\right) X\,dx < C(x_2 - x_1),$$

where $C$ is the greatest value of $\left(\frac{dY}{dy} + \frac{dZ}{dz}\right) X$ between $x = x_1$ and $x = x_2$.

Hence when $x_2 - x_1$ is diminished without limit this term must ultimately vanish, leaving

$$A = \iint \frac{1}{8\pi}(X_2^2 - X_1^2)\,dy\,dz, \tag{17}$$

where $X_1$ is the value of $X$ on the negative and $X_2$ on the positive side of the surface.

But by Art. 78 $b$, $\quad X_2 - X_1 = \frac{dV_1}{dx} - \frac{dV_2}{dx} = 4\pi\sigma, \qquad (18)$

so that we may write

$$A = \iint \tfrac{1}{2}(X_2 + X_1)\,\sigma\,dy\,dz. \tag{19}$$

Here $dy\,dz$ is the element of the surface, $\sigma$ is the surface-density, and $\frac{1}{2}(X_2 + X_1)$ is the arithmetical mean of the electromotive intensities on the two sides of the surface.

Hence an element of a charged surface is acted on by a force, the component of which normal to the surface is equal to the charge of the element into the arithmetical mean of the normal electromotive intensities on the two sides of the surface.

Since the other two components of the electromotive intensity are not discontinuous, there can be no ambiguity in estimating the corresponding components of the force acting on the surface.

We may now suppose the direction of the normal to the surface to be in any direction with respect to the axes, and write the general expressions for the components of the force on the element of surface $dS$,

$$\left.\begin{aligned} A &= \tfrac{1}{2}(X_1+X_2)\,\sigma dS, \\ B &= \tfrac{1}{2}(Y_1+Y_2)\,\sigma dS, \\ C &= \tfrac{1}{2}(Z_1+Z_2)\,\sigma dS. \end{aligned}\right\} \tag{20}$$

*Charged Surface of a Conductor.*

80.] We have already shewn (Art. 72) that throughout the substance of a conductor in electric equilibrium $X = Y = Z = 0$, and therefore $V$ is constant.

Hence $$\frac{dX}{dx} + \frac{dY}{dy} + \frac{dZ}{dz} = 4\pi\rho = 0,$$

and therefore $\rho$ must be zero throughout the substance of the conductor, or there can be no electricity in the interior of the conductor.

Hence a superficial distribution of electricity is the only possible distribution in a conductor in equilibrium.

A distribution throughout the mass of a body can exist only when the body is a non-conductor.

Since the resultant intensity within the conductor is zero, the resultant intensity just outside the conductor must be in the direction of the normal and equal to $4\pi\sigma$, acting outwards from the conductor.

This relation between the surface-density and the resultant intensity close to the surface of a conductor is known as Coulomb's Law, Coulomb having ascertained by experiment that the electromotive intensity near a given point of the surface of a conductor is normal to the surface and proportional to the surface-density at the given point. The numerical relation

$$R = 4\pi\sigma$$

was established by Poisson.

The force acting on an element, $dS$, of the charged surface of a conductor is, by Art. 79, (since the intensity is zero on the inner side of the surface,)

$$\tfrac{1}{2}R\sigma dS = 2\pi\sigma^2 dS = \frac{1}{8\pi}R^2 dS.$$

This force acts along the normal outwards from the conductor, whether the charge of the surface is positive or negative.

Its value in dynes per square centimetre is

$$\tfrac{1}{2} R\sigma = 2\pi\sigma^2 = \frac{1}{8\pi} R^2,$$

acting as a tension outwards from the surface of the conductor.

81.] If we now suppose an elongated body to be electrified, we may, by diminishing its lateral dimensions, arrive at the conception of an electrified line.

Let $ds$ be the length of a small portion of the elongated body, and let $c$ be its circumference, and $\sigma$ the surface-density of the electricity on its surface; then, if $\lambda$ is the charge per unit of length, $\lambda = c\sigma$, and the resultant electric intensity close to the surface will be

$$4\pi\sigma = 4\pi\frac{\lambda}{c}\cdot$$

If, while $\lambda$ remains finite, $c$ be diminished indefinitely, the intensity at the surface will be increased indefinitely. Now in every dielectric there is a limit beyond which the intensity cannot be increased without a disruptive discharge. Hence a distribution of electricity in which a finite quantity is placed on a finite portion of a line is inconsistent with the conditions existing in nature.

Even if an insulator could be found such that no discharge could be driven through it by an infinite force, it would be impossible to charge a linear conductor with a finite quantity of electricity, for {since a finite charge would make the potential infinite} an infinite electromotive force would be required to bring the electricity to the linear conductor.

In the same way it may be shewn that a point charged with a finite quantity of electricity cannot exist in nature. It is convenient, however, in certain cases, to speak of electrified lines and points, and we may suppose these represented by electrified wires, and by small bodies of which the dimensions are negligible compared with the principal distances concerned.

Since the quantity of electricity on any given portion of a wire at a given potential diminishes indefinitely when the diameter of the wire is indefinitely diminished, the distribution of electricity on bodies of considerable dimensions will not be sensibly affected by the introduction of very fine metallic wires into the field, such as are used to form electrical connexions between these bodies and the earth, an electrical machine, or an electrometer.

## *On Lines of Force.*

82.] If a line be drawn whose direction at every point of its course coincides with that of the resultant intensity at that point, the line is called a Line of Force.

In every part of the course of a line of force, it is proceeding from a place of higher potential to a place of lower potential.

Hence a line of force cannot return into itself, but must have a beginning and an end. The beginning of a line of force must, by § 80, be in a positively charged surface, and the end of a line of force must be in a negatively charged surface.

The beginning and the end of the line are called corresponding points on the positive and negative surface respectively.

If the line of force moves so that its beginning traces a closed curve on the positive surface, its end will trace a corresponding closed curve on the negative surface, and the line of force itself will generate a tubular surface called a tube of induction. Such a tube is called a Solenoid *.

Since the force at any point of the tubular surface is in the tangent plane, there is no induction across the surface. Hence if the tube does not contain any electrified matter, by Art. 77 the total induction through the closed surface formed by the tubular surface and the two ends is zero, and the values of $\iint R\cos\epsilon\, dS$ for the two ends must be equal in magnitude but opposite in sign.

If these surfaces are the surfaces of conductors

$$\epsilon = 0 \quad \text{and} \quad R = -4\pi\sigma,$$

and $\iint R\cos\epsilon\, dS$ becomes $-4\pi\iint\sigma\, dS$, or the charge of the surface multiplied by $4\pi$ †.

Hence the positive charge of the surface enclosed within the closed curve at the beginning of the tube is numerically equal to the negative charge enclosed within the corresponding closed curve at the end of the tube.

Several important results may be deduced from the properties of lines of force.

* From σωλήν, a tube. Faraday uses (3271) the term 'Sphondyloid' in the same sense.

† {$R$ here is drawn outwards from the *tube*.}

The interior surface of a closed conducting vessel is entirely free from charge, and the potential at every point within it is the same as that of the conductor, provided there is no insulated and charged body within the vessel.

For since a line of force must begin at a positively charged surface and end at a negatively charged surface, and since no charged body is within the vessel, a line of force, if it exists within the vessel, must begin and end on the interior surface of the vessel itself.

But the potential must be higher at the beginning of a line of force than at the end of the line, whereas we have proved that the potential at all points of a conductor is the same.

Hence no line of force can exist in the space within a hollow conducting vessel, provided no charged body be placed inside it.

If a conductor within a closed hollow conducting vessel is placed in communication with the vessel, its potential becomes the same as that of the vessel, and its surface becomes continuous with the inner surface of the vessel. The conductor is therefore free from charge.

If we suppose any charged surface divided into elementary portions such that the charge of each element is unity, and if solenoids having these elements for their bases are drawn through the field of force, then the surface-integral for any other surface will be represented by the number of solenoids which it cuts. It is in this sense that Faraday uses his conception of lines of force to indicate not only the direction but the amount of the force at any place in the field.

We have used the phrase Lines of Force because it has been used by Faraday and others. In strictness, however, these lines should be called Lines of Electric Induction.

In the ordinary cases the lines of induction indicate the direction and magnitude of the resultant electromotive intensity at every point, because the intensity and the induction are in the same direction and in a constant ratio. There are other cases, however, in which it is important to remember that these lines indicate primarily the induction, and that the intensity is directly indicated by the equipotential surfaces, being normal to these surfaces and inversely proportional to the distances of consecutive surfaces.

*On Specific Inductive Capacity.*

83 *a*.] In the preceding investigation of surface-integrals we have adopted the ordinary conception of direct action at a distance, and have not taken into consideration any effects depending on the nature of the dielectric medium in which the forces are observed.

But Faraday has observed that the quantity of electricity induced by a given electromotive force on the surface of a conductor which bounds a dielectric is not the same for all dielectrics. The induced electricity is greater for most solid and liquid dielectrics than for air and gases. Hence these bodies are said to have a greater specific inductive capacity than air, which he adopted as the standard medium.

We may express the theory of Faraday in mathematical language by saying that in a dielectric medium the induction across any surface is the product of the normal electric intensity into the coefficient of specific inductive capacity of that medium. If we denote this coefficient by $K$, then in every part of the investigation of surface-integrals we must multiply $X$, $Y$, and $Z$ by $K$, so that the equation of Poisson will become

$$\frac{d}{dx}\cdot K\frac{dV}{dx}+\frac{d}{dy}\cdot K\frac{dV}{dy}+\frac{d}{dz}\cdot K\frac{dV}{dz}+4\pi\rho=0^*. \tag{1}$$

At the surface of separation of two media whose inductive capacities are $K_1$ and $K_2$, and in which the potentials are $V_1$ and $V_2$, the characteristic equation may be written

$$K_1\frac{dV_1}{d\nu_1}+K_2\frac{dV_2}{d\nu_2}+4\pi\sigma=0; \tag{2}$$

where $\nu_1$, $\nu_2$, are the normals drawn in the two media, and $\sigma$ is the true surface-density on the surface of separation; that is to say, the quantity of electricity which is actually on the surface in the form of a charge, and which can be altered only by conveying electricity to or from the spot.

*Apparent distribution of Electricity.*

83 *b*.] If we begin with the actual distribution of the potential and deduce from it the volume-density $\rho'$ and the surface-density $\sigma'$ on the hypothesis that $K$ is everywhere equal to unity, we

* {See note at the end of this chapter.}

may call $\rho'$ the apparent volume-density and $\sigma'$ the apparent surface-density, because a distribution of electricity thus defined would account for the actual distribution of potential, on the hypothesis that the law of electric force as given in Art. 66 requires no modification on account of the different properties of dielectrics.

The apparent charge of electricity within a given region may increase or diminish without any passage of electricity through the bounding surface of the region. We must therefore distinguish it from the true charge, which satisfies the equation of continuity.

In a heterogeneous dielectric in which $K$ varies continuously, if $\rho'$ be the apparent volume-density,

$$\frac{d^2V}{dx^2}+\frac{d^2V}{dy^2}+\frac{d^2V}{dz^2}+4\pi\rho'=0. \tag{3}$$

Comparing this with the equation (1) above, we find

$$4\pi(\rho-K\rho')+\frac{dK}{dx}\frac{dV}{dx}+\frac{dK}{dy}\frac{dV}{dy}+\frac{dK}{dz}\frac{dV}{dz}=0. \tag{4}$$

The true electrification, indicated by $\rho$, in the dielectric whose variable inductive capacity is denoted by $K$, will produce the same potential at every point as the apparent electrification, denoted by $\rho'$, would produce in a dielectric whose inductive capacity is everywhere equal to unity.

The apparent surface charge, $\sigma'$, is that deduced from the electrical forces in the neighbourhood of the surface, using the ordinary characteristic equation

$$\frac{dV_1}{d\nu_1}+\frac{dV_2}{d\nu_2}+4\pi\sigma'=0. \tag{5}$$

If a solid dielectric of any form is a perfect insulator, and if its surface receives no charge, then the true electrification remains zero, whatever be the electrical forces acting on it.

Hence
$$K_1\frac{dV_1}{d\nu_1}+K_2\frac{dV_2}{d\nu_2}=0,$$

$$\frac{dV_1}{d\nu_1}=\frac{4\pi\sigma' K_2}{K_1-K_2}, \qquad \frac{dV_2}{d\nu_2}=\frac{4\pi\sigma' K_1}{K_2-K_2}.$$

The surface-density $\sigma'$ is that of the apparent electrification produced at the surface of the solid dielectric by induction. It disappears entirely when the inducing force is removed, but if during the action of the inducing force the apparent electrification of the surface is discharged by passing a flame over the surface, then, when the inducing force is taken away, there will appear a true electrification opposite to $\sigma'$*.

## APPENDIX TO CHAPTER II.

The equations

$$\frac{d}{dx}\left(K\frac{dV}{dx}\right) + \frac{d}{dy}\left(K\frac{dV}{dy}\right) + \frac{d}{dz}\left(K\frac{dV}{dz}\right) + 4\pi\rho = 0,$$

$$K_2\frac{dV}{d\nu_2} + K_1\frac{dV}{d\nu_1} + 4\pi\sigma = 0,$$

are the expressions of the condition that the displacement across any closed surface is $4\pi$ times the quantity of electricity inside it. The first equation follows at once if we apply this principle to a parallelepiped whose faces are at right angles to the co-ordinate axes, and the second if we apply it to a cylinder enclosing a portion of the charged surface.

If we anticipate the results of the next chapter, we can deduce these equations directly from Faraday's definition of specific inductive capacity. Let us take the case of a condenser consisting of two infinite parallel plates. Let $V_1$, $V_2$ be the potentials of the plates respectively, $d$ the distance between them, and $E$ the charge on an area $A$ of one of the plates, then, if $K$ is the specific inductive capacity of the dielectric separating them,

$$E = KA\,\frac{V_1 - V_2}{4\pi d}.$$

$Q$, the energy of the system, is by Art. 84 equal to

$$\tfrac{1}{2}E(V_1 - V_2) = \tfrac{1}{2}KA\,\frac{(V_1 - V_2)^2}{4\pi d},$$

or if $F$ is the electromotive intensity at any point between the plates

$$Q = \frac{1}{8\pi}KAdF^2.$$

If we regard the energy as resident in the dielectric there will be $Q/Ad$ units of energy per unit of volume, so that the energy per unit volume equals $KF^2/8\pi$. This result will be true when the field is not

* See Faraday's 'Remarks on Static Induction,' *Proceedings of the Royal Institution*, Feb. 12, 1858.

uniform, so that if $Q$ denotes the energy in any electric field

$$Q = \frac{1}{8\pi}\iiint KF^2 dx\,dy\,dz$$

$$= \frac{1}{8\pi}\iiint K\left\{\left(\frac{dV}{dx}\right)^2 + \left(\frac{dV}{dy}\right)^2 + \left(\frac{dV}{dz}\right)^2\right\} dx\,dy\,dz.$$

Let us suppose that the potential at any point of the field is increased by a small quantity $\delta V$ when $\delta V$ is an arbitrary function of $x$, $y$, $z$, then $\delta Q$, the variation in the energy, is given by the equation

$$\delta Q = \frac{1}{4\pi}\iint\left(K\left\{\frac{dV}{dx}\frac{d.\delta V}{dx} + \frac{dV}{dy}\frac{d.\delta V}{dy} + \frac{dV}{dz}\frac{d.\delta V}{dz}\right\}\right) dx\,dy\,dz;$$

this, by Green's Theorem,

$$= -\frac{1}{4\pi}\iint\left(K_1\frac{dV}{d\nu_1} + K_2\frac{dV}{d\nu_2}\right)\delta V dS$$

$$-\frac{1}{4\pi}\iint\left\{\frac{d}{dx}\left(K\frac{dV}{dx}\right) + \frac{d}{dy}\left(K\frac{dV}{dy}\right) + \frac{d}{dz}\left(K\frac{dV}{dz}\right)\right\}\delta V dx\,dy\,dz,$$

where $d\nu_2$ and $d\nu_1$ denote elements of the normal to the surface drawn from the first to the second and from the second to the first medium respectively.

But by (Arts. 85, 86)

$$\delta Q = \Sigma(e\delta V) = \iint \sigma\delta V dS + \iiint \rho\delta V dx\,dy\,dz,$$

and since $\delta V$ is arbitrary we must have

$$-\frac{1}{4\pi}\left(K_1\frac{dV}{d\nu_1} + K_2\frac{dV}{d\nu_2}\right) = \sigma,$$

$$-\frac{1}{4\pi}\left\{\frac{d}{dx}\left(K\frac{dV}{dx}\right) + \frac{d}{dy}\left(K\frac{dV}{dy}\right) + \frac{d}{dz}\left(K\frac{dV}{dz}\right)\right\} = \rho,$$

which are the equations in the text.

In Faraday's experiment the flame may be regarded as a conductor in connexion with the earth, the effect of the dielectric may be represented by an apparent electrification over its surface, this apparent electrification acting on the conducting flame will attract the electricity of the opposite sign, which will spread over the surface of the dielectric while it will drive the electricity of the same sign through the flame to earth. Thus over the surface of the dielectric there will be a real electrification masking the effect of the apparent one; when the inducing force is removed the apparent electrification will disappear but the real electrification will remain and will no longer be masked by the apparent electrification.

# CHAPTER III.

## ON ELECTRICAL WORK AND ENERGY IN A SYSTEM OF CONDUCTORS.

84.] *On the Work which must be done by an external agent in order to charge an electrified system in a given manner.*

The work spent in bringing a quantity of electricity $\delta e$ from an infinite distance (or from any place where the potential is zero) to a given part of the system where the potential is $V$, is, by the definition of potential (Art. 70), $V\delta e$.

The effect of this operation is to increase the charge of the given part of the system by $\delta e$, so that if it was $e$ before, it will become $e+\delta e$ after the operation.

We may therefore express the work done in producing a given alteration in the charges of the system by the integral

$$W = \Sigma\left(\int V\delta e\right); \tag{1}$$

where the summation, $(\Sigma)$, is to be extended to all parts of the electrified system.

It appears from the expression for the potential in Art. 73, that the potential at a given point may be considered as the sum of a number of parts, each of these parts being the potential due to a corresponding part of the charge of the system.

Hence if $V$ is the potential at a given point due to a system of charges which we may call $\Sigma(e)$, and $V'$ the potential at the same point due to another system of charges which we may call $\Sigma(e')$, the potential at the same point due to both systems of charges existing together would be $V+V'$.

If, therefore, every one of the charges of the system is altered in the ratio of $n$ to 1, the potential at any given point in the system will also be altered in the ratio of $n$ to 1.

Let us, therefore, suppose that the operation of charging the system is conducted in the following manner. Let the system be originally free from charge and at potential zero, and let the different portions of the system be charged simultaneously, each at a rate proportional to its final charge.

Thus if $e$ is the final charge, and $V$ the final potential of any part of the system, then, if at any stage of the operation the charge is $n\,e$, the potential will be $n\,V$, and we may represent the process of charging by supposing $n$ to increase continuously from 0 to 1.

While $n$ increases from $n$ to $n+\delta n$, any portion of the system whose final charge is $e$, and whose final potential is $V$, receives an increment of charge $e\,\delta n$, its potential being $n\,V$, so that the work done on it during this operation is $e\,V\,n\,\delta n$.

Hence the whole work done in charging the system is

$$\Sigma\,(eV)\int_0^1 n\,dn = \tfrac{1}{2}\Sigma(eV), \tag{2}$$

or half the sum of the products of the charges of the different portions of the system into their respective potentials.

This is the work which must be done by an external agent in order to charge the system in the manner described, but since the system is a conservative system, the work required to bring the system into the same state by any other process must be the same.

We may therefore call

$$W = \tfrac{1}{2}\Sigma(eV) \tag{3}$$

the electric energy of the system, expressed in terms of the charges of the different parts of the system and their potentials.

85 *a*.] Let us next suppose that the system passes from the state $(e,\ V)$ to the state $(e',\ V')$ by a process in which the different charges increase simultaneously at rates proportional for each to its total increment $e'-e$.

If at any instant the charge of a given portion of the system is $e+n\,(e'-e)$, its potential will be $V+n\,(V'-V)$, and the work done in altering the charge of this portion will be

$$\int_0^1 (e'-e)\,[V+n\,(V'-V)]\,dn = \tfrac{1}{2}\,(e'-e)\,(V'+V);$$

so that if we denote by $W'$ the energy of the system in the state $(e',\ V')$

$$W'-W = \tfrac{1}{2}\Sigma\,(e'-e)\,(V'+V) \tag{4}$$

But $$W = \tfrac{1}{2}\Sigma(eV),$$
and $$W' = \tfrac{1}{2}\Sigma(e'V').$$

Substituting these values in equation (4), we find

$$\Sigma(eV') = \Sigma(e'V). \tag{5}$$

Hence if, in the same fixed system of electrified conductors, we consider two different states of electrification, the sum of the products of the charges in the first state into the potentials of the corresponding portions of the conductors in the second state, is equal to the sum of the products of the charges in the second state into the potentials of the corresponding conductors in the first state.

This result corresponds, in the elementary theory of electricity, to Green's Theorem in the analytical theory. By properly choosing the initial and final states of the system, we may deduce a number of useful results.

85 *b*.] From (4) and (5) we find another expression for the increment of the energy, in which it is expressed in terms of the increments of potential,

$$W' - W = \tfrac{1}{2}\Sigma(e'+e)(V'-V). \tag{6}$$

If the increments are infinitesimal, we may write (4) and (6)

$$dW = \Sigma(V\delta e) = \Sigma(e\delta V); \tag{7}$$

and if we denote by $W_e$ and $W_V$ the expressions for $W$ in terms of the charges and the potentials of the system respectively, and by $A_r$, $e_r$, and $V_r$ a particular conductor of the system, its charge, and its potential, then

$$V_r = \frac{dW_e}{de_r}, \tag{8}$$

$$e_r = \frac{dW_V}{dV_r}. \tag{9}$$

86.] If in any fixed system of conductors, any one of them, which we may denote by $A_t$, is without charge, both in the initial and final state, then for that conductor $e_t = 0$, and $e_t' = 0$, so that the terms depending on $A_t$ vanish from both members of equation (5).

If another conductor, say $A_u$, is at potential zero in both states of the system, then $V_u = 0$ and $V_u' = 0$, so that the terms depending on $A_u$ vanish from both members of equation (5).

If, therefore, all the conductors except two, $A_r$ and $A_s$, are either insulated and without charge, or else connected to the earth, equation (5) is reduced to the form

$$e_r V_r' + e_s V_s' = e_r' V_r + e_s' V_s. \tag{10}$$

If in the initial state

$$e_r = 1 \quad \text{and} \quad e_s = 0,$$

and in the final state

$$e_r' = 0 \quad \text{and} \quad e_s' = 1,$$

equation (10) becomes $\qquad V_r' = V_s;$ $\hfill$ (11)

or if a unit charge communicated to $A_r$ raises $A_s$ when insulated to a potential $V$, then a unit charge communicated to $A_s$ will raise $A_r$ when insulated to the same potential $V$, provided that every one of the other conductors of the system is either insulated and without charge, or else connected to earth so that its potential is zero.

This is the first instance we have met with in electricity of a reciprocal relation. Such reciprocal relations occur in every branch of science, and often enable us to deduce the solutions of new problems from those of simpler problems already solved.

Thus from the fact that at a point outside a conducting sphere whose charge is 1 the potential is $r^{-1}$, where $r$ is the distance from the centre, we conclude that if a small body whose charge is 1 is placed at a distance $r$ from the centre of a conducting sphere without charge, it will raise the potential of the sphere to $r^{-1}$.

Let us next suppose that in the initial state

$$V_r = 1 \quad \text{and} \quad V_s = 0,$$

and in the final state

$$V_r' = 0 \quad \text{and} \quad V_s' = 1,$$

equation (10) becomes $\qquad e_s = e_r';$ $\hfill$ (12)

or if, when $A_r$ is raised to unit potential, a charge $e$ is induced on $A_s$ put to earth, then if $A_s$ is raised to unit potential, an equal charge $e$ will be induced on $A_r$ put to earth.

Let us suppose in the third place, that in the initial state

$$V_r = 1 \quad \text{and} \quad e_s = 0,$$

and that in the final state

$$V_r' = 0 \quad \text{and} \quad e_s' = 1,$$

equation (10) becomes in this case

$$e_r' + V_s = 0. \tag{13}$$

Hence if when $A_s$ is without charge, the operation of charging $A_r$ to potential unity raises $A_s$ to potential $V$, then if $A_r$ is kept at potential zero, a unit charge communicated to $A_s$ will induce on $A_r$ a negative charge, the numerical value of which is $V$.

In all these cases we may suppose some of the other conductors to be insulated and without charge, and the rest to be connected to earth.

The third case is an elementary form of one of Green's theorems. As an example of its use let us suppose that we have ascertained the distribution of electric charge on the different elements of a conducting system at potential zero, induced by a charge unity communicated to a given body $A_s$ of the system.

Let $\eta_r$ be the charge of $A_r$ under these circumstances. Then if we suppose $A_s$ without charge, and the other bodies raised each to a different potential, the potential of $A_s$ will be

$$V_s = -\Sigma\,(\eta_r V_r). \tag{14}$$

Thus if we have ascertained the surface-density at any given point of a hollow conducting vessel at zero potential due to a unit charge placed at a given point within it, then, if we know the value of the potential at every point of a surface of the same size and form as the interior surface of the vessel, we can deduce the potential at a point within it the position of which corresponds to that of the unit charge.

Hence if the potential is known for all points of a closed surface it may be determined for any point within the surface, if there be no electrified body within it, and for any point outside, if there be no electrified body outside.

*Theory of a system of conductors.*

87.] Let $A_1, A_2, \dots A_n$ be $n$ conductors of any form; let $e_1$, $e_2$, $\dots e_n$ be their charges; and $V_1, V_2, \dots V_n$ their potentials.

Let us suppose that the dielectric medium which separates the conductors remains the same, and does not become charged with electricity during the operations to be considered.

We have shown in Art. 84 that the potential of each conductor is a homogeneous linear function of the $n$ charges.

Hence since the electric energy of the system is half the sum of the products of the potential of each conductor into its charge, the electric energy must be a homogeneous quadratic function of the $n$ charges, of the form

$$W_e = \tfrac{1}{2}p_{11}e_1^2 + p_{12}e_1e_2 + \tfrac{1}{2}p_{22}e_2^2 + p_{13}e_1e_3 + p_{23}e_2e_3 + \tfrac{1}{2}p_{33}e_3^2 + \&c. \tag{15}$$

The suffix $e$ indicates that $W$ is to be expressed as a function of the charges. When $W$ is written without a suffix it denotes the expression (3), in which both charges and potentials occur.

From this expression we can deduce the potential of any one of the conductors. For since the potential is defined as the work which must be done to bring a unit of electricity from potential zero to the given potential, and since this work is spent in increasing $W$, we have only to differentiate $W_e$ with respect to the charge of the given conductor to obtain its potential. We thus obtain

$$\left.\begin{array}{l} V_1 = p_{11}e_1 \ldots + p_{r1}e_r \ldots + p_{n1}e_n, \\ - \quad - \quad - \quad - \quad - \quad - \quad - \quad - \\ V_s = p_{1s}e_1 \ldots + p_{rs}e_r \ldots + p_{ns}e_n, \\ - \quad - \quad - \quad - \quad - \quad - \quad - \quad - \\ V_n = p_{1n}e_1 \ldots + p_{rn}e_r \ldots + p_{nn}e_n, \end{array}\right\} \tag{16}$$

a system of $n$ linear equations which express the $n$ potentials in terms of the $n$ charges.

The coefficients $p_{rs}$ &c., are called coefficients of potential. Each has two suffixes, the first corresponding to that of the charge, and the second to that of the potential.

The coefficient $p_{rr}$, in which the two suffixes are the same, denotes the potential of $A_r$ when its charge is unity, that of all the other conductors being zero. There are $n$ coefficients of this kind, one for each conductor.

The coefficient $p_{rs}$, in which the two suffixes are different, denotes the potential of $A_s$ when $A_r$ receives a charge unity, the charge of each of the other conductors, except $A_r$, being zero.

We have already proved in Art. 86 that $p_{rs} = p_{sr}$, but we may prove it more briefly by considering that

$$p_{rs} = \frac{dV_s}{de_r} = \frac{d}{de_r}\frac{dW_e}{de_s} = \frac{d}{de_s}\frac{dW_e}{de_r} = \frac{dV_r}{de_s} = p_{sr}. \tag{17}$$

The number of *different* coefficients with two different suffixes is therefore $\frac{1}{2}n(n-1)$, being one for each pair of conductors.

By solving the equations (16) for $e_1$, $e_2$, &c., we obtain $n$ equations giving the charges in terms of the potentials

$$\left.\begin{array}{l} e_1 = q_{11}V_1 \ldots + q_{1s}V_s \ldots + q_{1n}V_n, \\ - \quad - \quad - \quad - \quad - \quad - \quad - \quad - \\ e_r = q_{r1}V_1 \ldots + q_{rs}V_s \ldots + q_{rn}V_n, \\ - \quad - \quad - \quad - \quad - \quad - \quad - \quad - \\ e_n = q_{n1}V_1 \ldots + q_{ns}V_s \ldots + q_{n}V_{nn}, \end{array}\right\} \tag{18}$$

We have in this case also $q_{rs} = q_{sr}$, for

$$q_{rs} = \frac{de_r}{dV_s} = \frac{d}{dV_s}\frac{dW_V}{dV_r} = \frac{d}{dV_r}\frac{dW_V}{dV_s} = \frac{de_s}{dV_r} = q_{sr}. \tag{19}$$

By substituting the values of the charges in the equation for the electric energy

$$W = \tfrac{1}{2}\left[e_1 V_1 + \ldots + e_r V_r \ldots + e_n V_n\right], \tag{20}$$

we obtain an expression for the energy in terms of the potentials

$$\begin{aligned} W_V = \tfrac{1}{2} q_{11} V_1^2 + q_{12} V_1 V_2 + \tfrac{1}{2} q_{22} V_2^2 \\ + q_{13} V_1 V_3 + q_{23} V_2 V_3 + \tfrac{1}{2} q_{33} V_3^2 + \&c. \end{aligned} \tag{21}$$

A coefficient in which the two suffixes are the same is called the Electric Capacity of the conductor to which it belongs.

*Definition.* The Capacity of a conductor is its charge when its own potential is unity, and that of all the other conductors is zero.

This is the proper definition of the capacity of a conductor when no further specification is made. But it is sometimes convenient to specify the condition of some or all of the other conductors in a different manner, as for instance to suppose that the charge of certain of them is zero, and we may then define the capacity of the conductor under these conditions as its charge when its potential is unity.

The other coefficients are called coefficients of induction. Any one of them, as $q_{rs}$, denotes the charge of $A_r$ when $A_s$ is raised to potential unity, the potential of all the conductors except $A_s$ being zero.

The mathematical calculation of the coefficients of potential and of capacity is in general difficult. We shall afterwards prove that they have always determinate values, and in certain special cases we shall calculate these values. We shall also show how they may be determined by experiment.

When the capacity of a conductor is spoken of without specifying the form and position of any other conductor in the same system, it is to be interpreted as the capacity of the conductor when no other conductor or electrified body is within a finite distance of the conductor referred to.

It is sometimes convenient, when we are dealing with capacities and coefficients of induction only, to write them in the form $[A.P]$, this symbol being understood to denote the charge on $A$ when

$P$ is raised to unit potential {the other conductors being all at zero potential}.

In like manner $[(A+B) \,.\, (P+Q)]$ would denote the charge on $A+B$ when $P$ and $Q$ are both raised to potential 1; and it is manifest that since

$$[(A+B) \,.\, (P+Q)] = [A \,.\, P]+[A \,.\, Q]+[B \,.\, P]+[B \,.\, Q]$$
$$= [(P+Q) \,.\, (A+B)],$$

the compound symbols may be combined by addition and multiplication as if they were symbols of quantity.

The symbol $[A \,.\, A]$ denotes the charge on $A$ when the potential of $A$ is 1, that is to say, the capacity of $A$.

In like manner $[(A+B) \,.\, (A+Q)]$ denotes the sum of the charges on $A$ and $B$ when $A$ and $Q$ are raised to potential 1, the potential of all the conductors except $A$ and $Q$ being zero.

It may be decomposed into

$$[A \,.\, A]+[A \,.\, B]+[A \,.\, Q]+[B \,.\, Q].$$

The coefficients of potential cannot be dealt with in this way. The coefficients of induction represent charges, and these charges can be combined by addition, but the coefficients of potential represent potentials, and if the potential of $A$ is $V_1$ and that of $B$ is $V_2$, the sum $V_1+V_2$ has no physical meaning bearing on the phenomena, though $V_1-V_2$ represents the electromotive force from $A$ to $B$.

The coefficients of induction between two conductors may be expressed in terms of the capacities of the conductors and that of the two conductors together, thus:

$$[A \,.\, B] = \tfrac{1}{2}[(A+B) \,.\, (A+B)]-\tfrac{1}{2}[A \,.\, A]-\tfrac{1}{2}[B \,.\, B].$$

*Dimensions of the coefficients.*

88.] Since the potential of a charge $e$ at a distance $r$ is $\frac{e}{r}$, the dimensions of a charge of electricity are equal to those of the product of a potential into a line.

The coefficients of capacity and induction have therefore the same dimensions as a line, and each of them may be represented by a straight line, the length of which is independent of the system of units which we employ.

For the same reason, any coefficient of potential may be represented as the reciprocal of a line.

*On certain conditions which the coefficients must satisfy.*

89 *a*.] In the first place, since the electric energy of a system is an essentially positive quantity, its expression as a quadratic function of the charges or of the potentials must be positive, whatever values, positive or negative, are given to the charges or the potentials.

Now the conditions that a homogeneous quadratic function of $n$ variables shall be always positive are $n$ in number, and may be written

$$\left.\begin{array}{c} p_{11} > 0, \\ \begin{vmatrix} p_{11}, p_{12} \\ p_{21}, p_{22} \end{vmatrix} > 0, \\ \text{- - - - - - -} \\ \begin{vmatrix} p_{11} \cdots p_{1n} \\ \text{- - - - -} \\ p_{n1} \cdots p_{nn} \end{vmatrix} > 0. \end{array}\right\} \tag{22}$$

These $n$ conditions are necessary and sufficient to ensure that $W_e$ shall be essentially positive *.

But since in equation (16) we may arrange the conductors in any order, every determinant must be positive which is formed symmetrically from the coefficients belonging to any combination of the $n$ conductors, and the number of these combinations is $2^n - 1$.

Only $n$, however, of the conditions so found can be independent.

The coefficients of capacity and induction are subject to conditions of the same form.

89 *b*.] *The coefficients of potential are all positive, but none of the coefficients* $p_{rs}$ *is greater than* $p_{rr}$ *or* $p_{ss}$.

For let a charge unity be communicated to $A_r$, the other conductors being uncharged. A system of equipotential surfaces will be formed. Of these one will be the surface of $A_r$, and its potential will be $p_{rr}$. If $A_s$ is placed in a hollow excavated in $A_r$ so as to be completely enclosed by it, then the potential of $A_s$ will also be $p_{rr}$.

If, however, $A_s$ is outside of $A_r$ its potential $p_{rs}$ will lie between $p_{rr}$ and zero.

* See Williamson's *Differential Calculus*, 3rd edition, p. 407.

For consider the lines of force issuing from the charged conductor $A_r$. The charge is measured by the excess of the number of lines which issue from it over those which terminate in it. Hence, if the conductor has no charge, the number of lines which enter the conductor must be equal to the number which issue from it. The lines which enter the conductor come from places of greater potential, and those which issue from it go to places of less potential. Hence the potential of an uncharged conductor must be intermediate between the highest and lowest potentials in the field, and therefore the highest and lowest potentials cannot belong to any of the uncharged bodies.

The highest potential must therefore be $p_{rr}$, that of the charged body $A_r$, the lowest must be that of space at an infinite distance, which is zero, and all the other potentials such as $p_{rs}$ must lie between $p_{rr}$ and zero.

If $A_s$ completely surrounds $A_t$, then $p_{rs} = p_{rt}$.

89 *c*.] *None of the coefficients of induction are positive, and the sum of all those belonging to a single conductor is not numerically greater than the coefficient of capacity of that conductor, which is always positive.*

For let $A_r$ be maintained at potential unity while all the other conductors are kept at potential zero, then the charge on $A_r$ is $q_{rr}$, and that on any other conductor $A_s$ is $q_{rs}$.

The number of lines of force which issue from $A_r$ is $q_{rr}$. Of these some terminate in the other conductors, and some may proceed to infinity, but no lines of force can pass between any of the other conductors or from them to infinity, because they are all at potential zero.

No line of force can issue from any of the other conductors such as $A_s$, because no part of the field has a lower potential than $A_s$. If $A_s$ is completely cut off from $A_r$ by the closed surface of one of the conductors, then $q_{rs}$ is zero. If $A_s$ is not thus cut off, $q_{rs}$ is a negative quantity.

If one of the conductors $A_t$ completely surrounds $A_r$, then all the lines of force from $A_r$ fall on $A_t$ and the conductors within it, and the sum of the coefficients of induction of these conductors with respect to $A_r$ will be equal to $q_{rr}$ with its sign changed. But if $A_r$ is not completely surrounded by a conductor the arithmetical sum of the coefficients of induction $q_{rs}$, &c. will be less than $q_{rr}$.

We have deduced these two theorems independently by means of electrical considerations. We may leave it to the mathematical student to determine whether one is a mathematical consequence of the other.

89 *d*.] When there is only one conductor in the field its coefficient of potential on itself is the reciprocal of its capacity.

The centre of mass of the electricity when there are no external forces is called the electric centre of the conductor. If the conductor is symmetrical about a centre of figure, this point is the electric centre. If the dimensions of the conductor are small compared with the distances considered, the position of the electric centre may be estimated sufficiently nearly by conjecture.

The potential at a distance $c$ from the electric centre must be between

$$\frac{e}{c}\left(1+\frac{a^2}{c^2}\right) \text{ and } \frac{e}{c}\left(1-\tfrac{1}{2}\frac{a^2}{c^2}\right)*;$$

where $e$ is the charge, and $a$ is the greatest distance of any part of the surface of the body from the electric centre.

For if the charge be concentrated in two points at distances $a$ on opposite sides of the electric centre, the first of these expressions is the potential at a point in the line joining the charges, and the second at a point in a line perpendicular to the line joining the charges. For all other distributions within the sphere whose radius is $a$ the potential is intermediate between those values.

If there are two conductors in the field, their mutual coefficient of potential is $\frac{1}{c'}$, where $c'$ cannot differ from $c$, the distance between the electric centres, by more than $\frac{a^2+b^2}{c}$; $a$ and $b$ being the greatest distances of any part of the surfaces of the bodies from their respective electric centres.

* {For let $\rho$ be the density of the electricity at any point, then if we take the line joining the electric centre to $P$ as the axis of $z$, the potential at $P$ is

$$\iiint\frac{\rho\,dx\,dy\,dz}{r}=\iiint\rho\left\{\frac{1}{c}+\frac{z}{c^2}+\frac{2z^2-(x^2+y^2)}{2c^3}+\dots\right\}dx\,dy\,dz,$$

where $c$ is the distance of $P$ from the electric centre. The first term equals $e/c$, the second vanishes since the origin is the electric centre, and the greatest value of the

89 *e*.] If a new conductor is brought into the field the coefficient of potential of any one of the others on itself is diminished.

For let the new body, $B$, be supposed at first to be a non-conductor {having the same specific inductive capacity as air} free from charge in any part, then when one of the conductors, $A_1$, receives a charge $e_1$, the distribution of the electricity on the conductors of the system will not be disturbed by $B$, as $B$ is still without charge in any part, and the electric energy of the system will be simply

$$\tfrac{1}{2} e_1 V_1 = \tfrac{1}{2} e_1^2 p_{11}.$$

Now let $B$ become a conductor. Electricity will flow from places of higher to places of lower potential, and in so doing will diminish the electric energy of the system, so that the quantity $\frac{1}{2} e_1^2 p_{11}$ must diminish.

But $e_1$ remains constant, therefore $p_{11}$ must diminish.

Also if $B$ increases by another body $b$ being placed in contact with it, $p_{11}$ will be further diminished.

For let us first suppose that there is no electric communication between $B$ and $b$; the introduction of the new body $b$ will diminish $p_{11}$. Now let a communication be opened between $B$ and $b$. If any electricity flows through it, it flows from a place of higher to a place of lower potential, and therefore, as we have shewn, still further diminishes $p_{11}$.

third is when the electricity is concentrated at the points for which the third term inside the bracket has its greatest value, which is $a^2/e^3$, thus the greatest value of the third term is $ea^2/c^3$; the least value of this term is when the electricity is concentrated at the points for which the third term inside the bracket has its greatest negative value which is $-\frac{1}{2}a^2/c^3$; thus the least value of the third term is $-\frac{1}{2}ea^2/c^3$.

The result at the end of Art. 89 *d* may be deduced as follows. Suppose the charge is on the first conductor, then the potential due to the electricity on this conductor by the above is less than

$$\frac{e}{R} + \frac{ea^2}{R^3},$$

where $R$ is the distance of the point from the electric centre of the first conductor; in the second term if we are only proceeding as far as $c^{-3}$, we may put $R=c$ for any point on the second conductor. The first term represents the potential to which the second conductor is raised by a charge $e$ at the electric centre of the first, but by Art. 86, this is the same as the potential at the electric centre of the first due to a charge $e$ on the second conductor, but we have just seen that this must be less than

$$\frac{e}{c} + \frac{eb^2}{c^3};$$

thus the potential of the second conductor due to a charge $e$ on the first must be less than

$$\frac{e}{c} + \frac{e(a^2+b^2)}{c^3}.$$

This however is not in general a very close approximation to the mutual potential of two conductors.}

Hence the diminution of $p_{11}$ by the body $B$ is greater than that which would be produced by any conductor the surface of which can be inscribed in $B$, and less than that produced by any conductor the surface of which can be described about $B$.

We shall shew in Chapter XI, that a sphere of diameter $b$ at a distance $r$, great compared with $b$, diminishes the value of $p_{11}$ by a quantity which is approximately $\frac{1}{8}\frac{b^3}{r^4}$ *.

Hence if the body $B$ is of any other figure, and if $b$ is its greatest diameter, the diminution of the value of $p_{11}$ must be less than $\frac{1}{8}\frac{b^3}{r^4}$.

Hence if the greatest diameter of $B$ is so small compared with its distance from $A_1$ that we may neglect quantities of the order $\frac{1}{8}\frac{b^3}{r^4}$, we may consider the reciprocal of the capacity of $A_1$ when alone in the field as a sufficient approximation to $p_{11}$.

90 *a*.] Let us therefore suppose that the capacity of $A_1$ when alone in the field is $K_1$, and that of $A_2$, $K_2$, and let the mean distance between $A_1$ and $A_2$ be $r$, where $r$ is very great compared with the greatest dimensions of $A_1$ and $A_2$, then we may write

$$p_{11} = \frac{1}{K_1}, \quad p_{12} = \frac{1}{r}, \quad p_{22} = \frac{1}{K_2};$$

$$V_1 = e_1 K_1^{-1} + e_2 r^{-1},$$

$$V_2 = e_1 r^{-1} \quad + e_2 K_2^{-1}.$$

Hence

$$q_{11} = \quad K_1 (1 - K_1 K_2 r^{-2})^{-1},$$

$$q_{12} = -K_1 K_2 r^{-1} (1 - K_1 K_2 r^{-2})^{-1},$$

$$q_{22} = \quad K_2 (1 - K_1 K_2 r^{-2})^{-1}.$$

Of these coefficients $q_{11}$ and $q_{22}$ are the capacities of $A_1$ and $A_2$ when, instead of being each alone at an infinite distance from any other body, they are brought so as to be at a distance $r$ from each other.

90 *b*.] When two conductors are placed so near together that their coefficient of mutual induction is large, the combination is called a Condenser.

Let $A$ and $B$ be the two conductors or electrodes of a condenser.

* {See equation (43), Art. 146.}

Let $L$ be the capacity of $A$, $N$ that of $B$, and $M$ the coefficient of mutual induction. (We must remember that $M$ is essentially negative, so that the numerical values of $L+M$ and $M+N$ are less than $L$ and $N$.)

Let us suppose that $a$ and $b$ are the electrodes of another condenser at a distance $R$ from the first, $R$ being very great compared with the dimensions of either condenser, and let the coefficients of capacity and induction of the condenser $a\,b$ when alone be $l$, $n$, $m$. Let us calculate the effect of one of the condensers on the coefficients of the other.

Let $\qquad D = LN - M^2, \quad$ and $\quad d = ln - m^2$;

then the coefficients of potential for each condenser by itself are

$$\begin{aligned} p_{AA} &= D^{-1}N, & p_{aa} &= d^{-1}n, \\ p_{AB} &= -D^{-1}M, & p_{ab} &= -d^{-1}m, \\ p_{BB} &= D^{-1}L, & p_{bb} &= d^{-1}l. \end{aligned}$$

The values of these coefficients will not be sensibly altered when the two condensers are at a distance $R$.

The coefficient of potential of any two conductors at distance $R$ is $R^{-1}$, so that

$$p_{Aa} = p_{Ab} = p_{Ba} = p_{Bb} = R^{-1}.$$

The equations of potential are therefore

$$\begin{aligned} V_A &= D^{-1}Ne_A - D^{-1}Me_B + R^{-1}e_a + R^{-1}e_b, \\ V_B &= -D^{-1}Me_A + D^{-1}Le_B + R^{-1}e_a + R^{-1}e_b, \\ V_a &= R^{-1}e_A + R^{-1}e_B + d^{-1}ne_a - d^{-1}me_b, \\ V_b &= R^{-1}e_A + R^{-1}e_B - d^{-1}me_a + d^{-1}le_b. \end{aligned}$$

Solving these equations for the charges, we find

$$q_{AA} = L' = L + \frac{(L+M)^2(l+2m+n)}{R^2 - (L+2M+N)(l+2m+n)},$$

$$q_{AB} = M' = M + \frac{(L+M)(M+N)(l+2m+n)}{R^2 - (L+2M+N)(l+2m+n)},$$

$$q_{Aa} = -\frac{R(L+M)(l+m)}{R^2 - (L+2M+N)(l+2m+n)},$$

$$q_{Ab} = -\frac{R(L+M)(m+n)}{R^2 - (L+2M+N)(l+2m+n)};$$

where $L'$, $M'$, $N'$ are what $L$, $M$, $N$ become when the second condenser is brought into the field.

If only one conductor, $a$, is brought into the field, $m = n = 0$, and

$$q_{AA} = L' = L + \frac{(L+M)^2 l}{R^2 - l(L+2M+N)},$$

$$q_{AB} = M' = M + \frac{(L+M)(M+N)l}{R^2 - l(L+2M+N)},$$

$$q_{Aa} = -\frac{Rl(L+M)}{R^2 - l(L+2M+N)}.$$

If there are only the two simple conductors, $A$ and $a$,

$$M = N = m = n = 0,$$

and
$$q_{AA} = L + \frac{L^2 l}{R^2 - Ll}, \qquad q_{Aa} = -\frac{RLl}{R^2 - Ll};$$

expressions which agree with those found in Art. 90 $a$.

The quantity $L + 2M + N$ is the total charge of the condenser when its electrodes are at potential 1. It cannot exceed half the greatest diameter of the condenser *.

$L + M$ is the charge of the first electrode, and $M + N$ that of the second when both are at potential 1. These quantities must be each of them positive and less than the capacity of the electrode by itself. Hence the corrections to be applied to the coefficients of capacity of a condenser are much smaller than those for a simple conductor of equal capacity.

Approximations of this kind are often useful in estimating the capacities of conductors of irregular form placed at a considerable distance from other conductors.

91.] When a round conductor, $A_3$, of small size compared with the distances between the conductors, is brought into the field, the coefficient of potential of $A_1$ on $A_2$ will be increased when $A_3$ is inside and diminished when $A_3$ is outside of a sphere whose diameter is the straight line $A_1 A_2$.

For if $A_1$ receives a unit positive charge there will be a distribution of electricity on $A_3$, $+e$ being on the side furthest from $A_1$, and $-e$ on the side nearest $A_1$. The potential at $A_2$ due to this distribution on $A_3$ will be positive or negative as $+e$ or $-e$ is nearest to $A_2$, and if the form of $A_3$ is not very elongated this will depend on whether the angle $A_1 A_3 A_2$ is obtuse or acute, and therefore on whether $A_3$ is inside or outside the sphere described on $A_1 A_2$ as diameter.

* {For we may prove, as in Art. 89 $e$, that the capacity of a condenser all of whose parts are at the same potential is less than that of the sphere circumscribing it, and the capacity of the sphere is equal to its radius.}

If $A_3$ is of an elongated form it is easy to see that if it is placed with its longest axis in the direction of the tangent to the circle drawn through the points $A_1$, $A_3$, $A_2$ it may increase the potential of $A_2$, even when it is entirely outside the sphere, and that if it is placed with its longest axis in the direction of the radius of the sphere, it may diminish the potential of $A_2$ even when entirely within the sphere. But this proposition is only intended for forming a rough estimate of the phenomena to be expected in a given arrangement of apparatus.

92.] If a new conductor, $A_3$, is introduced into the field, the capacities of all the conductors already there are increased, and the numerical values of the coefficients of induction between every pair of them are diminished.

Let us suppose that $A_1$ is at potential unity and all the rest at potential zero. Since the charge of the new conductor is negative it will induce a positive charge on every other conductor, and will therefore increase the positive charge of $A_1$ and diminish the negative charge of each of the other conductors.

93 *a*.] *Work done by the electric forces during the displacement of a system of insulated charged conductors.*

Since the conductors are insulated, their charges remain constant during the displacement. Let their potentials be $V_1$, $V_2, \ldots V_n$ before and $V_1'$, $V_2', \ldots V_n'$ after the displacement. The electric energy is

$$W = \tfrac{1}{2}\Sigma(eV)$$

before the displacement, and

$$W' = \tfrac{1}{2}\Sigma(eV')$$

after the displacement.

The work done by the electric forces during the displacement is the excess of the initial energy $W$ over the final energy $W'$, or

$$W - W' = \tfrac{1}{2}\Sigma\,[e(V - V')].$$

This expression gives the work done during any displacement, small or large, of an insulated system.

To find the force tending to produce a particular kind of displacement, let $\phi$ be the variable whose variation corresponds to the kind of displacement, and let $\Phi$ be the corresponding force, reckoned positive when the electric force tends to increase $\phi$, then

$$\Phi\, d\phi = -d\,W_e,$$

or

$$\Phi = -\frac{dW_e}{d\phi};$$

where $W_e$ denotes the expression for the electric energy as a quadratic function of the charges.

93 b.] To prove that $\dfrac{dW_e}{d\phi} + \dfrac{dW_V}{d\phi} = 0.$

We have three different expressions for the energy of the system,

(1) $$W = \tfrac{1}{2}\Sigma(eV),$$

a definite function of the $n$ charges and $n$ potentials,

(2) $$W_e = \tfrac{1}{2}\Sigma\Sigma(e_r e_s p_{rs}),$$

where $r$ and $s$ may be the same or different, and both $rs$ and $sr$ are to be included in the summation.

This is a function of the $n$ charges and of the variables which define the configuration. Let $\phi$ be one of these.

(3) $$W_V = \tfrac{1}{2}\Sigma\Sigma(V_r V_s q_{rs}),$$

where the summation is to be taken as before. This is a function of the $n$ potentials and of the variables which define the configuration of which $\phi$ is one.

Since $$W = W_e = W_V,$$
$$W_e + W_V - 2W = 0.$$

Now let the $n$ charges, the $n$ potentials, and $\phi$ vary in any consistent manner, and we must have

$$\Sigma\left[\left(\frac{dW_e}{de_r} - V_r\right)\delta e_r\right] + \Sigma\left[\left(\frac{dW_V}{dV_s} - e_s\right)\delta V_s\right] + \left(\frac{dW_e}{d\phi} + \frac{dW_V}{d\phi}\right)\delta\phi = 0.$$

Now the $n$ charges, the $n$ potentials, and $\phi$ are not all independent of each other, for in fact only $n+1$ of them can be independent. But we have already proved that

$$\frac{dW_e}{de_r} = V_r,$$

so that the first sum of terms vanishes identically, and it follows from this, even if we had not already proved it, that

$$\frac{dW_V}{dV_s} = e_s,$$

and that lastly, $$\frac{dW_e}{d\phi} + \frac{dW_V}{d\phi} = 0.$$

*Work done by the electric forces during the displacement of a system whose potentials are maintained constant.*

93 c.] It follows from the last equation that the force $\Phi = \dfrac{dW_V}{d\phi}$, and if the system is displaced under the condition that all the

potentials remain constant, the work done by the electric forces is

$$\int \Phi d\phi = \int dW_V = W_V' - W_V;$$

or the work done by the electric forces in this case is equal to the *increment* of the electric energy.

Here, then, we have an increase of energy together with a quantity of work done by the system. The system must therefore be supplied with energy from some external source, such as a voltaic battery, in order to maintain the potentials constant during the displacement.

The work done by the battery is therefore equal to the sum of the work done by the system and the increment of energy, or, since these are equal, the work done by the battery is twice the work done by the system of conductors during the displacement.

*On the comparison of similar electrified systems.*

94.] If two electrified systems are similar in a geometrical sense, so that the lengths of corresponding lines in the two systems are as $L$ to $L'$, then if the dielectric which separates the conducting bodies is the same in both systems, the coefficients of induction and of capacity will be in the proportion of $L$ to $L'$. For if we consider corresponding portions, $A$ and $A'$, of the two systems, and suppose the quantity of electricity on $A$ to be $e$, and that on $A'$ to be $e'$, then the potentials $V$ and $V'$ at corresponding points $B$ and $B'$, due to this electrification, will be

$$V = \frac{e}{AB}, \text{ and } V' = \frac{e'}{A'B'}.$$

But $AB$ is to $A'B'$ as $L$ to $L'$, so that we must have

$$e : e' :: LV : L'V'.$$

But if the inductive capacity of the dielectric is different in the two systems, being $K$ in the first and $K'$ in the second, then if the potential at any point of the first system is to that at the corresponding point of the second as $V$ to $V'$, and if the quantities of electricity on corresponding parts are as $e$ and $e'$, we shall have

$$e : e' :: LVK : L'V'K'.$$

By this proportion we may find the relation between the total charges of corresponding parts of two systems, which are in the first place geometrically similar, in the second place composed of dielectric media of which the specific inductive capacities at corresponding points are in the proportion of $K$ to $K'$, and in the third place so electrified that the potentials of corresponding points are as $V$ to $V'$.

From this it appears that if $q$ be any coefficient of capacity or induction in the first system, and $q'$ the corresponding one in the second,

$$q : q' :: LK : L'K';$$

and if $p$ and $p'$ denote corresponding coefficients of potential in the two systems,

$$p : p' :: \frac{1}{LK} : \frac{1}{L'K'}.$$

If one of the bodies be displaced in the first system, and the corresponding body in the second system receives a similar displacement, then these displacements are in the proportion of $L$ to $L'$, and if the forces acting on the two bodies are as $F$ to $F'$, then the work done in the two systems will be as $FL$ to $F'L'$.

But the total electric energy is half the sum of the charges of electricity multiplied each by the potential of the charged body, so that in the similar systems, if $W$ and $W'$ be the total electric energies in the two systems respectively,

$$W : W' :: eV : e'V',$$

and the differences of energy after similar displacements in the two systems will be in the same proportion. Hence, since $FL$ is proportional to the electrical work done during the displacement,

$$FL : F'L' :: eV : e'V'.$$

Combining these proportions, we find that the ratio of the resultant force on any body of the first system to that on the corresponding body of the second system is

$$F : F' :: V^2K : V'^2K',$$

or

$$F : F' :: \frac{e^2}{L^2K} : \frac{e'^2}{L'^2K'}.$$

The first of these proportions shews that in similar systems the force is proportional to the square of the electromotive force and to the inductive capacity of the dielectric, but is independent of the actual dimensions of the system.

Hence two conductors placed in a liquid whose inductive capacity is greater than that of air, and electrified to given potentials, will attract each other more than if they had been electrified to the same potentials in air.

The second proportion shews that if the quantity of electricity on each body is given, the forces are proportional to the squares of the charges and inversely to the squares of the distances, and also inversely to the inductive capacities of the media.

Hence, if two conductors with given charges are placed in a liquid whose inductive capacity is greater than that of air, they will attract each other less than if they had been surrounded by air and charged with the same quantities of electricity*.

* {It follows from the preceding investigation that the force between two electrified bodies surrounded by a medium whose specific inductive capacity is $K$ is $ee'/Kr^2$, where $e$ and $e'$ are the charges on the bodies and $r$ is the distance between them.}

# CHAPTER IV.

## GENERAL THEOREMS.

95 *a*.] IN the second chapter we have calculated the potential function and investigated some of its properties on the hypothesis that there is a direct action at a distance between electrified bodies, which is the resultant of the direct actions between the various electrified parts of the bodies.

If we call this the direct method of investigation, the inverse method will consist in assuming that the potential is a function characterised by properties the same as those which we have already established, and investigating the form of the function.

In the direct method the potential is calculated from the distribution of electricity by a process of integration, and is found to satisfy certain partial differential equations. In the inverse method the partial differential equations are supposed given, and we have to find the potential and the distribution of electricity.

It is only in problems in which the distribution of electricity is given that the direct method can be used. When we have to find the distribution on a conductor we must make use of the inverse method.

We have now to shew that the inverse method leads in every case to a determinate result, and to establish certain general theorems deduced from Poisson's partial differential equation,

$$\frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} + \frac{d^2V}{dz^2} + 4\pi\rho = 0.$$

The mathematical ideas expressed by this equation are of a different kind from those expressed by the definite integral

$$V = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{\rho}{r}\, dx'\, dy'\, dz'.$$

In the differential equation we express that the sum of the second derivatives of $V$ in the neighbourhood of any point is related to the density at that point in a certain manner, and no relation is expressed between the value of $V$ at that point and the value of $\rho$ at any point at a finite distance from it.

In the definite integral, on the other hand, the distance of the point $(x', y', z')$, at which $\rho$ exists, from the point $(x, y, z)$, at which $V$ exists, is denoted by $r$, and is distinctly recognised in the expression to be integrated.

The integral, therefore, is the appropriate mathematical expression for a theory of action between particles at a distance, whereas the differential equation is the appropriate expression for a theory of action exerted between contiguous parts of a medium.

We have seen that the result of the integration satisfies the differential equation. We have now to shew that it is the only solution of that equation satisfying certain conditions.

We shall in this way not only establish the mathematical equivalence of the two expressions, but prepare our minds to pass from the theory of direct action at a distance to that of action between contiguous parts of a medium.

95 *b*.] The theorems considered in this chapter relate to the properties of certain volume-integrals taken throughout a finite region of space which we may refer to as the electric field.

The element of these integrals, that is to say, the quantity under the integral sign, is either the square of a certain vector quantity whose direction and magnitude vary from point to point in the field, or the product of one vector into the resolved part of another in its own direction.

Of the different modes in which a vector quantity may be distributed in space, two are of special importance.

The first is that in which the vector may be represented as the space-variation [Art. 17] of a scalar function called the Potential.

Such a distribution may be called an Irrotational distribution. The resultant force arising from the attraction or repulsion of any combination of centres of force, the law of each being any given function of the distance, is distributed irrotationally.

The second mode of distribution is that in which the convergence [Art. 25] is zero at every point. Such a distribution may be called a Solenoidal distribution. The velocity of an incompressible fluid is distributed in a solenoidal manner.

When the central forces which, as we have said, give rise to an irrotational distribution of the resultant force, vary according to the inverse square of the distance, then, if these centres are outside the field, the distribution within the field will be solenoidal as well as irrotational.

When the motion of an incompressible fluid which, as we have said, is solenoidal, arises from the action of central forces depending on the distance, or of surface pressures, on a frictionless fluid originally at rest, the distribution of velocity is irrotational as well as solenoidal.

When we have to specify a distribution which is at once irrotational and solenoidal, we shall call it a Laplacian distribution; Laplace having pointed out some of the most important properties of such a distribution.

The volume integrals discussed in this chapter are, as we shall see, expressions for the energy of the electric field. In the first group of theorems, beginning with Green's Theorem, the energy is expressed in terms of the electromotive intensity, a vector which is distributed irrotationally in all cases of electric equilibrium. It is shewn that if the surface-potentials be given, then of all irrotational distributions, that which is also solenoidal has the least energy; whence it also follows that there can be only one Laplacian distribution consistent with the surface potentials.

In the second group of theorems, including Thomson's Theorem, the energy is expressed in terms of the electric displacement, a vector of which the distribution is solenoidal. It is shewn that if the surface-charges are given, then of all solenoidal distributions that has least energy which is also irrotational, whence it also follows that there can be only one Laplacian distribution consistent with the given surface-charges.

The demonstration of all these theorems is conducted in the same way. In order to avoid the repetition in every case of the steps of a surface integration conducted with reference to rectangular axes, we make use in each case of the result of Theorem III, Art. 21*, where the relation between a volume-integral and the corresponding surface-integral is fully worked out. All that

* This theorem seems to have been first given by Ostrogradsky in a paper read in 1828, but published in 1831 in the *Mém. de l'Acad. de St. Pétersbourg*, T. I. p. 39. It may be regarded, however, as a form of the equation of continuity.

we have to do, therefore, is to substitute for $X$, $Y$, and $Z$ in that Theorem the components of the vector on which the particular theorem depends.

In the first edition of this book the statement of each theorem was cumbered with a multitude of alternative conditions which were intended to shew the generality of the theorem and the variety of cases to which it might be applied, but which tended rather to confuse in the mind of the reader what was assumed with what was to be proved.

In the present edition each theorem is at first stated in a more definite, if more restricted, form, and it is afterwards shewn what further degree of generality the theorem admits of.

We have hitherto used the symbol $V$ for the potential, and we shall continue to do so whenever we are dealing with electrostatics only. In this chapter, however, and in those parts of the second volume in which the electric potential occurs in electro-magnetic investigations, we shall use $\Psi$ as a special symbol for the electric potential.

## *Green's Theorem.*

96 *a*.] The following important theorem was given by George Green, in his 'Essay on the Application of Mathematics to Electricity and Magnetism.'

The theorem relates to the space bounded by the closed surface $s$. We may refer to this finite space as the Field. Let $\nu$ be a normal drawn from the surface $s$ into the field, and let $l$, $m$, $n$ be the direction cosines of this normal, then

$$l\frac{d\Psi}{dx}+m\frac{d\Psi}{dy}+n\frac{d\Psi}{dz}=\frac{d\Psi}{d\nu} \tag{1}$$

will be the rate of variation of the function $\Psi$ in passing along the normal $\nu$. Let it be understood that the value of $\frac{d\Psi}{d\nu}$ is to be taken at the surface itself, where $\nu = 0$.

Let us also write, as in Arts. 26 and 77,

$$\frac{d^2\Psi}{dx^2}+\frac{d^2\Psi}{dy^2}+\frac{d^2\Psi}{dz^2}=-\nabla^2\Psi, \tag{2}$$

and when there are two functions, $\Psi$ and $\Phi$, let us write

$$\frac{d\Psi}{dx}\frac{d\Phi}{dx}+\frac{d\Psi}{dy}\frac{d\Phi}{dy}+\frac{d\Psi}{dz}\frac{d\Phi}{dz}=-S.\nabla\Psi\nabla\Phi. \tag{3}$$

The reader who is not acquainted with the method of Quaternions may, if it pleases him, regard the expressions $\nabla^2\Psi$ and $S.\nabla\Psi\nabla\Phi$ as mere conventional abbreviations for the quantities to which they are equated above, and as in what follows we shall employ ordinary Cartesian methods, it will not be necessary to remember the Quaternion interpretation of these expressions. The reason, however, why we use as our abbreviations these expressions and not single letters arbitrarily chosen, is, that in the language of Quaternions they represent fully the quantities to which they are equated. The operator $\nabla$ applied to the scalar function $\Psi$ gives the space-variation of that function, and the expression $-S.\nabla\Psi\nabla\Phi$ is the scalar part of the product of two space-variations, or the product of either space-variation into the resolved part of the other in its own direction. The expression $\frac{d\Psi}{d\nu}$ is usually written in Quaternions $S.U\nu\nabla\Psi$, $U\nu$ being a unit-vector in the direction of the normal. There does not seem much advantage in using this notation here, but we shall find the advantage of doing so when we come to deal with anisotropic {non-isotropic} media.

*Statement of Green's Theorem.*

Let $\Psi$ and $\Phi$ be two functions of $x, y, z$, which, with their first derivatives, are finite and continuous within the acyclic region $\mathfrak{s}$, bounded by the closed surface $s$, then

$$\iint \Psi \frac{d\Phi}{d\nu}\, ds - \iiint \Psi\nabla^2\Phi\, d\mathfrak{s} = \iiint S.\nabla\Psi\nabla\Phi\, d\mathfrak{s}$$

$$= \iint \Phi \frac{d\Psi}{d\nu}\, ds - \iiint \Phi\nabla^2\Psi\, d\mathfrak{s}; \qquad (4)$$

where the double integrals are to be extended over the whole closed surface $s$, and the triple integrals throughout the field, $\mathfrak{s}$, enclosed by that surface.

To prove this, let us write, in Art. 21, Theorem III,

$$X = \Psi\frac{d\Phi}{dx}, \qquad Y = \Psi\frac{d\Phi}{dy}, \qquad Z = \Psi\frac{d\Phi}{dz}, \qquad (5)$$

then $$R\cos\epsilon = -\Psi\left(l\frac{d\Phi}{dx} + m\frac{d\Phi}{dy} + n\frac{d\Phi}{dz}\right)$$

$$= -\Psi\frac{d\Phi}{d\nu}, \text{ by (1)}; \qquad (6)$$

and $$\frac{dX}{dx}+\frac{dY}{dy}+\frac{dZ}{dz}=\Psi\left(\frac{d^2\Phi}{dx^2}+\frac{d^2\Phi}{dy^2}+\frac{d^2\Phi}{dz^2}\right)$$
$$+\frac{d\Psi}{dx}\frac{d\Phi}{dx}+\frac{d\Psi}{dy}\frac{d\Phi}{dy}+\frac{d\Psi}{dz}\frac{d\Phi}{dz}$$
$$=-\Psi\nabla^2\Phi-S.\nabla\Psi\nabla\Phi,\ \text{by (2) and (3).} \tag{7}$$

But by Theorem III
$$\iint R\cos\epsilon\, ds=\iiint\left(\frac{dX}{dx}+\frac{dY}{dy}+\frac{dZ}{dz}\right)ds\,;$$
or by (6) and (7)
$$\iint\Psi\frac{d\Phi}{d\nu}ds-\iiint\Psi\nabla^2\Phi\, ds=\iiint S.\nabla\Psi\nabla\Phi\, ds. \tag{8}$$

Since in the second member of this equation $\Psi$ and $\Phi$ may be interchanged, we may do so in the first, and we thus obtain the complete statement of Green's Theorem, as given in equation (4).

96 *b*.] We have next to shew that Green's Theorem is true when one of the functions, say $\Psi$, is a many-valued one, provided that its first derivatives are single-valued, and do not become infinite within the acyclic region $s$.

Since $\nabla\Psi$ and $\nabla\Phi$ are single-valued, the second member of equation (4) is single-valued; but since $\Psi$ is many-valued, any one element of the first member, as $\Psi\nabla^2\Phi$, is many-valued. If, however, we select one of the many values of $\Psi$, as $\Psi_0$, at the point $A$ within the region $s$, then the value of $\Psi$ at any other point, $P$, will be definite. For, since the selected value of $\Psi$ is continuous within the region, the value of $\Psi$ at $P$ must be that which is arrived at by continuous variation along any path from $A$ to $P$, beginning with the value $\Psi_0$ at $A$. If the value at $P$ were different for two paths between $A$ and $P$, then these two paths must embrace between them a closed curve at which the first derivatives of $\Psi$ become infinite *. Now this is contrary to the specification, for since the first derivatives do not become infinite within the region $s$, the closed curve must be entirely without the region; and since the region is acyclic, two paths within the region cannot embrace anything outside the region.

* $\left\{\int_A^P\left(\frac{d\Psi}{dx}dx+\frac{d\Psi}{dy}dy+\frac{d\Psi}{dz}dz\right)\right.$ is the same for all reconcileable paths, and since the region is acyclic all paths are reconcileable.$\left.\right\}$

Hence, if $\Psi_0$ is given as the value of $\Psi$ at the point $A$, the value at $P$ is definite.

If any other value of $\Psi$, say $\Psi_0 + n\kappa$, had been chosen as the value at $A$, then the value at $P$ would have been $\Psi + n\kappa$. But the value of the first member of equation (4) would be the same as before, for the change amounts to increasing the first member by

$$n\kappa\left[\iint \frac{d\Phi}{d\nu} ds - \iiint \nabla^2 \Phi \, d\varsigma\right];$$

and this, by Theorem III, Art. 21, is zero.

96 c.] If the region $\varsigma$ is doubly or multiply connected, we may reduce it to an acyclic region by closing each of its circuits with a diaphragm, {we can then apply the theorem to the region bounded by the surface of $\varsigma$ and the positive and negative sides of the diaphragm}.

Let $s_1$ be one of these diaphragms, and $\kappa_1$ the corresponding cyclic constant, that is to say, the increment of $\Psi$ in going once round the circuit in the positive direction. Since the region $\varsigma$ lies on both sides of the diaphragm $s_1$, every element of $s_1$ will occur twice in the surface integral.

If we suppose the normal $\nu_1$ drawn towards the positive side of $ds_1$, and $\nu_1'$ drawn towards the negative side,

$$\frac{d\Phi}{d\nu_1'} = -\frac{d\Phi}{d\nu_1},$$

and

$$\Psi_1' = \Psi_1 + (\kappa_1)$$

so that the element of the surface-integral arising from $ds_1$ will be, since $d\nu_1$ is the element of the inward normal for the positive surface,

$$\Psi_1 \frac{d\Phi}{d\nu_1} ds_1 + \Psi_1' \frac{d\Phi}{d\nu'_1} ds_1 = -\kappa_1 \frac{d\Phi}{d\nu_1} ds_1.$$

Hence if the region $\varsigma$ is multiply connected, the first term of equation (4) must be written

$$\iint \Psi \frac{d\Phi}{d\nu} ds - \kappa_1 \iint \frac{d\Phi}{d\nu_1} ds_1 - \&c. - \kappa_n \iint \frac{d\Phi}{d\nu_n} ds_n - \iiint \Psi \nabla^2 \Phi \, d\varsigma; \quad (4_a)$$

where $d\nu$ is an element of the inward normal to the bounding surface and where the first surface-integral is to be taken over the bounding surface, and the others over the different diaphragms, each element of surface of a diaphragm being taken once only, and the normal being drawn in the positive direction of the circuit.

This modification of the theorem in the case of multiply-connected regions was first shewn to be necessary by Helmholtz*, and was first applied to the theorem by Thomson†.

96 *d*.] Let us now suppose, with Green, that one of the functions, say $\Phi$, does not satisfy the condition that it and its first derivatives do not become infinite within the given region, but that it becomes infinite at the point $P$, and at that point only, in that region, and that very near to $P$ the value of $\Phi$ is $\Phi_0 + e/r$‡, where $\Phi_0$ is a finite and continuous quantity, and $r$ is the distance from $P$. This will be the case if $\Phi$ is the potential of a quantity of electricity $e$ concentrated at the point $P$, together with any distribution of electricity the volume density of which is nowhere infinite within the region considered.

Let us now suppose a very small sphere whose radius is $a$ to be described about $P$ as centre; then since in the region outside this sphere, but within the surface $s$, $\Phi$ presents no singularity, we may apply Green's Theorem to this region, remembering that the surface of the small sphere is to be taken account of in forming the surface-integral.

In forming the volume-integrals we have to subtract from the volume-integral arising from the whole region that arising from the small sphere.

Now $\iiint \Phi \nabla^2 \Psi \, dx\,dy\,dz$ for the sphere cannot be numerically greater than

$$(\nabla^2\Psi)_g \iiint \Phi \, dx\,dy\,dz,$$

or

$$(\nabla^2\Psi)_g \left\{2\pi e a^2 + \tfrac{4}{3}\pi a^3 \Phi_0\right\},$$

where the suffix, $_g$, attached to any quantity, indicates that the greatest numerical value of that quantity within the sphere is to be taken.

This volume-integral, therefore, is of the order $a^2$, and may be neglected when $a$ diminishes and ultimately vanishes.

The other volume-integral

$$\iiint \Psi \nabla^2 \Phi \, dx\,dy\,dz$$

we shall suppose taken through the region between the small sphere and the surface $S$, so that the region of integration does not include the point at which $\phi$ becomes infinite.

* 'Ueber Integrale der hydrodynamischen Gleichungen welche den Wirbelbewegungen entsprechen,' *Crelle*, 1858. Translated by Prof. Tait, *Phil. Mag.*, 1867 (I).

† 'On Vortex Motion,' *Trans. R. S. Edin.* xxv. part. i. p. 241 (1867).

‡ The mark / separates the numerator from the denominator of a fraction.

The surface-integral $\iint \Phi \frac{d\Psi}{d\nu}\, ds'$ for the sphere cannot be numerically greater than $\Phi_g \iint \frac{d\Psi}{d\nu}\, ds'$.

Now by Theorem III, Art. 21,

$$\iint \frac{d\Psi}{d\nu}\, ds = -\iiint \nabla^2\Psi\, dx\, dy\, dz,$$

since $d\nu$ is here measured outwards from the sphere, and this cannot be numerically greater than $(\nabla^2\Psi)_g \frac{4}{3}\pi a^3$, and $\Phi_g$ at the surface is approximately $\frac{e}{a}$, so that $\iint \Phi \frac{d\Psi}{d\nu}\, ds$ cannot be numerically greater than

$$\tfrac{4}{3}\pi a^2 e\, (\nabla^2\Psi)_g,$$

and is therefore of the order $a^2$, and may be neglected when $a$ vanishes.

But the surface-integral for the sphere on the other side of the equation, namely,

$$\iint \Psi \frac{d\Phi}{d\nu}\, ds',$$

does not vanish, for

$$\iint \frac{d\Phi}{d\nu}\, ds' = -4\pi e;$$

$d\nu$ being measured outwards from the sphere, and if $\Psi_0$ be the value of $\Psi$ at the point $P$,

$$\iint \Psi \frac{d\Phi}{d\nu}\, ds = -4\pi e \Psi_0.$$

Equation (4) therefore becomes in this case

$$\iint \Psi \frac{d\Phi}{d\nu} ds - \iiint \Psi \nabla^2\Phi\, ds - 4\pi e \Psi_0 = \iint \Phi \frac{d\Psi}{d\nu}\, ds - \iiint \Phi \nabla^2\Psi\, ds^*. \quad (4_b)$$

97 *a.*] We may illustrate this case of Green's Theorem by employing it as Green does to determine the surface-density of a distribution which will produce a potential whose values inside and outside a given closed surface are given. These values must coincide at the surface, also within the surface $\nabla^2\Psi = 0$, and outside $\nabla^2\Psi' = 0$ where $\psi$ and $\psi'$ denote the potentials inside and outside the surface.

Green begins with the direct process, that is to say, the distri-

* {In this equation $d\nu$ is drawn to the inside of the surface and $\iiint \psi \nabla^2\phi\, dx\, dy\, dz$ is not taken through the space occupied by a small sphere whose centre is the point at which $\phi$ becomes infinite.}

bution of the surface density, $\sigma$, being given, the potentials at an internal point $P$ and an external point $P'$ are found by integrating the expressions

$$\Psi_P = \iint \frac{\sigma}{r}\, ds, \qquad \Psi'_{P'} = \iint \frac{\sigma}{r'}\, ds\,; \tag{9}$$

where $r$ and $r'$ are measured from the points $P$ and $P'$ respectively.

Now let $\Phi = 1/r$, then applying Green's Theorem to the space within the surface, and remembering that $\nabla^2\Phi = 0$ and $\nabla^2\Psi = 0$ throughout the limits of integration we find

$$\iint \Psi \frac{d\frac{1}{r}}{d\nu'}\, ds - 4\pi\Psi_P = \iint \frac{1}{r}\frac{d\Psi}{d\nu'}\, ds*, \tag{10}$$

where $\Psi_P$ is the value of $\Psi$ at $P$.

Again, if we apply the theorem to the space between the surface $s$ and a surface surrounding it at an infinite distance $a$, the part of the surface-integral belonging to the latter surface will be of the order $1/a$ and may be neglected, and we have

$$\iint \Psi' \frac{d\frac{1}{r}}{d\nu}\, ds = \iint \frac{1}{r}\frac{d\Psi'}{d\nu}\, ds. \tag{11}$$

Now at the surface, $\Psi = \Psi'$, and since the normals $\nu$ and $\nu'$ are drawn in opposite directions,

$$\frac{d\frac{1}{r}}{d\nu} + \frac{d\frac{1}{r}}{d\nu'} = 0.$$

Hence on adding equations (10) and (11), the left-hand members destroy each other, and we have

$$-4\pi\Psi_P = \iint \frac{1}{r}\left(\frac{d\Psi}{d\nu'} + \frac{d\Psi'}{d\nu}\right) ds. \tag{12}$$

97 *b*.] Green also proves that if the value of the potential $\Psi$ at every point of a closed surface $s$ be given arbitrarily, the potential at any point inside or outside the surface may be determined, provided $\nabla^2\Psi = 0$ inside or outside the surface.

For this purpose he supposes the function $\Phi$ to be such that near the point $P$ its value is sensibly $1/r$, while at the surface $s$ its value is zero, and at every point within the surface

$$\nabla^2\Phi = 0.$$

* {In equations 10 and 11 $d\nu'$ is drawn to the inside of the surface and $d\nu$ to the outside.}

That such a function must exist, Green proves from the physical consideration that if $s$ is a conducting surface connected to the earth, and if a unit of electricity is placed at the point $P$, the potential within $s$ must satisfy the above conditions. For since $s$ is connected to the earth the potential must be zero at every point of $s$, and since the potential arises from the electricity at $P$ and the electricity induced on $s$, $\nabla^2\Phi = 0$ at every point within the surface.

Applying Green's Theorem to this case, we find

$$4\pi\Psi_P = \iint \Psi \frac{d\Phi}{d\nu'} ds, \tag{13}$$

where, in the surface-integral, $\Psi$ is the given value of the potential at the element of surface $ds$; and since, if $\sigma_P$ is the density of the electricity induced on $s$ by unit of electricity at $P$,

$$4\pi\sigma_P + \frac{d\Phi}{d\nu'} = 0, \tag{14}$$

we may write equation (13)

$$\Psi_P = -\iint \Psi\sigma ds^*, \tag{15}$$

where $\sigma$ is the surface-density of the electricity induced on $ds$ by a charge equal to unity at the point $P$.

Hence if the value of $\sigma$ is known at every point of the surface for a particular position of $P$, then we can calculate by ordinary integration the potential at the point $P$, supposing the potential at every point of the surface to be given, and the potential within the surface to be subject to the condition

$$\nabla^2\Phi = 0.$$

We shall afterwards prove that if we have obtained a value of $\Psi$ which satisfies these conditions, it is the only value of $\Psi$ which satisfies them.

*Green's Function.*

98.] Let a closed surface $s$ be maintained at potential zero. Let $P$ and $Q$ be two points on the positive side of the surface $s$ (we may suppose either the inside or the outside positive), and let a small body charged with unit of electricity be placed at $P$; the potential at the point $Q$ will consist of two parts, of which one is due to the direct action of the electricity at $P$, while the

* {This is the same as equation (14), p. 107.}

other is due to the action of the electricity induced on $s$ by $P$. The latter part of the potential is called Green's Function, and is denoted by $G_{pq}$.

This quantity is a function of the positions of the two points $P$ and $Q$, the form of the function depending on the surface $s$. It has been calculated for the case in which $s$ is a sphere, and for a very few other cases. It denotes the potential at $Q$ due to the electricity induced on $s$ by unit of electricity at $P$.

The actual potential at any point $Q$ due to the electricity at $P$ and to the electricity induced on $s$ is $1/r_{pq} + G_{pq}$, where $r_{pq}$ denotes the distance between $P$ and $Q$.

At the surface $s$, and at all points on the negative side of $s$, the potential is zero, therefore

$$G_{pa} = -\frac{1}{r_{pa}}, \tag{1}$$

where the suffix $_a$ indicates that a point $A$ on the surface $s$ is taken instead of $Q$.

Let $\sigma_{pa'}$ denote the surface-density induced by $P$ at a point $A'$ of the surface $s$, then, since $G_{pq}$ is the potential at $Q$ due to the superficial distribution,

$$G_{pq} = \iint \frac{\sigma_{pa'}}{r_{qa'}}\, ds', \tag{2}$$

where $ds'$ is an element of the surface $s$ at $A'$, and the integration is to be extended over the whole surface $s$.

But if unit of electricity had been placed at $Q$, we should have had by equation (1),

$$\frac{1}{r_{qa'}} = -G_{qa'} \tag{3}$$

$$= -\iint \frac{\sigma_{qa}}{r_{aa'}}\, ds; \tag{4}$$

where $\sigma_{qa}$ is the density at $A$ of the electricity induced by $Q$, $ds$ is an element of surface, and $r_{aa'}$ is the distance between $A$ and $A'$. Substituting this value of $1/r_{qa'}$ in the expression for $G_{pq}$, we find

$$G_{pq} = -\iiiint \frac{\sigma_{qa}\sigma_{pa'}}{r_{aa'}}\, ds\, ds'. \tag{5}$$

Since this expression is not altered by changing $_p$ into $_q$ and $_q$ into $_p$, we find that

$$G_{pq} = G_{qp}; \tag{6}$$

a result which we have already shewn to be necessary in Art. 86, but which we now see to be deducible from the mathematical process by which Green's function may be calculated.

If we assume any distribution of electricity whatever, and place in the field a point charged with unit of electricity, and if the surface of potential zero completely separates the point from the assumed distribution, then if we take this surface for the surface $s$, and the point for $P$, Green's function, for any point on the same side of the surface as $P$, will be the potential of the assumed distribution on the other side of the surface. In this way we may construct any number of cases in which Green's function can be found for a particular position of $P$. To find the form of the function when the form of the surface is given and the position of $P$ is arbitrary, is a problem of far greater difficulty, though, as we have proved, it is mathematically possible.

Let us suppose the problem solved, and that the point $P$ is taken within the surface. Then for all external points the potential of the superficial distribution is equal and opposite to that of $P$. The superficial distribution is therefore *centrobaric* *, and its action on all external points is the same as that of a unit of negative electricity placed at $P$.

99 *a*.] If in Green's Theorem we make $\Psi = \Phi$, we find

$$\iint \Psi \frac{d\Psi}{d\nu}\, ds - \iiint \Psi \nabla^2 \Psi\, ds = \iiint (\nabla\Psi)^2 ds. \tag{16}$$

If $\Psi$ is the potential of a distribution of electricity in space with a volume-density $\rho$ and on conductors whose surfaces are $s_1$, $s_2$, &c., and whose potentials are $\Psi_1$, $\Psi_2$, &c., with surface-densities $\sigma_1$, $\sigma_2$, &c., then

$$\nabla^2 \Psi = 4\pi\rho, \tag{17}$$

$$\frac{d\Psi}{d\nu} = -4\pi\sigma, \tag{18}$$

since $d\nu$ is drawn outwards from the conductor, and

$$\iint \frac{d\Psi}{d\nu_1} ds_1 = -4\pi e_1, \tag{19}$$

where $e_1$ is the charge of the surface $s_1$.

Dividing (16) by $-8\pi$, we find

$$\frac{1}{2}(\Psi_1 e_1 + \Psi_2 e_2 + \text{\&c.}) + \frac{1}{2}\iiint \Psi\rho\, dx\, dy\, dz$$
$$= \frac{1}{8\pi}\iiint \left[\left(\frac{d\Psi}{dx}\right)^2 + \left(\frac{d\Psi}{dy}\right)^2 + \left(\frac{d\Psi}{dz}\right)^2\right] dx\, dy\, dz. \tag{20}$$

* Thomson and Tait's *Natural Philosophy*, § 526.

The first term is the electric energy of the system arising from the surface-distributions, and the second is that arising from the distribution of electricity through the field, if such a distribution exists.

Hence the second member of the equation expresses the whole electric energy of the system *, the potential $\Psi$ being a given function of $x$, $y$, $z$.

As we shall often have occasion to employ this volume-integral, we shall denote it by the abbreviation $W_\Psi$, so that

$$W_\Psi = \frac{1}{8\pi}\iiint\left[\left(\frac{d\Psi}{dx}\right)^2 + \left(\frac{d\Psi}{dy}\right)^2 + \left(\frac{d\Psi}{dz}\right)^2\right]dx\,dy\,dz. \tag{21}$$

If the only charges are those on the surfaces of the conductors, $\rho = 0$, and the second term of the first member of equation (20) disappears.

The first term is the expression for the energy of the charged system expressed, as in Art. 84, in terms of the charges and the potentials of the conductors, and this expression for the energy we denote by $W$.

99 *b*.] Let $\Psi$ be a function of $x$, $y$, $z$, subject to the condition that its value at the closed surface $s$ is $\bar{\Psi}$, a known quantity for every point of the surface. The value of $\Psi$ at points not on the surface $s$ is perfectly arbitrary.

Let us also write

$$W = \frac{1}{8\pi}\iiint\left[\left(\frac{d\Psi}{dx}\right)^2 + \left(\frac{d\Psi}{dy}\right)^2 + \left(\frac{d\Psi}{dz}\right)^2\right]dx\,dy\,dz, \tag{22}$$

the integration being extended throughout the space within the surface; then we shall prove that if $\Psi_1$ is a particular form of $\Psi$ which satisfies the surface condition and also satisfies Laplace's Equation

$$\nabla^2\Psi_1 = 0 \tag{23}$$

at every point within the surface, then $W_1$, the value of $W$ corresponding to $\Psi_1$, is less than that corresponding to any function which differs from $\Psi_1$ at any point within the surface.

For let $\Psi$ be any function coinciding with $\Psi_1$ at the surface but not at every point within it, and let us write

$$\Psi = \Psi_1 + \Psi_2; \tag{24}$$

then $\Psi_2$ is a function which is zero at every point of the surface.

* {The expression on the right-hand side of (20) does not represent the energy where the conductors are surrounded by any dielectric other than air.}

The value of $W$ for $\Psi$ will be evidently

$$W = W_1 + W_2 + \frac{1}{4\pi}\iiint\left(\frac{d\Psi_1}{dx}\frac{d\Psi_2}{dx} + \frac{d\Psi_1}{dy}\frac{d\Psi_2}{dy} + \frac{d\Psi_1}{dz}\frac{d\Psi_2}{dz}\right)dx\,dy\,dz. \quad (25)$$

By Green's Theorem the last term may be written

$$\frac{1}{4\pi}\iiint\Psi_2\nabla^2\Psi_1 ds - \frac{1}{4\pi}\iint\Psi_2\frac{d\Psi_1}{d\nu}ds. \quad (26)$$

The volume-integral vanishes because $\nabla^2\Psi_1 = 0$ within the surface, and the surface-integral vanishes because at the surface $\Psi_2 = 0$. Hence equation (25) is reduced to the form

$$W = W_1 + W_2. \quad (27)$$

Now the elements of the integral $W_2$ being sums of three squares, are incapable of negative values, so that the integral itself can only be positive or zero. Hence if $W_2$ is not zero it must be positive, and therefore $W$ greater than $W_1$. But if $W_2$ is zero, every one of its elements must be zero, and therefore

$$\frac{d\Psi_2}{dx} = 0, \quad \frac{d\Psi_2}{dy} = 0, \quad \frac{d\Psi_2}{dz} = 0$$

at every point within the surface, and $\Psi_2$ must be a constant within the surface. But at the surface $\Psi_2 = 0$, therefore $\Psi_2 = 0$ at every point within the surface, and $\Psi = \Psi_1$, so that if $W$ is not greater than $W_1$, $\Psi$ must be identical with $\Psi_1$ at every point within the surface.

It follows from this that $\Psi_1$ is the only function of $x$, $y$, $z$ which becomes equal to $\overline{\Psi}$ at the surface, and which satisfies Laplace's Equation at every point within the surface.

For if these conditions are satisfied by any other function $\Psi_3$, then $W_3$ must be less than any other value of $W$. But we have already proved that $W_1$ is less than any other value, and therefore than $W_3$. Hence no function different from $\Psi_1$ can satisfy the conditions.

The case which we shall find most useful is that in which the field is bounded by one exterior surface, $s$, and any number of interior surfaces, $s_1$, $s_2$, &c., and when the conditions are that the value of $\Psi$ shall be zero at $s$, $\Psi_1$ at $s_1$, $\Psi_2$ at $s_2$, and so on, where $\Psi_1$, $\Psi_2$, &c. are constant for each surface, as in a system of conductors, the potentials of which are given.

Of all values of $\Psi$ satisfying these conditions, that gives the minimum value of $W_\psi$ for which $\nabla^2\Psi = 0$ at every point in the field.

*Thomson's Theorem.*

*Lemma.*

100 *a*.] Let $\Psi$ be any function of $x$, $y$, $z$ which is finite and continuous within the closed surface $s$, and which at certain closed surfaces, $s_1, s_2, \dots, s_p$, &c., has the values $\Psi_1, \Psi_2, \dots, \Psi_p$, &c. constant for each surface.

Let $u$, $v$, $w$ be functions of $x$, $y$, $z$, which we may consider as the components of a vector $\mathfrak{C}$ subject to the solenoidal condition

$$-S.\nabla\mathfrak{C} = \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0, \tag{28}$$

and let us put in Theorem III

$$X = \Psi u, \quad Y = \Psi v, \quad Z = \Psi w; \tag{29}$$

we find as the result of these substitutions

$$\Sigma_p \iint \Psi_p (l_p u + m_p v + n_p w)\, ds_p + \iiint \Psi \left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) dx\,dy\,dz$$
$$+ \iiint \left(u\frac{d\Psi}{dx} + v\frac{d\Psi}{dy} + w\frac{d\Psi}{dz}\right) dx\,dy\,dz = 0, \tag{30}$$

the surface-integrals being extended over the different surfaces and the volume-integrals being taken throughout the whole field, and where $l_p$, $m_p$, $n_p$ are the direction cosines of the normal to $s_p$ drawn from the surface into the field. Now the first volume-integral vanishes in virtue of the solenoidal condition for $u$, $v$, $w$, and the surface-integrals vanish in the following cases:—

(1) When at every point of the surface $\Psi = 0$.

(2) When at every point of the surface $lu + mv + nw = 0$.

(3) When the surface is entirely made up of parts which satisfy either (1) or (2).

(4) When $\Psi$ is constant over each of the closed surfaces, and

$$\iint (lu + mv + nw)\, ds = 0.$$

Hence in these four cases the volume-integral

$$M = \iiint \left(u\frac{d\Psi}{dx} + v\frac{d\Psi}{dy} + w\frac{d\Psi}{dz}\right) dx\,dy\,dz = 0. \tag{31}$$

100 *b*.] Now consider a field bounded by the external closed surface $s$, and the internal closed surfaces $s_1$, $s_2$, &c.

Let $\Psi$ be a function of $x$, $y$, $z$, which within the field is finite and continuous and satisfies Laplace's Equation

$$\nabla^2\Psi = 0, \tag{32}$$

and has the constant, but not given, values $\Psi_1$, $\Psi_2$, &c. at the surfaces $s_1$, $s_2$, &c. respectively, and is zero at the external surface $s$.

The charge of any of the conducting surfaces, as $s_1$, is given by the surface integral

$$e_1 = -\frac{1}{4\pi}\iint \frac{d\Psi}{d\nu_1}\, ds_1, \tag{33}$$

the normal $\nu_1$ being drawn from the surface $s_1$ into the electric field.

100 *c*.] Now let $f$, $g$, $h$ be functions of $x$, $y$, $z$, which we may consider as the components of a vector $\mathfrak{D}$, subject only to the conditions that at every point of the field they must satisfy the solenoidal equation

$$\frac{df}{dx} + \frac{dg}{dy} + \frac{dh}{dz} = 0, \tag{34}$$

and that at any one of the internal closed surfaces, as $s_1$, the surface-integral

$$\iint (l_1 f + m_1 g + n_1 h)\, ds = e_1, \tag{35}$$

where $l_1$, $m_1$, $n_1$ are the direction cosines of the normal $\nu_1$ drawn outwards from the surface $s_1$ into the electric field, and $e_1$ is the same quantity as in equation (33), being, in fact, the electric charge of the conductor whose surface is $s_1$.

We have to consider the value of the volume-integral

$$W_{\mathfrak{D}} = 2\pi \iiint (f^2 + g^2 + h^2)\, dx\, dy\, dz, \tag{36}$$

extended throughout the whole of the field within $s$ and without $s_1$, $s_2$, &c., and to compare it with

$$W_\Psi = \frac{1}{8\pi}\iiint \left[\left(\frac{d\Psi}{dx}\right)^2 + \left(\frac{d\Psi}{dy}\right)^2 + \left(\frac{d\Psi}{dz}\right)^2\right] dx\, dy\, dz, \tag{37}$$

the limits of integration being the same.

Let us write

$$u = f + \frac{1}{4\pi}\frac{d\Psi}{dx}, \quad v = g + \frac{1}{4\pi}\frac{d\Psi}{dy}, \quad w = h + \frac{1}{4\pi}\frac{d\Psi}{dz}, \tag{38}$$

$$\text{and} \quad W_{\mathfrak{G}} = 2\pi \iiint (u^2 + v^2 + w^2)\, dx\, dy\, dz; \tag{39}$$

then since

$$f^2+g^2+h^2 = \frac{1}{16\pi^2}\left[\left(\frac{d\Psi}{dx}\right)^2+\left(\frac{d\Psi}{dy}\right)^2+\left(\frac{d\Psi}{dz}\right)^2\right]$$

$$+u^2+v^2+w^2-\frac{1}{2\pi}\left[u\frac{d\Psi}{dx}+v\frac{d\Psi}{dy}+w\frac{d\Psi}{dz}\right],$$

$$W_{\mathfrak{D}} = W_{\Psi}+W_{\mathfrak{G}}-\iiint\left(u\frac{d\Psi}{dx}+v\frac{d\Psi}{dy}+w\frac{d\Psi}{dz}\right)dx\,dy\,dz. \quad (40)$$

Now in the first place, $u$, $v$, $w$ satisfy the solenoidal condition at every point of the field, for by equations (38)

$$\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}=\frac{df}{dx}+\frac{dg}{dy}+\frac{dh}{dz}-\frac{1}{4\pi}\nabla^2\Psi, \quad (41)$$

and by the conditions expressed in equations (34) and (32), both parts of the second member of (41) are zero.

In the second place, the surface-integral

$$\iint(l_1u+m_1v+n_1w)\,ds_1$$

$$=\iint(l_1f+m_1g+n_1h)\,ds_1+\frac{1}{4\pi}\iint\frac{d\Psi}{d\nu_1}ds_1, \quad (42)$$

but by (35) the first term of the second member is $e_1$, and by (33) the second term is $-e_1$, so that

$$\iint(l_1u+m_1v+n_1w)\,ds_1=0. \quad (43)$$

Hence, since $\Psi_1$ is constant, the fourth condition of Art. 100 *a* is satisfied, and the last term of equation (40) is zero, so that the equation is reduced to the form

$$W_{\mathfrak{D}} = W_{\Psi}+W_{\mathfrak{G}}. \quad (44)$$

Now since the element of the integral $W_{\mathfrak{G}}$ is the sum of three squares, $u^2+v^2+w^2$, it must be either positive or zero. If at any point within the field $u$, $v$, and $w$ are not each of them equal to zero, the integral $W_{\mathfrak{G}}$ must have a positive value, and $W_{\mathfrak{D}}$ must therefore be greater than $W_{\Psi}$. But the values $u=v=w=0$ at every point satisfy the conditions.

Hence, if at every point

$$f=-\frac{1}{4\pi}\frac{d\Psi}{dx},\quad g=-\frac{1}{4\pi}\frac{d\Psi}{dy},\quad h=-\frac{1}{4\pi}\frac{d\Psi}{dz}, \quad (45)$$

then

$$W_{\mathfrak{D}} = W_{\Psi}, \quad (46)$$

and the value of $W_{\mathfrak{D}}$ corresponding to these values of $f$, $g$, $h$, is less than the value corresponding to any values of $f$, $g$, $h$, differing from these.

Hence the problem of determining the displacement and potential, at every point of the field, when the charge on each conductor is given, has one and only one solution.

This theorem in one of its more general forms was first stated by Sir W. Thomson *. We shall afterwards show of what generalization it is capable.

100 *d.*] This theorem may be modified by supposing that the vector $\mathfrak{D}$, instead of satisfying the solenoidal condition at every point of the field, satisfies the condition

$$\frac{df}{dx}+\frac{dg}{dy}+\frac{dh}{dz}=\rho, \tag{47}$$

where $\rho$ is a finite quantity, whose value is given at every point in the field, and which may be positive or negative, continuous or discontinuous, its volume-integral within a finite region being, however, finite.

We may also suppose that at certain surfaces in the field

$$lf+mg+nh+l'f'+m'g'+n'h'=\sigma, \tag{48}$$

where $l, m, n$ and $l', m', n'$ are the direction cosines of the normals drawn from a point of the surface towards those regions in which the components of the displacement are $f, g, h$ and $f', g', h'$ respectively, and $\sigma$ is a quantity given at all points of the surface, the surface-integral of which, over a finite surface, is finite.

100 *e.*] We may also alter the condition at the bounding surfaces by supposing that at every point of these surfaces

$$lf+mg+nh=\sigma, \tag{49}$$

where $\sigma$ is given for every point.

(In the original statement we supposed only the value of the *integral* of $\sigma$ over each of the surfaces to be given. Here we suppose its value given for every element of surface, which comes to the same thing as if, in the original statement, we had considered every element as a separate surface.)

None of these modifications will affect the truth of the theorem provided we remember that $\Psi$ must satisfy the corresponding conditions, namely, the general condition,

$$\frac{d^2\Psi}{dx^2}+\frac{d^2\Psi}{dy^2}+\frac{d^2\Psi}{dz^2}+4\pi\rho=0, \tag{50}$$

and the surface condition

$$\frac{d\Psi}{d\nu}+\frac{d\Psi'}{d\nu'}+4\pi\sigma=0. \tag{51}$$

* *Cambridge and Dublin Mathematical Journal, February,* 1848.

For if, as before

$$f+\frac{1}{4\pi}\frac{d\Psi}{dx}=u,\quad g+\frac{1}{4\pi}\frac{d\Psi}{dy}=v,\quad h+\frac{1}{4\pi}\frac{d\Psi}{dz}=w,$$

then $u$, $v$, $w$ will satisfy the general solenoidal condition

$$\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}=0,$$

and the surface condition

$$lu+mv+nw+l'u'+m'v'+n'w'=0,$$

and at the bounding surface

$$lu+mv+nw=0,$$

whence we find as before that

$$M=\iiint\left(u\frac{d\Psi}{dx}+v\frac{d\Psi}{dy}+w\frac{d\Psi}{dz}\right)dxdydz=0,$$

and that $$W_{\mathfrak{D}}=W_{\Psi}+W_{\mathfrak{G}}.$$

Hence as before it is shewn that $W_{\mathfrak{D}}$ is a unique minimum when $W_{\mathfrak{G}}=0$, which implies that $u^2+v^2+w^2$ is everywhere zero, and therefore

$$f=-\frac{1}{4\pi}\frac{d\Psi}{dx},\quad g=-\frac{1}{4\pi}\frac{d\Psi}{dy},\quad h=-\frac{1}{4\pi}\frac{d\Psi}{dz}.$$

101 *a*.] In our statement of these theorems we have hitherto confined ourselves to that theory of electricity which assumes that the properties of an electric system depend on the form and relative position of the conductors, and on their charges, but takes no account of the nature of the dielectric medium between the conductors.

According to that theory, for example, there is an invariable relation between the surface density of a conductor and the electromotive intensity just outside it, as expressed in the law of Coulomb $$R=4\pi\sigma.$$

But this is true only in the standard medium, which we may take to be air. In other media the relation is different, as was proved experimentally, though not published, by Cavendish, and afterwards rediscovered independently by Faraday.

In order to express the phenomenon completely, we find it necessary to consider two vector quantities, the relation between which is different in different media. One of these is the electromotive intensity, the other is the electric displacement. The electromotive intensity is connected by equations of invariable form with the potential, and the electric displacement is connected by equations of invariable form with the distribution of electricity, but the relation between the electromotive intensity and the electric displacement depends on the nature of the dielectric medium, and must be expressed by equations, the most general form of which is as yet not fully determined, and can be determined only by experiments on dielectrics.

101 *b*.] The electromotive intensity is a vector defined in Art. 68, as the mechanical force on a small quantity $e$ of electricity divided by $e$. We shall denote its components by the letters $P$, $Q$, $R$, and the vector itself by $\mathfrak{E}$.

In electrostatics, the line integral of $\mathfrak{E}$ is always independent of the path of integration, or in other words $\mathfrak{E}$ is the space-variation of a potential. Hence

$$P = -\frac{d\Psi}{dx}, \quad Q = -\frac{d\Psi}{dy}, \quad R = -\frac{d\Psi}{dz},$$

or more briefly, in the language of Quaternions

$$\mathfrak{E} = -\nabla\Psi.$$

101 *c*.] The electric displacement in any direction is defined in Art. 60, as the quantity of electricity carried through a small area $A$, the plane of which is normal to that direction, divided by $A$. We shall denote the rectangular components of the electric displacement by the letters $f$, $g$, $h$, and the vector itself by $\mathfrak{D}$.

The volume-density at any point is determined by the equation

$$\rho = \frac{df}{dx} + \frac{dg}{dy} + \frac{dh}{dz},$$

or in the language of Quaternions

$$\rho = -S.\nabla\mathfrak{D}.$$

The surface-density at any point of a charged surface is determined by the equation

$$\sigma = lf + mg + nh + l'f' + m'g' + n'h',$$

where $f$, $g$, $h$ are the components of the displacement on one side of the surface, the direction cosines of the normal drawn from the surface on that side being $l$, $m$, $n$, and $f'$, $g'$, $h'$ and $l'$, $m'$, $n'$ are the components of the displacements, and the direction cosines of the normal on the other side.

This is expressed in Quaternions by the equation

$$\sigma = -\left[S.U\nu\mathfrak{D} + S.U\nu'\mathfrak{D}'\right],$$

where $U\nu$, $U\nu'$ are unit normals on the two sides of the surface, and $S$ indicates that the scalar part of the product is to be taken.

When the surface is that of a conductor, $\nu$ being the normal drawn outwards, then since $f'$, $g'$, $h'$ and $\mathfrak{D}'$ are zero, the equation is reduced to the form

$$\sigma = lf + mg + nh;$$
$$= -S\,.\,U\nu\mathfrak{D}.$$

The whole charge of the conductor is therefore

$$e = \iint (lf + mg + nh)\,ds;$$
$$= -\iint S\,.\,U\nu\mathfrak{D}\,ds.$$

101 *d*.] The electric energy of the system is, as was shewn in Art. 84, half the sum of the products of the charges into their respective potentials. Calling this energy W,

$$W = \tfrac{1}{2}\Sigma(e\Psi)$$
$$= \frac{1}{2}\iiint \rho\Psi\,dx\,dy\,dz + \frac{1}{2}\iint \sigma\Psi\,ds,$$
$$= \frac{1}{2}\iiint \Psi\left(\frac{df}{dx} + \frac{dg}{dy} + \frac{dh}{dz}\right) dx\,dy\,dz$$
$$+ \frac{1}{2}\iint \Psi\,(lf + mg + nh)\,ds;$$

where the volume-integral is to be taken throughout the electric field, and the surface-integral over the surfaces of the conductors.

Writing in Theorem III, Art. 21,

$$X = \Psi f, \quad Y = \Psi g, \quad Z = \Psi h,$$

we find, if $l$, $m$, $n$ are the direction cosines of the normal facing the surface into the field,

$$\iint \Psi\,(lf + mg + nh)\,ds = -\iiint \Psi\left(\frac{df}{dx} + \frac{dg}{dy} + \frac{dh}{dz}\right) dx\,dy\,dz,$$
$$-\iiint \left(f\frac{d\Psi}{dx} + g\frac{d\Psi}{dy} + h\frac{d\Psi}{dz}\right) dx\,dy\,dz.$$

Substituting this value for the surface-integral in $W$ we find

$$W = -\frac{1}{2}\iiint \left(f\frac{d\Psi}{dx} + g\frac{d\Psi}{dy} + h\frac{d\Psi}{dz}\right) dx\,dy\,dz;$$

or $$W = \frac{1}{2}\iiint (fP + gQ + hR)\,dx\,dy\,dz.$$

101 *e*.] We now come to the relation between $\mathfrak{D}$ and $\mathfrak{E}$.

The unit of electricity is usually defined with reference to experiments conducted in air. We now know from the experiments of Boltzmann that the dielectric constant of air is somewhat greater than that of a vacuum, and that it varies with the density. Hence, strictly speaking, all measurements of electric quantity require to be corrected to reduce them either to air of standard pressure and temperature, or, what would be more scientific, to a vacuum, just as indices of refraction measured in air require a similar correction, the correction in both cases being so small that it is sensible only in measurements of extreme accuracy.

In the standard medium

$$4\pi\mathfrak{D} = \mathfrak{E},$$

$$\text{or} \quad 4\pi f = P, \qquad 4\pi g = Q, \qquad 4\pi h = R.$$

In an isotropic medium whose dielectric constant is $K$

$$4\pi\mathfrak{D} = K\mathfrak{E},$$

$$4\pi f = KP, \qquad 4\pi g = KQ, \qquad 4\pi h = KR.$$

There are some media, however, of which glass has been the most carefully investigated, in which the relation between $\mathfrak{D}$ and $\mathfrak{E}$ is more complicated, and involves the time variation of one or both of these quantities, so that the relation must be of the form

$$F(\mathfrak{D}, \mathfrak{E}, \dot{\mathfrak{D}}, \dot{\mathfrak{E}}, \ddot{\mathfrak{D}}, \ddot{\mathfrak{E}}, \&c.) = 0.$$

We shall not attempt to discuss relations of this more general kind at present, but shall confine ourselves to the case in which $\mathfrak{D}$ is a linear and vector function of $\mathfrak{E}$.

The most general form of such a relation may be written

$$4\pi\mathfrak{D} = \phi(\mathfrak{E}),$$

where $\phi$ during the present investigation always denotes a linear and vector function. The components of $\mathfrak{D}$ are therefore homogeneous linear functions of those of $\mathfrak{E}$, and may be written in the form

$$\begin{aligned} 4\pi f &= K_{xx}P + K_{xy}Q + K_{xz}R, \\ 4\pi g &= K_{yx}P + K_{yy}Q + K_{yz}R, \\ 4\pi h &= K_{zx}P + K_{zy}Q + K_{zz}R; \end{aligned}$$

where the first suffix of each coefficient $K$ indicates the direction of the displacement, and the second that of the electromotive intensity.

The most general form of a linear and vector function involves nine independent coefficients. When the coefficients which have the same pair of suffixes are equal, the function is said to be self-conjugate.

If we express $\mathfrak{E}$ in terms of $\mathfrak{D}$ we shall have

$$\mathfrak{E} = 4\pi\phi^{-1}(\mathfrak{D}),$$

or

$$P = 4\pi(k_{xx}f + k_{yx}g + k_{zx}h),$$
$$Q = 4\pi(k_{xy}f + k_{yy}g + k_{zy}h),$$
$$R = 4\pi(k_{xz}f + k_{yz}g + k_{zz}h).$$

101 *f*.] The work done by the electromotive intensity whose components are $P, Q, R$, in producing a displacement whose components are $df$, $dg$, and $dh$, in unit of volume of the medium, is

$$dW = Pdf + Qdg + Rdh.$$

Since a dielectric {in a steady state} under electric displacement is a conservative system, $W$ must be a function of $f, g, h$, and since $f, g, h$ may vary independently, we have

$$P = \frac{dW}{df}, \qquad Q = \frac{dW}{dg}, \qquad R = \frac{dW}{dh}.$$

Hence

$$\frac{dP}{dg} = \frac{d^2W}{dg\,df} = \frac{d^2W}{df\,dg} = \frac{dQ}{df}.$$

But $\dfrac{dP}{dg} = 4\pi k_{yx}$, the coefficient of $g$ in the expression for $P$,

and $\dfrac{dQ}{df} = 4\pi k_{xy}$, the coefficient of $f$ in the expression for $Q$.

Hence if a dielectric is a conservative system (and we know that it is so, because it can retain its energy for an indefinite time),

$$k_{xy} = k_{yx},$$

and $\phi^{-1}$ is a self-conjugate function.

Hence it follows that $\phi$ also is self-conjugate, and $K_{xy} = K_{yx}$.

101 *g*.] The expression for the energy may therefore be written in either of the forms

$$W_{\mathfrak{E}} = \frac{1}{8\pi}\iiint[K_{xx}P^2 + K_{yy}Q^2 + K_{zz}R^2 + 2K_{yz}QR + 2K_{zx}RP + 2K_{xy}PQ]\,dx\,dy\,dz,$$

or

$$W_{\mathfrak{D}} = 2\pi\iiint[k_{xx}f^2 + k_{yy}g^2 + k_{zz}h^2 + 2k_{yz}gh + 2k_{zx}hf + 2k_{xy}fg]\,dx\,dy\,dz,$$

where the suffix denotes the vector in terms of which $W$ is to be expressed. When there is no suffix, the energy is understood to be expressed in terms of both vectors.

We have thus, in all, six different expressions for the energy of the electric field. Three of these involve the charges and potentials of the surfaces of conductors, and are given in Art. 87.

The other three are volume-integrals taken throughout the electric field, and involve the components of electromotive intensity or of electric displacement, or of both.

The first three therefore belong to the theory of action at a distance, and the last three to the theory of action by means of the intervening medium.

These three expressions for $W$ may be written,

$$W = -\frac{1}{2}\iiint S.\mathfrak{D}\mathfrak{E}\,d\varsigma,$$

$$W_{\mathfrak{E}} = -\frac{1}{8\pi}\iiint S.\mathfrak{E}\phi(\mathfrak{E})\,d\varsigma,$$

$$W_{\mathfrak{D}} = -2\pi\iiint S.\mathfrak{D}\phi^{-1}(\mathfrak{D})\,d\varsigma.$$

101 *h.*] To extend Green's Theorem to the case of a heterogeneous anisotropic {non-isotropic} medium, we have only to write in Theorem III, Art. 21,

$$X = \Psi\left[K_{xx}\frac{d\Phi}{dx} + K_{xy}\frac{d\Phi}{dy} + K_{xz}\frac{d\Phi}{dz}\right],$$

$$Y = \Psi\left[K_{yx}\frac{d\Phi}{dx} + K_{yy}\frac{d\Phi}{dy} + K_{yz}\frac{d\Phi}{dz}\right],$$

$$Z = \Psi\left[K_{zx}\frac{d\Phi}{dx} + K_{zy}\frac{d\Phi}{dy} + K_{zz}\frac{d\Phi}{dz}\right],$$

and we obtain, if $l$, $m$, $n$ are the direction cosines of the outward normal to the surface (remembering that the order of the suffixes of the coefficients is indifferent),

$$\begin{aligned}\iint \Psi\Big[(K_{xx}l + K_{yx}m + K_{zx}n)\frac{d\Phi}{dx} &+ (K_{xy}l + K_{yy}m + K_{zy}n)\frac{d\Phi}{dy} \\ &+ (K_{xz}l + K_{yz}m + K_{zz}n)\frac{d\Phi}{dz}\Big]ds \\ -\iiint \Psi\Big[\frac{d}{dx}\Big(K_{xx}\frac{d\Phi}{dx} &+ K_{xy}\frac{d\Phi}{dy} + K_{xz}\frac{d\Phi}{dz}\Big) \\ + \frac{d}{dy}\Big(K_{yx}\frac{d\Phi}{dx} &+ K_{yy}\frac{d\Phi}{dy} + K_{yz}\frac{d\Phi}{dz}\Big) \\ + \frac{d}{dz}\Big(K_{zx}\frac{d\Phi}{dx} &+ K_{zy}\frac{d\Phi}{dy} + K_{zz}\frac{d\Phi}{dz}\Big)\Big]dx\,dy\,dz\end{aligned}$$

$$=\iiint\Big[K_{xx}\frac{d\Psi}{dx}\frac{d\Phi}{dx}+K_{yy}\frac{d\Psi}{dy}\frac{d\Phi}{dy}+K_{zz}\frac{d\Psi}{dz}\frac{d\Phi}{dz}$$

$$+K_{yz}\Big(\frac{d\Psi}{dy}\frac{d\Phi}{dz}+\frac{d\Psi}{dz}\frac{d\Phi}{dy}\Big)+K_{zx}\Big(\frac{d\Psi}{dz}\frac{d\Phi}{dx}+\frac{d\Psi}{dx}\frac{d\Phi}{dz}\Big)$$

$$+K_{xy}\Big(\frac{d\Psi}{dx}\frac{d\Phi}{dy}+\frac{d\Psi}{dy}\frac{d\Phi}{dx}\Big)\Big]dx\,dy\,dz$$

$$=\iint\Phi\Big[(K_{xx}l+K_{yx}m+K_{zx}n)\frac{d\Psi}{dx}+(K_{xy}l+K_{yy}m+K_{zy}n)\frac{d\Psi}{dy}$$

$$+(K_{xz}l+K_{yz}m+K_{zz}n)\frac{d\Psi}{dz}\Big]ds$$

$$-\iiint\Phi\Big[\frac{d}{dx}\Big(K_{xx}\frac{d\Psi}{dx}+K_{xy}\frac{d\Psi}{dy}+K_{xz}\frac{d\Psi}{dz}\Big)$$

$$+\frac{d}{dy}\Big(K_{yx}\frac{d\Psi}{dx}+K_{yy}\frac{d\Psi}{dy}+K_{yz}\frac{d\Psi}{dz}\Big)$$

$$+\frac{d}{dz}\Big(K_{zx}\frac{d\Psi}{dx}+K_{zy}\frac{d\Psi}{dy}+K_{zz}\frac{d\Psi}{dz}\Big)\Big]dx\,dy\,dz.$$

Using quaternion notation, the result may be written more briefly,

$$\iint\Psi S\,.\,Uv\phi(\nabla\Phi)\,ds-\iiint\Psi S\,.\,\{\nabla\phi(\nabla\Psi)\}\,d\varsigma$$

$$=-\iiint S\,.\,\nabla\Psi\phi(\nabla\Phi)\,d\varsigma=-\iiint S\,.\,\nabla\Phi\phi(\nabla\Psi)\,d\varsigma$$

$$=\iint\Phi S\,.\,Uv\phi(\nabla\Psi)ds-\iiint\Phi S\,.\,\{\nabla\phi(\nabla\Psi)\}\,d\varsigma.$$

*Limits between which the electric capacity of a conductor must lie.*

102 *a*.] The capacity of a conductor or system of conductors has been already defined as the charge of that conductor or system of conductors when raised to potential unity, all the other conductors in the field being at potential zero.

The following method of determining limiting values between which the capacity must lie, was suggested by a paper 'On the Theory of Resonance,' by the Hon. J. W. Strutt, Phil. Trans. 1871. See Art. 306.

Let $s_1$ denote the surface of the conductor, or system of conductors, whose capacity is to be determined, and $s_0$ the surface of all other conductors. Let the potential of $s_1$ be $\Psi_1$, and that of $s_0$, $\Psi_0$. Let the charge of $s_1$ be $e_1$. That of $s_0$ will be $-e_1$.

Then if $q$ is the capacity of $s_1$,

$$q = \frac{e_1}{\Psi_1 - \Psi_0}, \tag{1}$$

and if $W$ is the energy of the system with its actual distribution of electricity $$W = \tfrac{1}{2} e_1 (\Psi_1 - \Psi_0), \tag{2}$$

and $$q = \frac{2W}{(\Psi_1 - \Psi_0)^2} = \frac{e_1^{\;2}}{2W}. \tag{3}$$

To find an upper limit of the value of the capacity: assume any value of $\Psi$ which is equal to 1 at $s_1$ and equal to zero at $s_0$, and calculate the value of the volume-integral

$$W_\Psi = \frac{1}{8\pi} \iiint \left[ \left(\frac{d\Psi}{dx}\right)^2 + \left(\frac{d\Psi}{dy}\right)^2 + \left(\frac{d\Psi}{dz}\right)^2 \right] dx\, dy\, dz \tag{4}$$

extended over the whole field.

Then as we have proved (Art. 99 *b*) that $W$ cannot be greater than $W_\Psi$, the capacity, $q$, cannot be greater than $2W_\Psi$.

To find a lower limit of the value of the capacity: assume any system of values of $f$, $g$, $h$, which satisfies the equation

$$\frac{df}{dx} + \frac{dg}{dy} + \frac{dh}{dz} = 0, \tag{5}$$

and let it make $$\iint (l_1 f + m_1 g + n_1 h)\, ds_1 = e_1. \tag{6}$$

Calculate the value of the volume-integral

$$W_{\mathfrak{D}} = 2\pi \iiint (f^2 + g^2 + h^2)\, dx\, dy\, dz, \tag{7}$$

extended over the whole field; then as we have proved (Art. 100 *c*) that $W$ cannot be greater than $W_{\mathfrak{D}}$, the capacity, $q$, cannot be less than

$$\frac{e_1^{\;2}}{2W_{\mathfrak{D}}}. \tag{8}$$

The simplest method of obtaining a system of values of $f$, $g$, $h$, which will satisfy the solenoidal condition, is to assume a distribution of electricity on the surface of $s_1$, and another on $s_0$, the sum of the charges being zero, then to calculate the potential, $\Psi$, due to this distribution, and the electric energy of the system thus arranged.

If we then make

$$f = -\frac{1}{4\pi}\frac{d\Psi}{dx}, \quad g = -\frac{1}{4\pi}\frac{d\Psi}{dy}, \quad h = -\frac{1}{4\pi}\frac{d\Psi}{dz},$$

these values of $f$, $g$, $h$ will satisfy the solenoidal condition.

But in this case we can determine $W_{\mathfrak{D}}$ without going through the process of finding the volume-integral. For since this solution makes $\nabla^2\Psi = 0$ at all points in the field, we can obtain $W_{\mathfrak{D}}$ in the form of the surface-integrals,

$$W_{\mathfrak{D}} = \frac{1}{2}\iint \Psi \sigma_1 ds_1 + \frac{1}{2}\iint \Psi \sigma_0 ds_0, \tag{9}$$

where the first integral is extended over the surface $s_1$ and the second over the surface $s_0$.

If the surface $s_0$ is at an infinite distance from $s_1$, the potential at $s_0$ is zero and the second term vanishes.

102 *b*.] An approximation to the solution of any problem of the distribution of electricity on conductors whose potentials are given may be made in the following manner:—

Let $s_1$ be the surface of a conductor or system of conductors maintained at potential 1, and let $s_0$ be the surface of all the other conductors, including the hollow conductor which surrounds the rest, which last, however, may in certain cases be at an infinite distance from the others.

Begin by drawing a set of lines, straight or curved, from $s_1$ to $s_0$.

Along each of these lines, assume $\Psi$ so that it is equal to 1 at $s_1$, and equal to 0 at $s_0$. Then if $P$ is a point on one of these lines {$s_1$ and $s_0$ the points where the line cuts the surfaces} we may take $\Psi_1 = \dfrac{Ps_0}{s_1 s_0}$ as a first approximation.

We shall thus obtain a first approximation to $\Psi$ which satisfies the condition of being equal to unity at $s_1$ and equal to zero at $s_0$.

The value of $W_{\Psi}$ calculated from $\Psi_1$ would be greater than $W$.

Let us next assume as a second approximation to the lines of force

$$f = -p\frac{d\Psi_1}{dx}, \quad g = -p\frac{d\Psi_1}{dy}, \quad h = -p\frac{d\Psi_1}{dz}. \tag{10}$$

The vector whose components are $f, g, h$ is normal to the surfaces for which $\Psi_1$ is constant. Let us determine $p$ so as to make $f, g, h$ satisfy the solenoidal condition. We thus get

$$p\left(\frac{d^2\Psi_1}{dx^2} + \frac{d^2\Psi_1}{dy^2} + \frac{d^2\Psi_1}{dz^2}\right) + \frac{dp}{dx}\frac{d\Psi_1}{dx} + \frac{dp}{dy}\frac{d\Psi_1}{dy} + \frac{dp}{dz}\frac{d\Psi_1}{dz} = 0. \tag{11}$$

If we draw a line from $s_1$ to $s_0$ whose direction is always normal to the surfaces for which $\Psi_1$ is constant, and if we denote the length of this line measured from $s_0$ by $s$, then

$$R\frac{dx}{ds} = -\frac{d\Psi_1}{dx}, \quad R\frac{dy}{ds} = -\frac{d\Psi_1}{dy}, \quad R\frac{dz}{ds} = -\frac{d\Psi_1}{dz}, \tag{12}$$

where $R$ is the resultant intensity $= -\dfrac{d\Psi_1}{ds}$, so that

$$\frac{dp}{dx}\frac{d\Psi_1}{dx} + \frac{dp}{dy}\frac{d\Psi_1}{dy} + \frac{dp}{dz}\frac{d\Psi_1}{dz} = -R\frac{dp}{ds},$$

$$= R^2\frac{dp}{d\Psi_1}, \tag{13}$$

and equation (11) becomes

$$p\nabla^2\Psi = R^2\frac{dp}{d\Psi_1}, \tag{14}$$

whence
$$p = C\,exp.\int_0^{\Psi_1}\frac{\nabla^2\Psi_1}{R^2}\,d\Psi_1, \tag{15}$$

the integral being a line integral taken along the line $s$.

Let us next assume that along the line $s$,

$$-\frac{d\Psi_2}{ds} = f\frac{dx}{ds} + g\frac{dy}{ds} + h\frac{dz}{ds},$$

$$= -p\frac{d\Psi_1}{ds}, \tag{16}$$

then
$$\Psi_2 = C\int_0^{\Psi}\left(exp.\int\frac{\nabla^2\Psi_1}{R^2}\,d\Psi_1\right)d\Psi_1, \tag{17}$$

the integration being always understood to be performed along the line $s$.

The constant $C$ is now to be determined from the condition that $\Psi_2 = 1$ at $s_1$ when also $\Psi_1 = 1$, so that

$$C\int_0^1\left\{exp.\int_0^{\Psi}\frac{\nabla^2\Psi}{R^2}\,d\Psi\right\}d\Psi = 1. \tag{18}$$

This gives a second approximation to $\Psi$, and the process may be repeated.

The results obtained from calculating $W_{\Psi_1}$, $W_{\mathfrak{D}_2}$, $W_{\Psi_2}$, &c., give capacities alternately above and below the true capacity and continually approximating thereto.

The process as indicated above involves the calculation of the form of the line $s$ and integration along this line, operations which are in general too difficult for practical purposes.

In certain cases however we may obtain an approximation by a simpler process.

102 *c*.] As an illustration of this method, let us apply it to obtain successive approximations to the equipotential surfaces and lines of induction in the electric field between two surfaces which are nearly but not exactly plane and parallel, one of which is maintained at potential zero, and the other at potential unity.

Let the equations of the two surfaces be

$$z_1 = f_1(x, y) = a \tag{19}$$

for the surface whose potential is zero, and

$$z_2 = f_2(x, y) = b \tag{20}$$

for the surface whose potential is unity, $a$ and $b$ being given functions of $x$ and $y$, of which $b$ is always greater than $a$. The first derivatives of $a$ and $b$ with respect to $x$ and $y$ are small quantities of which we may neglect powers and products of more than two dimensions.

We shall begin by supposing that the lines of induction are parallel to the axis of $z$, in which case

$$f = 0, \quad g = 0, \quad \frac{dh}{dz} = 0. \tag{21}$$

Hence $h$ is constant along each individual line of induction, and

$$\Psi = -4\pi\int_a^z h\,dz = -4\pi h(z-a). \tag{22}$$

When $z = b$, $\Psi = 1$, hence

$$h = -\frac{1}{4\pi(b-a)}, \tag{23}$$

and

$$\Psi = \frac{z-a}{b-a}, \tag{24}$$

which gives a first approximation to the potential, and indicates a series of equipotential surfaces the intervals between which, measured parallel to $z$, are equal.

To obtain a second approximation to the lines of induction, let us assume that they are everywhere normal to the equipotential surfaces as given by equation (24).

This is equivalent to the conditions

$$4\pi f = \lambda\frac{d\Psi}{dx}, \quad 4\pi g = \lambda\frac{d\Psi}{dy}, \quad 4\pi h = \lambda\frac{d\Psi}{dz}, \tag{25}$$

where $\lambda$ is to be determined so that at every point of the field

$$\frac{df}{dx} + \frac{dg}{dy} + \frac{dh}{dz} = 0, \tag{26}$$

and also so that the line-integral

$$4\pi \int \left(f\frac{dx}{ds} + g\frac{dy}{ds} + h\frac{dz}{ds}\right) ds, \tag{27}$$

taken along any line of induction from the surface $a$ to the surface $b$, shall be equal to $-1$.

Let us assume

$$\lambda = 1 + A + B(z-a) + C(z-a)^2, \tag{28}$$

and let us neglect powers and products of $A$, $B$, $C$, and at this stage of our work powers and products of the first derivatives of $a$ and $b$.

The solenoidal condition then gives

$$B = -\nabla^2 a, \qquad C = -\tfrac{1}{2}\frac{\nabla^2(b-a)}{b-a}, \tag{29}$$

where

$$\nabla^2 = -\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2}\right). \tag{30}$$

If instead of taking the line-integral along the new line of induction, we take it along the old line of induction, parallel to $z$, the second condition gives

$$1 = 1 + A + \tfrac{1}{2}B(b-a) + \tfrac{1}{3}C(b-a)^2.$$

Hence

$$A = \tfrac{1}{6}(b-a)\nabla^2(2a+b), \tag{31}$$

and

$$\lambda = 1 + \tfrac{1}{6}(b-a)\nabla^2(2a+b) - (z-a)\nabla^2 a - \tfrac{1}{2}\frac{(z-a)^2}{b-a}\nabla^2(b-a). \tag{32}$$

We thus find for the second approximation to the components of displacement,

$$\left.\begin{aligned} -4\pi f &= \frac{\lambda}{b-a}\left[\frac{da}{dx} + \frac{d(b-a)}{dx}\frac{z-a}{b-a}\right], \\ -4\pi g &= \frac{\lambda}{b-a}\left[\frac{da}{dy} + \frac{d(b-a)}{dy}\frac{z-a}{b-a}\right], \\ 4\pi h &= \frac{\lambda}{b-a}, \end{aligned}\right\} \tag{33}$$

and for the second approximation to the potential,

$$\Psi = \frac{z-a}{b-a} + \tfrac{1}{6}\nabla^2(2a+b)(z-a) - \tfrac{1}{2}\nabla^2 a\frac{(z-a)^2}{b-a} - \tfrac{1}{6}\nabla^2(b-a)\frac{(z-a)^3}{(b-a)^2}. \tag{34}$$

If $\sigma_a$ and $\sigma_b$ are the surface-densities and $\Psi_a$ and $\Psi_b$ the potentials of the surfaces $a$ and $b$ respectively,

$$\sigma_a = \frac{1}{4\pi}(\Psi_a - \Psi_b)\left[\frac{1}{b-a} + \tfrac{1}{3}\nabla^2 a + \tfrac{1}{6}\nabla^2 b\right],$$

$$\sigma_b = \frac{1}{4\pi}(\Psi_b - \Psi_a)\left[\frac{1}{b-a} - \tfrac{1}{6}\nabla^2 a - \tfrac{1}{3}\nabla^2 b\right]*.$$

* {This investigation is not very rigorous, and the expressions for the surface density do not agree with the results obtained by rigorous methods for the cases of two spheres, two cylinders, a sphere and plane, or a cylinder and plane placed close together. We can obtain an expression for the surface density as follows. Let us assume that the axis of $z$ is an axis of symmetry, then the axis will cut all the equipotential surfaces at right angles, and if $V$ is the potential, $R_1$ $R_2$ the principal radii of curvature of an equipotential surface where it is cut by the axis of $z$, the solenoidal condition along the axis of $z$ may easily be shown to be

$$\frac{d^2V}{dz^2} + \left(\frac{1}{R_1} + \frac{1}{R_2}\right)\frac{dV}{dz} = 0.$$

If $V_A$, $V_B$ are the potentials of the two surfaces respectively, $t$ the distance between the surfaces along the axis of $z$,

$$V_B = V_A + t\left(\frac{dV}{dz}\right)_A + \tfrac{1}{2}t^2\left(\frac{d^2V}{dz^2}\right)_A + \dots;$$

or if $R_{A_1}$, $R_{A_2}$ denote the principal radii of curvature of the first surfaces, substituting for $\frac{d^2V}{dz^2}$ from the differential equation, we get

$$V_B - V_A = t\left(\frac{dV}{dz}\right)_A\left\{1 - \tfrac{1}{2}t\left\{\frac{1}{R_{A_1}} + \frac{1}{R_{A_2}}\right\}\right\} + \dots;$$

but

$$\left(\frac{dV}{dz}\right)_A = -4\pi\sigma_A$$

when $\sigma_A$ is the surface density where the axis of $z$ cut the first surface, hence

$$\sigma_A = \frac{1}{4\pi}\frac{(V_A - V_B)}{t}\left\{1 + \tfrac{1}{2}t\left\{\frac{1}{R_{A_1}} + \frac{1}{R_{A_2}}\right\}\right\} \quad \text{approximately,}$$

similarly $$\sigma_B = \frac{1}{4\pi}\frac{(V_B - V_A)}{t}\left\{1 + \tfrac{1}{2}t\left\{\frac{1}{R_{B_1}} + \frac{1}{R_{B_2}}\right\}\right\} \quad \text{approximately,}$$

and these expressions agree in the cases before mentioned with those obtained by rigorous methods.}

# CHAPTER V.

## MECHANICAL ACTION BETWEEN TWO ELECTRICAL SYSTEMS.

103.] LET $E_1$ and $E_2$ be two electrical systems the mutual action between which we propose to investigate. Let the distribution of electricity in $E_1$ be defined by the volume-density, $\rho_1$, of the element whose coordinates are $x_1, y_1, z_1$. Let $\rho_2$ be the volume-density of the element of $E_2$, whose coordinates are $x_2, y_2, z_2$.

Then the $x$-component of the force acting on the element of $E_1$ on account of the repulsion of the element of $E_2$ will be

$$\rho_1 \rho_2 \frac{x_1 - x_2}{r^3} dx_1 dy_1 dz_1 dx_2 dy_2 dz_2,$$

where $$r^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2,$$

and if $A$ denotes the $x$-component of the whole force acting on $E_1$ on account of the presence of $E_2$

$$A = \iiiiiint \frac{x_1 - x_2}{r^3} \rho_1 \rho_2 dx_1 dy_1 dz_1 dx_2 dy_2 dz_2, \tag{1}$$

where the integration with respect to $x_1$, $y_1$, $z_1$ is extended throughout the region occupied by $E_1$, and the integration with respect to $x_2, y_2, z_2$ is extended throughout the region occupied by $E_2$.

Since, however, $\rho_1$ is zero except in the system $E_1$, and $\rho_2$ is zero except in the system $E_2$, the value of the integral will not be altered by extending the limits of the integrations, so that we may suppose the limits of every integration to be $\pm\infty$.

This expression for the force is a literal translation into mathematical symbols of the theory which supposes the electric force to act directly between bodies at a distance, no attention being bestowed on the intervening medium.

If we now define $\Psi_2$, the potential at the point $x_1$, $y_1$, $z_1$, arising from the presence of the system $E_2$, by the equation

$$\Psi_2 = \iiint \frac{\rho_2}{r} dx_2 dy_2 dz_2, \tag{2}$$

$\Psi_2$ will vanish at an infinite distance, and will everywhere satisfy the equation
$$\nabla^2 \Psi_2 = 4\pi\rho_2. \tag{3}$$

We may now express $A$ in the form of a triple integral
$$A = -\iiint \frac{d\Psi_2}{dx_1}\rho_1 dx_1 dy_1 dz_1. \tag{4}$$

Here the potential $\Psi_2$ is supposed to have a definite value at every point of the field, and in terms of this, together with the distribution, $\rho_1$, of electricity in the first system $E_1$, the force $A$ is expressed, no explicit mention being made of the distribution of electricity in the second system $E_2$.

Now let $\Psi_1$ be the potential arising from the first system, expressed as a function of $x$, $y$, $z$, and defined by the equation
$$\Psi_1 = \iiint \frac{\rho_1}{r} dx_1 dy_1 dz_1, \tag{5}$$
$\Psi_1$ will vanish at an infinite distance, and will everywhere satisfy the equation
$$\nabla^2 \Psi_1 = 4\pi\rho_1. \tag{6}$$

We may now eliminate $\rho_1$ from $A$ and obtain
$$A = -\frac{1}{4\pi}\iiint \frac{d\Psi_2}{dx_1}\nabla^2\Psi_1 dx_1 dy_1 dz_1, \tag{7}$$
in which the force is expressed in terms of the two potentials only.

104.] In all the integrations hitherto considered, it is indifferent what limits are prescribed, provided they include the whole of the system $E_1$. In what follows we shall suppose the systems $E_1$ and $E_2$ to be such that a certain closed surface $s$ contains within it the whole of $E_1$ but no part of $E_2$.

Let us also write
$$\rho = \rho_1 + \rho_2, \quad \Psi = \Psi_1 + \Psi_2, \tag{8}$$
then within $s$, $\qquad \rho_2 = 0, \;\; \rho = \rho_1,$

and without $s$, $\qquad \rho_1 = 0, \;\; \rho = \rho_2. \qquad (9)$

Now
$$A_{11} = -\iiint \frac{d\Psi_1}{dx_1}\rho_1 dx_1 dy_1 dz_1 \tag{10}$$
represents the resultant force, in the direction $x$, on the system $E_1$ arising from the electricity in the system itself. But on the theory of direct action this must be zero, for the action of any particle $P$ on another $Q$ is equal and opposite to that of $Q$ on $P$, and since the components of both actions enter into the integral, they will destroy each other.

We may therefore write

$$A = -\frac{1}{4\pi}\iiint \frac{d\Psi}{dx}\nabla^2\Psi\, dx_1 dy_1 dz_1, \tag{11}$$

where $\Psi$ is the potential arising from both systems, the integration being now limited to the space within the closed surface $s$, which includes the whole of the system $E_1$ but none of $E_2$.

105.] If the action of $E_2$ on $E_1$ is effected, not by direct action at a distance, but by means of a distribution of stress in a medium extending continuously from $E_2$ to $E_1$, it is manifest that if we know the stress at every point of any closed surface $s$ which completely separates $E_1$ from $E_2$, we shall be able to determine completely the mechanical action of $E_2$ on $E_1$. For if the force on $E_1$ is not completely accounted for by the stress through $s$, there must be direct action between something outside of $s$ and something inside of $s$.

Hence if it is possible to account for the action of $E_2$ on $E_1$ by means of a distribution of stress in the intervening medium, it must be possible to express this action in the form of a surface-integral extended over any surface $s$ which completely separates $E_2$ from $E_1$.

Let us therefore endeavour to express

$$A = \frac{1}{4\pi}\iiint \frac{d\Psi}{dx}\left[\frac{d^2\Psi}{dx^2} + \frac{d^2\Psi}{dy^2} + \frac{d^2\Psi}{dz^2}\right] dx\, dy\, dz \tag{12}$$

in the form of a surface integral.

By Theorem III, Art. 21, we may do so if we can determine $X$, $Y$ and $Z$, so that

$$\frac{d\Psi}{dx}\left(\frac{d^2\Psi}{dx^2} + \frac{d^2\Psi}{dy^2} + \frac{d^2\Psi}{dz^2}\right) = \frac{dX}{dx} + \frac{dY}{dy} + \frac{dZ}{dz}. \tag{13}$$

Taking the terms separately,

$$\frac{d\Psi}{dx}\frac{d^2\Psi}{dx^2} = \frac{1}{2}\frac{d}{dx}\left(\frac{d\Psi}{dx}\right)^2,$$

$$\frac{d\Psi}{dx}\frac{d^2\Psi}{dy^2} = \frac{d}{dy}\left(\frac{d\Psi}{dx}\frac{d\Psi}{dy}\right) - \frac{d\Psi}{dy}\frac{d^2\Psi}{dx\,dy},$$

$$= \frac{d}{dy}\left(\frac{d\Psi}{dx}\frac{d\Psi}{dy}\right) - \frac{1}{2}\frac{d}{dx}\left(\frac{d\Psi}{dy}\right)^2.$$

Similarly $$\frac{d\Psi}{dx}\frac{d^2\Psi}{dz^2} = \frac{d}{dz}\left(\frac{d\Psi}{dx}\frac{d\Psi}{dz}\right) - \frac{1}{2}\frac{d}{dx}\left(\frac{d\Psi}{dz}\right)^2.$$

If, therefore, we write

$$\left.\begin{aligned}\left(\frac{d\Psi}{dx}\right)^2-\left(\frac{d\Psi}{dy}\right)^2-\left(\frac{d\Psi}{dz}\right)^2&=8\pi p_{xx},\\ \left(\frac{d\Psi}{dy}\right)^2-\left(\frac{d\Psi}{dz}\right)^2-\left(\frac{d\Psi}{dx}\right)^2&=8\pi p_{yy},\\ \left(\frac{d\Psi}{dz}\right)^2-\left(\frac{d\Psi}{dx}\right)^2-\left(\frac{d\Psi}{dy}\right)^2&=8\pi p_{zz},\\ \frac{d\Psi}{dy}\frac{d\Psi}{dz}=4\pi p_{yz}&=4\pi p_{zy},\\ \frac{d\Psi}{dz}\frac{d\Psi}{dx}=4\pi p_{zx}&=4\pi p_{xz},\\ \frac{d\Psi}{dx}\frac{d\Psi}{dy}=4\pi p_{xy}&=4\pi p_{yx};\end{aligned}\right\} \tag{14}$$

then $$A=\iiint\left(\frac{dp_{xx}}{dx}+\frac{dp_{yx}}{dy}+\frac{dp_{zx}}{dz}\right)dx\,dy\,dz, \tag{15}$$

the integration being extended throughout the space within $s$. Transforming the volume-integral by Theorem III, Art. 21,

$$A=\iint(lp_{xx}+mp_{yx}+np_{zx})\,ds, \tag{16}$$

where $ds$ is an element of any closed surface including the whole of $E_1$ but none of $E_2$, and $l$, $m$, $n$ are the direction cosines of the normal drawn from $ds$ outwards.

For the components of the force on $E_1$ in the directions of $y$ and $z$, we obtain in the same way

$$B=\iint(lp_{xy}+mp_{yy}+np_{zy})\,ds, \tag{17}$$

$$C=\iint(lp_{xz}+mp_{yz}+np_{zz})\,ds. \tag{18}$$

If the action of the system $E_2$ on $E_1$ does in reality take place by direct action at a distance, without the intervention of any medium, we must consider the quantities $p_{xx}$ &c. as mere abbreviated forms for certain symbolical expressions, and as having no physical significance.

But if we suppose that the mutual action between $E_2$ and $E_1$ is kept up by means of stress in the medium between them, then since the equations (16), (17), (18) give the components of the resultant force arising from the action, on the outside of the surface $s$, of the stress whose six components are $p_{xx}$ &c., we must consider $p_{xx}$ &c. as the components of a stress actually existing in the medium.

106.] To obtain a clearer view of the nature of this stress let us alter the form of part of the surface $s$ so that the element $ds$ may become part of an equipotential surface. (This alteration of the surface is legitimate provided we do not thereby exclude any part of $E_1$ or include any part of $E_2$.)

Let $\nu$ be a normal to $ds$ drawn outwards.

Let $R = -\frac{d\Psi}{d\nu}$ be the intensity of the electromotive intensity in the direction of $\nu$, then

$$\frac{d\Psi}{dx} = -Rl, \quad \frac{d\Psi}{dy} = -Rm, \quad \frac{d\Psi}{dz} = -Rn.$$

Hence the six components of the stress are

$$p_{xx} = \frac{1}{8\pi} R^2 (l^2 - m^2 - n^2), \quad p_{yz} = \frac{1}{4\pi} R^2 mn,$$

$$p_{yy} = \frac{1}{8\pi} R^2 (m^2 - n^2 - l^2), \quad p_{zx} = \frac{1}{4\pi} R^2 nl,$$

$$p_{zz} = \frac{1}{8\pi} R^2 (n^2 - l^2 - m^2), \quad p_{xy} = \frac{1}{4\pi} R^2 lm.$$

If $a, b, c$ are the components of the force on $ds$ per unit of area,

$$a = lp_{xx} + mp_{yx} + np_{zx} = \frac{1}{8\pi} R^2 l,$$

$$b = \frac{1}{8\pi} R^2 m,$$

$$c = \frac{1}{8\pi} R^2 n.$$

Hence the force exerted by the part of the medium outside $ds$ on the part of the medium inside $ds$ is normal to the element and directed outwards, that is to say, it is a tension like that of a rope, and its value per unit of area is $\frac{1}{8\pi} R^2$.

Let us next suppose that the element $ds$ is at right angles to the equipotential surfaces which cut it, in which case

$$l \frac{d\Psi}{dx} + m \frac{d\Psi}{dy} + n \frac{d\Psi}{dz} = 0. \tag{19}$$

Now $$8\pi (lp_{xx} + mp_{yx} + np_{zx}) = l\left[\left(\frac{d\Psi}{dx}\right)^2 - \left(\frac{d\Psi}{dy}\right)^2 - \left(\frac{d\Psi}{dz}\right)^2\right]$$

$$+ 2m \frac{d\Psi}{dx}\frac{d\Psi}{dy} + 2n \frac{d\Psi}{dx}\frac{d\Psi}{dz}. \tag{20}$$

Multiplying (19) by $2\frac{d\Psi}{dx}$ and subtracting from (20), we find

$$8\pi(lp_{xx}+mp_{yx}+np_{zx}) = -l\left[\left(\frac{d\Psi}{dx}\right)^2+\left(\frac{d\Psi}{dy}\right)^2+\left(\frac{d\Psi}{dz}\right)^2\right]$$

$$= -lR^2. \tag{21}$$

Hence the components of the tension per unit of area of $ds$ are

$$a = -\frac{1}{8\pi}R^2 l,$$

$$b = -\frac{1}{8\pi}R^2 m,$$

$$c = -\frac{1}{8\pi}R^2 n.$$

Hence if the element $ds$ is at right angles to an equipotential surface, the force which acts on it is normal to the surface, and its numerical value per unit of area is the same as in the former case, but the direction of the force is different, for it is a pressure instead of a tension.

We have thus completely determined the type of the stress at any given point of the medium.

The direction of the electromotive intensity at the point is a principal axis of stress, and the stress in this direction is a tension whose numerical value is

$$p = \frac{1}{8\pi}R^2, \tag{22}$$

where $R$ is the electromotive intensity.

Any direction at right angles to this is also a principal axis of stress, and the stress along such an axis is a pressure whose numerical magnitude is also $p$.

The stress as thus defined is not of the most general type, for it has two of its principal stresses equal to each other, and the third has the same value with the sign reversed.

These conditions reduce the number of independent variables which determine the stress from six to three, accordingly it is completely determined by the three components of the electromotive intensity

$$-\frac{d\Psi}{dx}, \quad -\frac{d\Psi}{dy}, \quad -\frac{d\Psi}{dz}.$$

The three relations between the six components of stress are

$$\left.\begin{aligned} p^2_{yz} &= (p_{xx}+p_{yy})(p_{zz}+p_{xx}),\\ p^2_{zx} &= (p_{yy}+p_{zz})(p_{xx}+p_{yy}),\\ p^2_{xy} &= (p_{zz}+p_{xx})(p_{yy}+p_{zz}). \end{aligned}\right\} \tag{23}$$

107.] Let us now examine whether the results we have obtained will require modification when a finite quantity of electricity is collected on a finite surface so that the volume-density becomes infinite at the surface.

In this case, as we have shewn in Arts. 78 *a*, 78 *b*, the components of the electromotive intensity are discontinuous at the surface. Hence the components of stress will also be discontinuous at the surface.

Let $l$, $m$, $n$ be the direction cosines of the normal to $ds$. Let $P$, $Q$, $R$ be the components of the electromotive intensity on the side on which the normal is drawn, and $P'$, $Q'$, $R'$ their values on the other side.

Then by Arts. 78 *a* and 78 *b*, if $\sigma$ is the surface-density

$$\left.\begin{aligned} P-P' &= 4\pi\sigma l,\\ Q-Q' &= 4\pi\sigma m,\\ R-R' &= 4\pi\sigma n. \end{aligned}\right\} \tag{24}$$

Let $a$ be the $x$-component of the resultant force acting on the surface per unit of area, arising from the stress on both sides, then

$$\begin{aligned} a &= l(p_{xx}-p'_{xx})+m(p_{xy}-p'_{xy})+n(p_{xz}-p'_{xz}),\\ &= \frac{1}{8\pi}l\{(P^2-P'^2)-(Q^2-Q'^2)-(R^2-R'^2)\}\\ &\qquad\qquad + \frac{1}{4\pi}m(PQ-P'Q')+\frac{1}{4\pi}n(PR-P'R'),\\ &= \frac{1}{8\pi}l\{(P-P')(P+P')-(Q-Q')(Q+Q')-(R-R')(R+R')\}\\ &\qquad + \frac{1}{8\pi}m\{(P-P')(Q+Q')+(P+P')(Q-Q')\}\\ &\qquad + \frac{1}{8\pi}n\{(P-P')(R+R')+(P+P')(R-R')\},\\ &= \frac{1}{2}l\sigma\{l(P+P')-m(Q+Q')-n(R+R')\}\\ &+\frac{1}{2}m\sigma\{l(Q+Q')+m(P+P')\}+\frac{1}{2}n\sigma\{l(R+R')+n(P+P')\},\\ &= \frac{1}{2}\sigma(P+P'). \end{aligned} \tag{25}$$

Hence, assuming that the stress at any point is given by equations (14), we find that the resultant force in the direction of $x$ on a charged surface per unit of volume is equal to the surface-density multiplied into the arithmetical mean of the $x$-components of the electromotive intensities on the two sides of the surface.

This is the same result as we obtained in Art. 79 by a process essentially similar.

Hence the hypothesis of stress in the surrounding medium is applicable to the case in which a finite quantity of electricity is collected on a finite surface.

The resultant force on an element of surface is usually deduced from the theory of action at a distance by considering a portion of the surface, the dimensions of which are very small compared with the radii of curvature of the surface *.

On the normal to the middle point of this portion of the surface take a point $P$ whose distance from the surface is very small compared with the dimensions of the portion of the surface. The electromotive intensity at this point, due to the small portion of the surface, will be approximately the same as if the surface had been an infinite plane, that is to say $2\pi\sigma$ in the direction of the normal drawn from the surface. For a point $P'$ just on the other side of the surface the intensity will be the same, but in the opposite direction.

Now consider the part of the electromotive intensity arising from the rest of the surface and from other electrified bodies at a finite distance from the element of surface. Since the points $P$ and $P'$ are infinitely near one another, the components of the electromotive intensity arising from electricity at a finite distance will be the same for both points.

Let $P_0$ be the $x$-component of the electromotive intensity on $A$ or $A'$ arising from electricity at a finite distance, then the total value of the $x$-component for $A$ will be

$$P = P_0 + 2\pi\sigma l,$$

and for $A'$

$$P' = P_0 - 2\pi\sigma l.$$

Hence

$$P_0 = \tfrac{1}{2}(P+P').$$

Now the resultant mechanical force on the element of surface must arise entirely from the action of electricity at a finite distance,

* This method is due to Laplace. See Poisson, 'Sur la Distribution de l'électricité &c.' *Mém. de l'Institut*, 1811, p. 30.

since the action of the element on itself must have a resultant zero. Hence the $x$-component of this force per unit of area must be

$$\begin{aligned} a &= \sigma P_0, \\ &= \tfrac{1}{2}\sigma(P+P'). \end{aligned}$$

108.] If we define the potential (as in equation (2)) in terms of a distribution of electricity supposed to be given, then it follows from the fact that the action and reaction between any pair of electric particles are equal and opposite, that the $x$-component of the force arising from the action of a system on itself must be zero, and we may write this in the form

$$\frac{1}{4\pi}\iiint\frac{d\Psi}{dx}\nabla^2\Psi\,dx\,dy\,dz = 0. \tag{26}$$

But if we define $\Psi$ as a function of $x$, $y$, $z$ which satisfies the equation

$$\nabla^2\Psi = 0$$

at every point outside the closed surface $s$, and is zero at an infinite distance, the fact, that the volume-integral extended throughout any space including $s$ is zero, would seem to require proof.

One method of proof is founded on the theorem (Art. 100 $c$), that if $\nabla^2\Psi$ is given at every point, and $\Psi = 0$ at an infinite distance, then the value of $\Psi$ at every point is determinate and equal to

$$\Psi' = \frac{1}{4\pi}\iiint\frac{1}{r}\nabla^2\Psi\,dx\,dy\,dz, \tag{27}$$

where $r$ is the distance between the element $dx\,dy\,dz$ at which the concentration of $\Psi$ is given $= \nabla^2\Psi$ and the point $x', y', z'$ at which $\Psi'$ is to be found.

This reduces the theorem to what we deduced from the first definition of $\Psi$.

But when we consider $\Psi$ as the primary function of $x$, $y$, $z$, from which the others are derived, it is more appropriate to reduce (26) to the form of a surface-integral,

$$A = \iint(lp_{xx} + mp_{xy} + np_{xz})\,dS, \tag{28}$$

and if we suppose the surface $S$ to be everywhere at a great distance $a$ from the surface $s$, which includes every point where $\nabla^2\Psi$ differs from zero, then we know that $\Psi$ cannot be numerically greater than $e/a$, where $4\pi e$ is the volume-integral of $\nabla^2\Psi$, and that $R$ cannot be greater than $-d\Psi/da$ or $e/a^2$, and that the quantities $p_{xx}$, $p_{xy}$, $p_{xz}$ can none of them be greater than $p$, i.e. $R^2/8\pi$ or $e^2/8\pi a^4$. Hence the surface-integral taken over a sphere whose radius is very great and equal to $a$ cannot exceed $e^2/2a^2$, and when $a$ is increased without limit, the surface-integral must become ultimately zero.

But this surface-integral is equal to the volume-integral (26), and the value of this volume-integral is the same whatever be the size of the space enclosed within $S$, provided $S$ encloses every point at which $\nabla^2\Psi$ differs from zero. Hence, since the integral is zero when $a$ is infinite, it must also be zero when the limits of integration are defined by any surface which includes every point at which $\nabla^2\Psi$ differs from zero.

109.] The distribution of stress considered in this chapter is precisely that to which Faraday was led in his investigation of induction through dielectrics. He sums up in the following words:—

'(1297) The direct inductive force, which may be conceived to be exerted in lines between the two limiting and charged conducting surfaces, is accompanied by a lateral or transverse force equivalent to a dilatation or repulsion of these representative lines (1224); or the attractive force which exists amongst the particles of the dielectric in the direction of the induction is accompanied by a repulsive or a diverging force in the transverse direction.

'(1298) Induction appears to consist in a certain polarized state of the particles, into which they are thrown by the electrified body sustaining the action, the particles assuming positive and negative points or parts, which are symmetrically arranged with respect to each other and the inducting surfaces or particles. The state must be a forced one, for it is originated and sustained only by force, and sinks to the normal or quiescent state when that force is removed. It can be *continued* only in insulators by the same portion of electricity, because they only can retain this state of the particles.'

This is an exact account of the conclusions to which we have been conducted by our mathematical investigation. At every point of the medium there is a state of stress such that there is tension along the lines of force and pressure in all directions at right angles to these lines, the numerical magnitude of the pressure being equal to that of the tension, and both varying as the square of the resultant force at the point.

The expression 'electric tension' has been used in various senses by different writers. I shall always use it to denote the tension along the lines of force, which, as we have seen, varies from point to point, and is always proportional to the square of the resultant force at the point.

110.] The hypothesis that a state of stress of this kind exists in a fluid dielectric, such as air or turpentine, may at first sight appear at variance with the established principle that at any point in a fluid the pressures in all directions are equal. But in the deduction of this principle from a consideration of the mobility and equilibrium of the parts of the fluid it is taken for granted that no action such as that which we here suppose to take place along the lines of force exists in the fluid. The state of stress which we have been studying is perfectly consistent with the mobility and equilibrium of the fluid, for we have seen that, if any portion of the fluid is devoid of electric charge, it experiences no resultant force from the stresses on its surface, however intense these may be. It is only when a portion of the fluid becomes charged that its equilibrium is disturbed by the stresses on its surface, and we know that in this case it actually tends to move. Hence the supposed state of stress is not inconsistent with the equilibrium of a fluid dielectric.

The quantity $W$, which was investigated in Chapter IV, Art. 99$a$, may be interpreted as the energy in the medium due to the distribution of stress. It appears from the theorems of that chapter that the distribution of stress which satisfies the conditions there given also makes $W$ an absolute minimum. Now when the energy is a minimum for any configuration, that configuration is one of equilibrium, and the equilibrium is stable. Hence the dielectric, when subjected to the inductive action of electrified bodies, will of itself take up a state of stress distributed in the way we have described*.

It must be carefully borne in mind that we have made only one step in the theory of the action of the medium. We have supposed it to be in a state of stress, but we have not in any way accounted for this stress, or explained how it is maintained. This step, however, seems to me to be an important one, as it

* {The subject of the stress in the medium will be further considered in the Supplementary Volume, it may however be noticed here that the problem of finding a system of stresses which will produce the same forces as those existing in the electric field is one which has an infinite number of solutions. That adopted by Maxwell is one that could not in general be produced by strains in an elastic solid.}

explains, by the action of the consecutive parts of the medium, phenomena which were formerly supposed to be explicable only by direct action at a distance.

111.] I have not been able to make the next step, namely, to account by mechanical considerations for these stresses in the dielectric. I therefore leave the theory at this point, merely stating what are the other parts of the phenomenon of induction in dielectrics.

I. *Electric Displacement.* When induction is transmitted through a dielectric, there is in the first place a displacement of electricity in the direction of the induction. For instance, in a Leyden jar, of which the inner coating is charged positively and the outer coating negatively, the direction of the displacement of positive electricity in the substance of the glass is from within outwards.

Any increase of this displacement is equivalent, during the time of increase, to a current of positive electricity from within outwards, and any diminution of the displacement is equivalent to a current in the opposite direction.

The whole quantity of electricity displaced through any area of a surface fixed in the dielectric is measured by the quantity which we have already investigated (Art. 75) as the surface-integral of induction through that area, multiplied by $K/4\pi$, where $K$ is the specific inductive capacity of the dielectric.

II. *Surface charge of the particles of the dielectric.* Conceive any portion of the dielectric, large or small, to be separated (in imagination) from the rest by a closed surface, then we must suppose that on every elementary portion of this surface there is a charge measured by the total displacement of electricity through that element of surface *reckoned inwards.*

In the case of the Leyden jar of which the inner coating is charged positively, any portion of the glass will have its inner side charged positively and its outer side negatively. If this portion be entirely in the interior of the glass, its surface charge will be neutralized by the opposite charge of the parts in contact with it, but if it be in contact with a conducting body, which is incapable of maintaining in itself the inductive state, the surface charge will not be neutralized, but will constitute that apparent charge which is commonly called the Charge of the Conductor.

The charge therefore at the bounding surface of a conductor and the surrounding dielectric, which on the old theory was called the charge of the conductor, must be called in the theory of induction the surface charge of the surrounding dielectric.

According to this theory, all charge is the residual effect of the polarization of the dielectric. The polarization exists throughout the interior of the substance, but it is there neutralized by the juxtaposition of oppositely charged parts, so that it is only at the surface of the dielectric that the effects of the charge become apparent.

The theory completely accounts for the theorem of Art. 77, that the total induction through a closed surface is equal to the total quantity of electricity within the surface multiplied by $4\pi$. For what we have called the induction through the surface is simply the electric displacement multiplied by $4\pi$, and the total displacement outwards is necessarily equal to the total charge within the surface.

The theory also accounts for the impossibility of communicating an 'absolute charge' to matter. For every particle of the dielectric has equal and opposite charges on its opposite sides, if it would not be more correct to say that these charges are only the manifestations of a single phenomenon, which we may call Electric Polarization.

A dielectric medium, when thus polarized, is the seat of electric energy, and the energy in unit of volume of the medium is numerically equal to the electric tension on unit of area, both quantities being equal to half the product of the displacement and the resultant electromotive intensity, or

$$p = \tfrac{1}{2}\mathfrak{D}\mathfrak{E} = \frac{1}{8\pi}K\mathfrak{E}^2 = \frac{2\pi}{K}\mathfrak{D}^2,$$

where $p$ is the electric tension, $\mathfrak{D}$ the displacement, $\mathfrak{E}$ the electromotive intensity, and $K$ the specific inductive capacity.

If the medium is not a perfect insulator, the state of constraint, which we call electric polarization, is continually giving way. The medium yields to the electromotive force, the electric stress is relaxed, and the potential energy of the state of constraint is converted into heat. The rate at which this decay of the state of polarization takes place depends on the nature of the medium. In some kinds of glass, days or years may elapse before the polarization sinks to half its original value. In copper, a similar change is effected in less than the billionth of a second.

We have supposed the medium after being polarized to be simply left to itself. In the phenomenon called the electric current the constant passage of electricity through the medium tends to restore the state of polarization as fast as the conductivity of the medium allows it to decay. Thus the external agency which maintains the current is always doing work in restoring the polarization of the medium, which is continually becoming relaxed, and the potential energy of this polarization is continually becoming transformed into heat, so that the final result of the energy expended in maintaining the current is to gradually raise the temperature of the conductor, till as much heat is lost by conduction and radiation from its surface as is generated in the same time by the electric current.

# CHAPTER VI.

## ON POINTS AND LINES OF EQUILIBRIUM.

112.] If at any point of the electric field the resultant force is zero, the point is called a Point of equilibrium.

If every point on a certain line is a point of equilibrium, the line is called a Line of equilibrium.

The conditions that a point shall be a point of equilibrium are that at that point

$$\frac{dV}{dx} = 0, \quad \frac{dV}{dy} = 0, \quad \frac{dV}{dz} = 0.$$

At such a point, therefore, the value of $V$ is a maximum, or a minimum, or is stationary, with respect to variations of the coordinates. The potential, however, can have a maximum or a minimum value only at a point charged with positive or with negative electricity, or throughout a finite space bounded by a surface charged positively or negatively. If, therefore, a point of equilibrium occurs in an uncharged part of the field the potential must be stationary, and not a maximum or a minimum.

In fact, a condition for a maximum or minimum is that

$$\frac{d^2V}{dx^2}, \quad \frac{d^2V}{dy^2}, \quad \text{and} \quad \frac{d^2V}{dz^2}$$

must be all negative or all positive, if they have finite values.

Now, by Laplace's equation, at a point where there is no charge, the sum of these three quantities is zero, and therefore this condition cannot be satisfied.

Instead of investigating the analytical conditions for the cases in which the components of the force simultaneously vanish, we shall give a general proof by means of the equipotential surfaces.

If at any point, $P$, there is a true maximum value of $V$, then, at all other points in the immediate neighbourhood of $P$, the value of $V$ is less than at $P$. Hence $P$ will be surrounded by a series of closed equipotential surfaces, each outside the one before it, and at all points of any one of these surfaces the electrical force will be directed outwards. But we have proved, in Art. 76, that the surface-integral of the electromotive intensity taken over any closed surface gives the total charge within that surface multiplied by $4\pi$. Now, in this case the force is everywhere outwards, so that the surface-integral is necessarily positive, and therefore there is a positive charge within the surface, and, since we may take the surface as near to $P$ as we please, there is a positive charge at the point $P$.

In the same way we may prove that if $V$ is a minimum at $P$, then $P$ is negatively charged.

Next, let $P$ be a point of equilibrium in a region devoid of charge, and let us describe a sphere of very small radius round $P$, then, as we have seen, the potential at this surface cannot be everywhere greater or everywhere less than at $P$. It must therefore be greater at some parts of the surface and less at others. These portions of the surface are bounded by lines in which the potential is equal to that at $P$. Along lines drawn from $P$ to points at which the potential is less than that at $P$ the electrical force is from $P$, and along lines drawn to points of greater potential the force is towards $P$. Hence the point $P$ is a point of stable equilibrium for some displacements, and of unstable equilibrium for other displacements.

113.] To determine the number of the points and lines of equilibrium, let us consider the surface or surfaces for which the potential is equal to $C$, a given quantity. Let us call the regions in which the potential is less than $C$ the negative regions, and those in which it is greater than $C$ the positive regions. Let $V_0$ be the lowest, and $V_1$ the highest potential existing in the electric field. If we make $C = V_0$, the negative region will include only the point or conductor of lowest potential, and this is necessarily charged negatively. The positive region consists of the rest of space, and since it surrounds the negative region it is periphractic. See Art. 18.

If we now increase the value of $C$, the negative region will expand, and new negative regions will be formed round negatively charged bodies. For every negative region thus formed the surrounding positive region acquires one degree of periphraxy.

As the different negative regions expand, two or more of them may meet in a point or a line. If $n+1$ negative regions meet, the positive region loses $n$ degrees of periphraxy, and the point or the line in which they meet is a point or line of equilibrium of the $n$th degree.

When $C$ becomes equal to $V_1$ the positive region is reduced to the point or the conductor of highest potential, and has therefore lost all its periphraxy. Hence, if each point or line of equilibrium counts for one, two, or $n$, according to its degree, the number so made up by the points or lines now considered will be less by one than the number of negatively charged bodies.

There are other points or lines of equilibrium which occur where the positive regions become separated from each other, and the negative region acquires periphraxy. The number of these, reckoned according to their degrees, is less by one than the number of positively charged bodies.

If we call a point or line of equilibrium positive when it is the meeting-place of two or more positive regions, and negative when the regions which unite there are negative, then, if there are $p$ bodies positively and $n$ bodies negatively charged, the sum of the degrees of the positive points and lines of equilibrium will be $p-1$, and that of the negative ones $n-1$. The surface which surrounds the electrical system at an infinite distance from it is to be reckoned as a body whose charge is equal and opposite to the sum of the charges of the system.

But, besides this definite number of points and lines of equilibrium arising from the junction of different regions, there may be others, of which we can only affirm that their number must be even. For if, as any one of the negative regions expands, it meets itself, it becomes a cyclic region, and it may acquire, by repeatedly meeting itself, any number of degrees of cyclosis, each of which corresponds to the point or line of equilibrium at which the cyclosis was established. As the negative region continues to expand till it fills all space, it loses every degree of cyclosis it has acquired, and becomes at last acyclic. Hence there is a set of points or lines of equilibrium at which cyclosis is lost, and these are equal in number of degrees to those at which it is acquired.

If the form of the charged bodies or conductors is arbitrary, we can only assert that the number of these additional points or lines is even, but if they are charged points or spherical conductors, the number arising in this way cannot exceed $(n-1)(n-2)$, where $n$ is the number of bodies *.

114.] The potential close to any point $P$ may be expanded in the series

$$V = V_0 + H_1 + H_2 + \&c.;$$

where $H_1$, $H_2$, &c. are homogeneous functions of $x$, $y$, $z$, whose dimensions are 1, 2, &c. respectively.

Since the first derivatives of $V$ vanish at a point of equilibrium, $H_1 = 0$, if $P$ be a point of equilibrium.

Let $H_n$ be the first function which does not vanish, then close to the point $P$ we may neglect all functions of higher degrees as compared with $H_n$.

Now $$H_n = 0$$

is the equation of a cone of the degree $n$, and this cone is the cone of closest contact with the equipotential surface at $P$.

It appears, therefore, that the equipotential surface passing through $P$ has, at that point, a conical point touched by a cone of the second or of a higher degree. The intersection of this cone with a sphere whose centre is the vertex is called the Nodal line.

If the point $P$ is not on a line of equilibrium the nodal line does not intersect itself, but consists of $n$ or some smaller number of closed curves.

If the nodal line intersects itself, then the point $P$ is on a line of equilibrium, and the equipotential surface through $P$ cuts itself in that line.

If there are intersections of the nodal line not on opposite points of the sphere, then $P$ is at the intersection of three or more lines of equilibrium. For the equipotential surface through $P$ must cut itself in each line of equilibrium.

115.] If $n$ sheets of the same equipotential surface intersect, they must intersect at angles each equal to $\pi/n$.

For let the tangent to the line of intersection be taken as the axis of $z$, then $d^2V/dz^2 = 0$. Also let the axis of $x$ be a tangent to one of the sheets, then $d^2V/dx^2 = 0$. It follows from this, by Laplace's equation, that $d^2V/dy^2 = 0$, or the axis of $y$ is a tangent to the other sheet.

This investigation assumes that $H_2$ is finite. If $H_2$ vanishes, let the tangent to the line of intersection be taken as the axis

* {I have not been able to find any place where this result is proved.}

of $z$, and let $x = r\cos\theta$, and $y = r\sin\theta$, then, since

$$\frac{d^2 V}{dz^2} = 0, \quad \frac{d^2 V}{dx^2} + \frac{d^2 V}{dy^2} = 0;$$

or
$$\frac{d^2 V}{dr^2} + \frac{1}{r}\frac{dV}{dr} + \frac{1}{r^2}\frac{d^2 V}{d\theta^2} = 0;$$

the solution of which equation in ascending powers of $r$ is

$$V = V_0 + A_1 r\cos(\theta + a_1) + A_2 r^2 \cos(2\theta + a_2) + \&c. + A_n r^n \cos(n\theta + a_n).$$

At a point of equilibrium $A_1$ is zero. If the first term that does not vanish is that in $r^n$, then

$$V - V_0 = A_n r^n \cos(n\theta + a_n) + \text{terms in higher powers of } r.$$

This equation shews that $n$ sheets of the equipotential surface $V = V_0$ intersect at angles each equal to $\pi/n$. This theorem was given by Rankine *.

It is only under certain conditions that a line of equilibrium can exist in free space, but there must be a line of equilibrium on the surface of a conductor whenever the surface density of the conductor is positive in one portion and negative in another.

In order that a conductor may be charged oppositely on different portions of its surface, there must be in the field some places where the potential is higher than that of the body and others where it is lower.

Let us begin with two conductors electrified positively to the same potential. There will be a point of equilibrium between the two bodies. Let the potential of the first body be gradually diminished. The point of equilibrium will approach it, and, at a certain stage of the process, will coincide with a point on its surface. During the next stage of the process, the equipotential surface round the second body which has the same potential as the first body will cut the surface of the second body at right angles in a closed curve, which is a line of equilibrium. This

* 'Summary of the Properties of certain Stream Lines,' *Phil. Mag.*, Oct. 1864. See also, Thomson and Tait's *Natural Philosophy*, § 780; and Rankine and Stokes, in the *Proc. R. S.*, 1867, p. 468; also W. R. Smith, *Proc. R. S. Edin.* 1869–70, p. 79.

{This investigation is not satisfactory as $d^2V/dz^2$ only vanishes along the axis of $z$. Rankine's original proof is rigid. $Hm$ may be written as

$$u_n z^{m-n} + u_{n+1} z^{m-n-1} + \dots u_m,$$

where $u_n, u_{n+1} \dots$ are homogeneous functions of $x, y$ of degrees $n, n+1$ respectively, the axis of $z$ is a singular line of degree $n$. Since $Hm$ satisfies $\nabla^2 Hm = 0$, we must have

$$\frac{d^2 u_n}{dx^2} + \frac{d^2 u_n}{dy^2} = 0,$$

or $u_n = Ar^n \cos(n\theta + a)$; but $u_n = 0$ is the equation of the tangent planes from the axis of $z$ to the cone $Hm = 0$, that is of the $n$ sheets of the equipotential surface, hence these cut at angle $\pi/n$.}

closed curve, after sweeping over the entire surface of the conductor, will again contract to a point; and then the point of equilibrium will move off on the other side of the first body, and will be at an infinite distance when the charges of the two bodies are equal and opposite.

### *Earnshaw's Theorem.*

116.] A charged body placed in a field of electric force cannot be in stable equilibrium.

First, let us suppose the electricity of the moveable body $A$, and also that of the system of surrounding bodies $B$, to be fixed in those bodies.

Let $V$ be the potential at any point of the moveable body due to the action of the surrounding bodies $B$, and let $e$ be the electricity on a small portion of the moveable body $A$ surrounding this point. Then the potential energy of $A$ with respect to $B$ will be

$$M = \Sigma (Ve),$$

where the summation is to be extended to every charged portion of $A$.

Let $a$, $b$, $c$ be the coordinates of any charged part of $A$ with respect to axes fixed in $A$, and parallel to those of $x$, $y$, $z$. Let the absolute coordinates of the origin of these axes be $\xi$, $\eta$, $\zeta$.

Let us suppose for the present that the body $A$ is constrained to move parallel to itself, then the absolute coordinates of the point $a$, $b$, $c$ will be

$$x = \xi + a, \qquad y = \eta + b, \qquad z = \zeta + c.$$

The potential of the body $A$ with respect to $B$ may now be expressed as the sum of a number of terms, in each of which $V$ is expressed in terms of $a$, $b$, $c$ and $\xi$, $\eta$, $\zeta$, and the sum of these terms is a function of the quantities $a$, $b$, $c$, which are constant for each point of the body, and of $\xi$, $\eta$, $\zeta$, which vary when the body is moved.

Since Laplace's equation is satisfied by each of these terms it is satisfied by their sum, or

$$\frac{d^2M}{d\xi^2} + \frac{d^2M}{d\eta^2} + \frac{d^2M}{d\zeta^2} = 0.$$

Now let a small displacement be given to $A$, so that

$$d\xi = l\,dr, \qquad d\eta = m\,dr, \qquad d\zeta = n\,dr;$$

and let $dM$ be the increment of the potential of $A$ with respect to the surrounding system $B$.

If this be positive, work will have to be done to increase $r$, and there will be a force $R = dM/dr$ tending to diminish $r$ and to restore $A$ to its former position, and for this displacement therefore the equilibrium will be stable. If, on the other hand, this quantity is negative, the force will tend to increase $r$, and the equilibrium will be unstable.

Now consider a sphere whose centre is the origin and whose radius is $r$, and so small that when the point fixed in the body lies within this sphere no part of the moveable body $A$ can coincide with any part of the external system $B$. Then, since within the sphere $\nabla^2 M = 0$, the surface-integral

$$\iint \frac{dM}{dr}\, dS,$$

taken over the surface of the sphere, is zero.

Hence, if at any part of the surface of the sphere $dM/dr$ is positive, there must be some other part of the surface where it is negative, and if the body $A$ be displaced in a direction in which $dM/dr$ is negative, it will tend to move from its original position, and its equilibrium is therefore necessarily unstable.

The body therefore is unstable even when constrained to move parallel to itself, and *à fortiori* it is unstable when altogether free.

Now let us suppose that the body $A$ is a conductor. We might treat this as a case of equilibrium of a system of bodies, the moveable electricity being considered as part of that system, and we might argue that as the system is unstable when deprived of so many degrees of freedom by the fixture of its electricity, it must *à fortiori* be unstable when this freedom is restored to it.

But we may consider this case in a more particular way, thus—

First, let the electricity be fixed in $A$, and let $A$ move parallel to itself through the small distance $dr$. The increment of the potential of $A$ due to this cause has been already considered.

Next, let the electricity be allowed to move within $A$ into its position of equilibrium, which is always stable. During this motion the potential will necessarily be *diminished* by a quantity which we may call $Cdr$.

Hence the total increment of the potential when the electricity is free to move will be

$$\left(\frac{dM}{dr} - C\right) dr\,;$$

and the force tending to bring $A$ back towards its original position will be

$$\frac{dM}{dr} - C,$$

where $C$ is always positive.

Now we have shewn that $dM/dr$ is negative for certain directions of $r$, hence when the electricity is free to move the instability in these directions will be increased.

# CHAPTER VII.

## FORMS OF THE EQUIPOTENTIAL SURFACES AND LINES OF INDUCTION IN SIMPLE CASES.

117.] WE have seen that the determination of the distribution of electricity on the surface of conductors may be made to depend on the solution of Laplace's equation

$$\frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} + \frac{d^2V}{dz^2} = 0,$$

$V$ being a function of $x$, $y$, and $z$, which is always finite and continuous, which vanishes at an infinite distance, and which has a given constant value at the surface of each conductor.

It is not in general possible by known mathematical methods to solve this equation so as to fulfil arbitrarily given conditions, but it is easy to write down any number of expressions for the function $V$ which shall satisfy the equation, and to determine in each case the forms of the conducting surfaces, so that the function $V$ shall be the true solution.

It appears, therefore, that what we should naturally call the inverse problem of determining the forms of the conductors when the expression for the potential is given is more manageable than the direct problem of determining the potential when the form of the conductors is given.

In fact, every electrical problem of which we know the solution has been constructed by this inverse process. It is therefore of great importance to the electrician that he should know what results have been obtained in this way, since the only method by which he can expect to solve a new problem is by reducing it to one of the cases in which a similar problem has been constructed by the inverse process.

This historical knowledge of results can be turned to account in two ways. If we are required to devise an instrument for making electrical measurements with the greatest accuracy, we may select those forms for the electrified surfaces which correspond to cases of which we know the accurate solution. If, on the other hand, we are required to estimate what will be the electrification of bodies whose forms are given, we may begin with some case in which one of the equipotential surfaces takes a form somewhat resembling the given form, and then by a tentative method we may modify the problem till it more nearly corresponds to the given case. This method is evidently very imperfect considered from a mathematical point of view, but it is the only one we have, and if we are not allowed to choose our conditions, we can make only an approximate calculation of the electrification. It appears, therefore, that what we want is a knowledge of the forms of equipotential surfaces and lines of induction in as many different cases as we can collect together and remember. In certain classes of cases, such as those relating to spheres, there are known mathematical methods by which we may proceed. In other cases we cannot afford to despise the humbler method of actually drawing tentative figures on paper, and selecting that which appears least unlike the figure we require.

This latter method I think may be of some use, even in cases in which the exact solution has been obtained, for I find that an eye-knowledge of the forms of the equipotential surfaces often leads to a right selection of a mathematical method of solution.

I have therefore drawn several diagrams of systems of equipotential surfaces and lines of induction, so that the student may make himself familiar with the forms of the lines. The methods by which such diagrams may be drawn will be explained in Art. 123.

118.] In the first figure at the end of this volume we have the sections of the equipotential surfaces surrounding two points charged with quantities of electricity of the same kind and in the ratio of 20 to 5.

Here each point is surrounded by a system of equipotential surfaces which become more nearly spheres as they become smaller, though none of them are accurately spheres. If two of these surfaces, one surrounding each point, be taken to represent the surfaces of two conducting bodies, nearly but not quite spherical, and if these bodies be charged with the same kind of electricity, the charges being as 4 to 1, then the diagram will represent the equipotential surfaces, provided we expunge all those which are drawn inside the two bodies. It appears from the diagram that the action between the bodies will be the same as that between two points having the same charges, these points being not exactly in the middle of the axis of each body, but each somewhat more remote than the middle point from the other body.

The same diagram enables us to see what will be the distribution of electricity on one of the oval figures, larger at one end than the other, which surround both centres. Such a body, if charged with 25 units of electricity and free from external influence, will have the surface-density greatest at the small end, less at the large end, and least in a circle somewhat nearer the smaller than the larger end *.

There is one equipotential surface, indicated by a dotted line, which consists of two lobes meeting at the conical point $P$. That point is a point of equilibrium, and the surface-density on a body of the form of this surface would be zero at this point.

The lines of force in this case form two distinct systems, divided from one another by a surface of the sixth degree, indicated by a dotted line, passing through the point of equilibrium, and somewhat resembling one sheet of the hyperboloid of two sheets.

This diagram may also be taken to represent the lines of force and equipotential surfaces belonging to two spheres of gravitating matter whose masses are as 4 to 1.

119.] In the second figure we have again two points whose charges are as 20 to 5, but the one positive and the other negative. In this case one of the equipotential surfaces, that, namely, corresponding to potential zero, is a sphere. It is marked in the diagram by the dotted circle $Q$. The importance of this spherical surface will be seen when we come to the theory of Electrical Images.

We may see from this diagram that if two round bodies are charged with opposite kinds of electricity they will attract each other as much as two points having the same charges but placed somewhat nearer together than the middle points of the round bodies.

* {This can be seen by comparing the distances between the equipotential surfaces in various parts of the field.}

Here, again, one of the equipotential surfaces, indicated by a dotted line, has two lobes, an inner one surrounding the point whose charge is 5 and an outer one surrounding both bodies, the two lobes meeting in a conical point $P$ which is a point of equilibrium.

If the surface of a conductor is of the form of the outer lobe, a roundish body having, like an apple, a conical dimple at one end of its axis, then, if this conductor be electrified, we shall be able to determine the surface-density at any point. That at the bottom of the dimple will be zero.

Surrounding this surface we have others having a rounded dimple which flattens and finally disappears in the equipotential surface passing through the point marked $M$.

The lines of force in this diagram form two systems divided by a surface which passes through the point of equilibrium.

If we consider points on the axis on the further side of the point $B$, we find that the resultant force diminishes to the double point $P$, where it vanishes. It then changes sign, and reaches a maximum at $M$, after which it continually diminishes.

This maximum, however, is only a maximum relatively to other points on the axis, for if we consider a surface through $M$ perpendicular to the axis, $M$ is a point of minimum force relatively to neighbouring points on that surface.

120.] Figure III represents the equipotential surfaces and lines of induction due to a point whose charge is 10 placed at $A$, and surrounded by a field of force, which, before the introduction of the charged point, was uniform in direction and magnitude at every part *.

The equipotential surfaces have each of them an asymptotic plane. One of them, indicated by a dotted line, has a conical point and a lobe surrounding the point $A$. Those below this surface have one sheet with a depression near the axis. Those above have a closed portion surrounding $A$ and a separate sheet with a slight depression near the axis.

If we take one of the surfaces below $A$ as the surface of a conductor, and another a long way below $A$ as the surface of another conductor at a different potential, the system of lines

* {Maxwell does not give the strength of the field. M. Cornu however has calculated the strength of the uniform field from the diagram of the lines of force, and finds that its electromotive intensity before the introduction of the charged body was 1·5.}

and surfaces between the two conductors will indicate the distribution of electric force. If the lower conductor is very far from $A$ its surface will be very nearly plane, so that we have here the solution of the distribution of electricity on two surfaces, both of them nearly plane and parallel to each other, except that the upper one has a protuberance near its middle point, which is more or less prominent according to the particular equipotential surface we choose.

121.] Figure IV represents the equipotential surfaces and lines of induction due to three points $A$, $B$ and $C$, the charge of $A$ being 15 units of positive electricity, that of $B$ 12 units of negative electricity, and that of $C$ 20 units of positive electricity. These points are placed in one straight line, so that

$$AB = 9, \quad BC = 16, \quad AC = 25.$$

In this case, the surface for which the potential is zero consists of two spheres whose centres are $A$ and $C$ and whose radii are 15 and 20. These spheres intersect in the circle which cuts the plane of the paper at right angles in $D$ and $D'$, so that $B$ is the centre of this circle and its radius is 12. This circle is an example of a line of equilibrium, for the resultant force vanishes at every point of this line.

If we suppose the sphere whose centre is $A$ to be a conductor with a charge of 3 units of positive electricity, placed under the influence of 20 units of positive electricity at $C$, the state of the case will be represented by the diagram if we leave out all the lines within the sphere $A$. The part of this spherical surface below the small circle $DD'$ will be negatively charged by the influence of $C$. All the rest of the sphere will be positively charged, and the small circle $DD'$ itself will be a line of no charge.

We may also consider the diagram to represent the sphere whose centre is $C$, charged with 8 units of positive electricity, and influenced by 15 units of positive electricity placed at $A$.

The diagram may also be taken to represent a conductor whose surface consists of the larger segments of the two spheres meeting in $DD'$, charged with 23 units of positive electricity.

We shall return to the consideration of this diagram as an illustration of Thomson's *Theory of Electrical Images*. See Art. 168.

122.] These diagrams should be studied as illustrations of the language of Faraday in speaking of 'lines of force,' the 'forces of an electrified body,' &c.

The word Force denotes a restricted aspect of that action between two material bodies by which their motions are rendered different from what they would have been in the absence of that action. The whole phenomenon, when both bodies are contemplated at once, is called Stress, and may be described as a transference of momentum from one body to the other. When we restrict our attention to the first of the two bodies, we call the stress acting on it the Moving Force, or simply the Force on that body, and it is measured by the momentum which that body is receiving per unit of time.

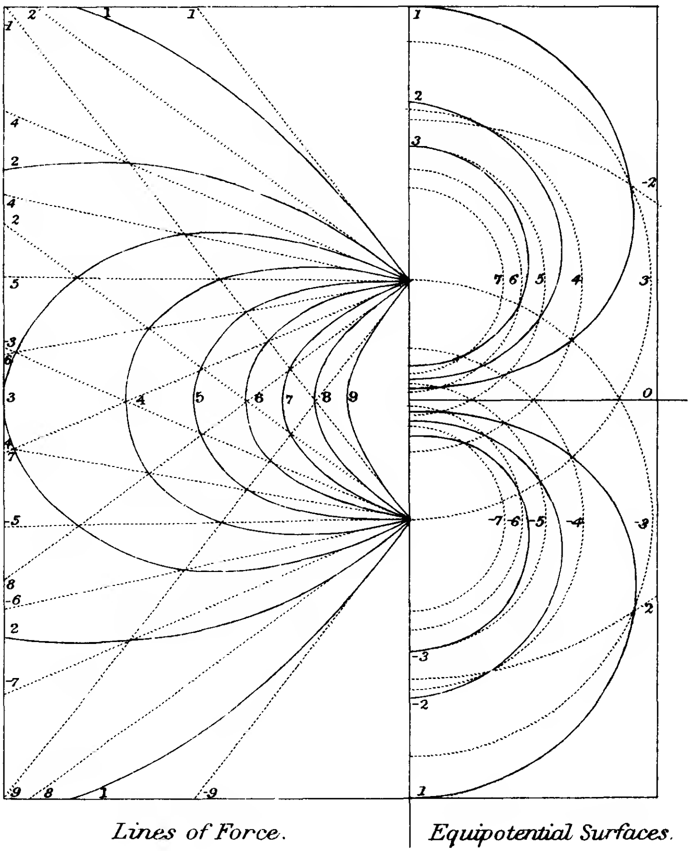
The mechanical action between two charged bodies is a stress, and that on one of them is a force. The force on a small charged body is proportional to its own charge, and the force per unit of charge is called the Intensity of the force.

The word Induction was employed by Faraday to denote the mode in which the charges of electrified bodies are related to each other, every unit of positive charge being connected with a unit of negative charge by a line, the direction of which, in fluid dielectrics, coincides at every part of its course with that of the electric intensity. Such a line is often called a line of Force, but it is more correct to call it a line of Induction.

Now the quantity of electricity in a body is measured, according to Faraday's ideas, by the *number* of lines of force, or rather of induction, which proceed from it. These lines of force must all terminate somewhere, either on bodies in the neighbourhood, or on the walls and roof of the room, or on the earth, or on the heavenly bodies, and wherever they terminate there is a quantity of electricity exactly equal and opposite to that on the part of the body from which they proceeded. By examining the diagrams this will be seen to be the case. There is therefore no contradiction between Faraday's views and the mathematical results of the old theory, but, on the contrary, the idea of lines of force throws great light on these results, and seems to afford the means of rising by a continuous process from the somewhat rigid conceptions of the old theory to notions which may be capable of greater expansion, so as to provide room for the increase of our knowledge by further researches.

To face P. 183.

FIG. 6.




*Method of drawing*

*Lines of Force and Equipotential Surfaces.*

123.] These diagrams are constructed in the following manner:—

First, take the case of a single centre of force, a small electrified body with a charge $e$. The potential at a distance $r$ is $V = e/r$; hence, if we make $r = e/V$, we shall find $r$, the radius of the sphere for which the potential is $V$. If we now give to $V$ the values 1, 2, 3, &c., and draw the corresponding spheres, we shall obtain a series of equipotential surfaces, the potentials corresponding to which are measured by the natural numbers. The sections of these spheres by a plane passing through their common centre will be circles, each of which we may mark with the number denoting its potential. These are indicated by the dotted semicircles on the right hand of Fig. 6.

If there be another centre of force, we may in the same way draw the equipotential surfaces belonging to it, and if we now wish to find the form of the equipotential surfaces due to both centres together, we must remember that if $V_1$ be the potential due to one centre, and $V_2$ that due to the other, the potential due to both will be $V_1 + V_2 = V$. Hence, since at every intersection of the equipotential surfaces belonging to the two series we know both $V_1$ and $V_2$, we also know the value of $V$. If therefore we draw a surface which passes through all those intersections for which the value of $V$ is the same, this surface will coincide with a true equipotential surface at all these intersections, and if the original systems of surfaces are drawn sufficiently close, the new surface may be drawn with any required degree of accuracy. The equipotential surfaces due to two points whose charges are equal and opposite are represented by the continuous lines on the right hand side of Fig. 6.

This method may be applied to the drawing of any system of equipotential surfaces when the potential is the sum of two potentials, for which we have already drawn the equipotential surfaces.

The lines of force due to a single centre of force are straight lines radiating from that centre. If we wish to indicate by these lines the intensity as well as the direction of the force at any point, we must draw them so that they mark out on the equipotential surfaces portions over which the surface-integral of induction has definite values. The best way of doing this is to suppose our plane figure to be the section of a figure in space formed by the revolution of the plane figure about an axis passing through the centre of force. Any straight line radiating from the centre and making an angle $\theta$ with the axis will then trace out a cone, and the surface-integral of the induction through that part of any surface which is cut off by this cone on the side next the positive direction of the axis is $2\pi e\,(1-\cos\theta)$.

If we further suppose this surface to be bounded by its intersection with two planes passing through the axis, and inclined at the angle whose arc is equal to half the radius, then the induction through the surface so bounded is

$$\tfrac{1}{2}\,e\,(1-\cos\theta) = \Phi, \text{ say};$$

$$\text{and} \quad \theta = \cos^{-1}\left(1-2\frac{\Phi}{e}\right).$$

If we now give to $\Phi$ a series of values 1, 2, 3...$e$, we shall find a corresponding series of values of $\theta$, and if $e$ be an integer, the number of corresponding lines of force, including the axis, will be equal to $e$.

We have thus a method of drawing lines of force so that the charge of any centre is indicated by the number of lines which diverge from it, and the induction through any surface cut off in the way described is measured by the number of lines of force which pass through it. The dotted straight lines on the left-hand side of Fig. 6 represent the lines of force due to each of two electrified points whose charges are 10 and $-10$ respectively.

If there are two centres of force on the axis of the figure we may draw the lines of force for each axis corresponding to values of $\Phi_1$ and $\Phi_2$, and then, by drawing lines through the consecutive intersections of these lines for which the value of $\Phi_1+\Phi_2$ is the same, we may find the lines of force due to both centres, and in the same way we may combine any two systems of lines of force which are symmetrically situated about the same axis. The continuous curves on the left-hand side of Fig. 6 represent the lines of force due to the two charged points acting at once.

After the equipotential surfaces and lines of force have been constructed by this method, the accuracy of the drawing may be tested by observing whether the two systems of lines are everywhere orthogonal, and whether the distance between consecutive equipotential surfaces is to the distance between consecutive lines of force as half the mean distance from the axis is to the assumed unit of length.

In the case of any such system of finite dimensions the line of force whose index number of $\Phi$ has an asymptote which passes through the electric centre (Art. 89 $d$) of the system, and is inclined to the axis at an angle whose cosine is $1-2\Phi/e$, where $e$ is the total electrification of the system, provided $\Phi$ is less than $e$. Lines of force whose index is greater than $e$ are finite lines. If $e$ is zero, they are all finite.

The lines of force corresponding to a field of uniform force parallel to the axis are lines parallel to the axis, the distances from the axis being the square roots of an arithmetical series.

The theory of equipotential surfaces and lines of force in two dimensions will be given when we come to the theory of conjugate functions *.

* See a paper 'On the Flow of Electricity in Conducting Surfaces,' by Prof. W. R. Smith, *Proc. R. S. Edin.*, 1869-70, p. 79.

# CHAPTER VIII.

## SIMPLE CASES OF ELECTRIFICATION.

### *Two Parallel Planes.*

124.] WE shall consider, in the first place, two parallel plane conducting surfaces of infinite extent, at a distance $c$ from each other, maintained respectively at potentials $A$ and $B$.

It is manifest that in this case the potential $V$ will be a function of the distance $z$ from the plane $A$, and will be the same for all points of any parallel plane between $A$ and $B$, except near the boundaries of the electrified surfaces, which by the supposition are at an infinitely great distance from the point considered.

Hence, Laplace's equation becomes reduced to

$$\frac{d^2 V}{dz^2} = 0,$$

the integral of which is

$$V = C_1 + C_2 z;$$

and since when $z = 0$, $V = A$, and when $z = c$, $V = B$,

$$V = A + (B - A)\frac{z}{c}.$$

For all points between the planes, the resultant intensity is normal to the planes, and its magnitude is

$$R = \frac{A - B}{c}.$$

In the substance of the conductors themselves, $R = 0$. Hence the distribution of electricity on the first plane has a surface-density $\sigma$, where

$$4\pi\sigma = R = \frac{A - B}{c}.$$

On the other surface, where the potential is $B$, the surface-density $\sigma'$ will be equal and opposite to $\sigma$, and

$$4\pi\sigma' = -R = \frac{B-A}{c}.$$

Let us next consider a portion of the first surface whose area is $S$, taken so that no part of $S$ is near the boundary of the surface.

The quantity of electricity on this surface is $e_1 = S\sigma$, and, by Art. 79, the force acting on every unit of electricity is $\frac{1}{2}R$, so that the whole force acting on the area $S$, and attracting it towards the other plane, is

$$F = \tfrac{1}{2}RS\sigma = \frac{1}{8\pi}R^2S = \frac{S}{8\pi}\frac{(B-A)^2}{c^2}.$$

Here the attraction is expressed in terms of the area $S$, the difference of potentials of the two surfaces $(A-B)$, and the distance between them $c$. The attraction, expressed in terms of the charge $e_1$, on the area $S$, is

$$F = \frac{2\pi}{S}e_1^2.$$

The electric energy due to the distribution of electricity on the area $S$, and that on the corresponding area $S'$ on the surface $B$ defined by projecting $S$ on the surface $B$ by a system of lines of force, which in this case are normals to the plane, is

$$\begin{aligned} W &= \tfrac{1}{2}(e_1A + e_2B), \\ &= \tfrac{1}{2}\frac{S}{4\pi}\frac{(A-B)^2}{c}, \\ &= \frac{R^2}{8\pi}Sc, \\ &= \frac{2\pi}{S}e_1^2c, \\ &= Fc. \end{aligned}$$

The first of these expressions is the general expression of electric energy (Art. 84).

The second gives the energy in terms of the area, the distance, and difference of potentials.

The third gives it in terms of the resultant force $R$, and the volume $Sc$ included between the areas $S$ and $S'$, and shews that the energy in unit of volume is $p$ where $8\pi p = R^2$.

The attraction between the planes is $pS$, or in other words, there is an electrical tension (or negative pressure) equal to $p$ on every unit of area.

The fourth expression gives the energy in terms of the charge.

The fifth shews that the electrical energy is equal to the work which would be done by the electric force if the two surfaces were to be brought together, moving parallel to themselves, with their electric charges constant.

To express the charge in terms of the difference of potentials, we have

$$e_1 = \frac{1}{4\pi}\frac{S}{c}(A-B) = q(A-B).$$

The coefficient $q$ represents the charge due to a difference of potentials equal to unity. This coefficient is called the Capacity of the surface $S$, due to its position relatively to the opposite surface.

Let us now suppose that the medium between the two surfaces is no longer air but some other dielectric substance whose specific inductive capacity is $K$, then the charge due to a given difference of potentials will be $K$ times as great as when the dielectric is air, or

$$e_1 = \frac{KS}{4\pi c}(A-B).$$

The total energy will be

$$W = \frac{KS}{8\pi c}(A-B)^2,$$

$$= \frac{2\pi}{KS}e_1^{\,2}c.$$

The force between the surfaces will be

$$F = pS = \frac{KS}{8\pi}\frac{(A-B)^2}{c^2},$$

$$= \frac{2\pi}{KS}e_1^{\,2}.$$

Hence the force between two surfaces kept at given potentials varies directly as $K$, the specific inductive capacity of the dielectric, but the force between two surfaces charged with given quantities of electricity varies inversely as $K$.

### *Two Concentric Spherical Surfaces.*

125.] Let two concentric spherical surfaces of radii $a$ and $b$, of which $b$ is the greater, be maintained at potentials $A$ and $B$ respectively, then it is manifest that the potential $V$ is a function of $r$ the distance from the centre. In this case, Laplace's equation becomes

$$\frac{d^2V}{dr^2} + \frac{2}{r}\frac{dV}{dr} = 0.$$

The solution of this is

$$V = C_1 + C_2 r^{-1};$$

and the conditions that $V = A$ when $r = a$, and $V = B$ when $r = b$, give for the space between the spherical surfaces,

$$V = \frac{Aa - Bb}{a - b} + \frac{A - B}{a^{-1} - b^{-1}} r^{-1};$$

$$R = -\frac{dV}{dr} = \frac{A - B}{a^{-1} - b^{-1}} r^{-2}.$$

If $\sigma_1$, $\sigma_2$ are the surface-densities on the opposed surfaces of a solid sphere of radius $a$, and a spherical hollow of radius $b$, then

$$\sigma_1 = \frac{1}{4\pi a^2} \frac{A - B}{a^{-1} - b^{-1}}, \qquad \sigma_2 = \frac{1}{4\pi b^2} \frac{B - A}{a^{-1} - b^{-1}}.$$

If $e_1$ and $e_2$ are the whole charges of electricity on these surfaces,

$$e_1 = 4\pi a^2 \sigma_1 = \frac{A - B}{a^{-1} - b^{-1}} = -e_2.$$

The capacity of the enclosed sphere is therefore $\dfrac{ab}{b - a}$.

If the outer surface of the shell is also spherical and of radius $c$, then, if there are no other conductors in the neighbourhood, the charge on the outer surface is

$$e_3 = Bc.$$

Hence the whole charge on the inner sphere is

$$e_1 = \frac{ab}{b - a}(A - B),$$

and that on the outer shell

$$e_2 + e_3 = \frac{ab}{b - a}(B - A) + Bc.$$

If we put $b = \infty$, we have the case of a sphere in an infinite space. The electric capacity of such a sphere is $a$, or it is numerically equal to its radius.

The electric tension on the inner sphere per unit of area is

$$p = \frac{1}{8\pi} \frac{b^2}{a^2} \frac{(A - B)^2}{(b - a)^2}.$$

The resultant of this tension over a hemisphere is $\pi a^2 p = F$ normal to the base of the hemisphere, and if this is balanced by a surface tension exerted across the circular boundary of the hemisphere, the tension on unit of length being $T$, we have

$$F = 2\pi a T.$$

Hence $$F = \frac{b^2}{8}\frac{(A-B)^2}{(b-a)^2} = \frac{e_1^{\,2}}{8a^2},$$
$$T = \frac{b^2}{16\pi a}\frac{(A-B)^2}{(b-a)^2}.$$

If a spherical soap bubble is electrified to a potential $A$, then, if its radius is $a$, the charge will be $Aa$, and the surface-density will be
$$\sigma = \frac{1}{4\pi}\frac{A}{a}.$$

The resultant intensity just outside the surface will be $4\pi\sigma$, and inside the bubble it is zero, so that by Art. 79 the electric force on unit of area of the surface will be $2\pi\sigma^2$, acting outwards. Hence the electrification will diminish the pressure of the air within the bubble by $2\pi\sigma^2$, or
$$\frac{1}{8\pi}\frac{A^2}{a^2}.$$

But it may be shewn that if $T_0$ is the tension which the liquid film exerts across a line of unit length, then the pressure from within required to keep the bubble from collapsing is $2T_0/a$. If the electric force is just sufficient to keep the bubble in equilibrium when the air within and without is at the same pressure,
$$A^2 = 16\pi a T_0.$$

### *Two Infinite Coaxal Cylindric Surfaces.*

126.] Let the radius of the outer surface of a conducting cylinder be $a$, and let the radius of an inner surface of a hollow cylinder, having the same axis with the first, be $b$. Let their potentials be $A$ and $B$ respectively. Then, since the potential $V$ is in this case a function only of $r$, the distance from the axis, Laplace's equation becomes
$$\frac{d^2V}{dr^2} + \frac{1}{r}\frac{dV}{dr} = 0,$$
whence $$V = C_1 + C_2 \log r.$$

Since $V = A$ when $r = a$, and $V = b$ when $r = b$,
$$V = \frac{A\log\frac{b}{r} + B\log\frac{r}{a}}{\log\frac{b}{a}}.$$

If $\sigma_1$, $\sigma_2$ are the surface-densities on the inner and outer surfaces,

$$4\pi\sigma_1 = \frac{A-B}{a\log\frac{b}{a}}, \qquad 4\pi\sigma_2 = \frac{B-A}{b\log\frac{b}{a}}.$$

If $e_1$ and $e_2$ are the charges on the portions of the two cylinders between two sections transverse to the axis at a distance $l$ from each other,

$$e_1 = 2\pi a l\sigma_1 = \tfrac{1}{2}\frac{A-B}{\log\frac{b}{a}}l = -e_2.$$

The capacity of a length $l$ of the interior cylinder is therefore

$$\tfrac{1}{2}\frac{l}{\log\frac{b}{a}}.$$

If the space between the cylinders is occupied by a dielectric of specific inductive capacity $K$ instead of air, then the capacity of a length $l$ of the inner cylinder is

$$\tfrac{1}{2}\frac{lK}{\log\frac{b}{a}}.$$

The energy of the electrical distribution on the part of the infinite cylinder which we have considered is

$$\tfrac{1}{4}\frac{lK(A-B)^2}{\log\frac{b}{a}}.$$

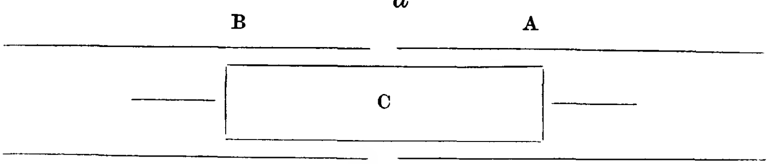



Fig. 5.

127.] Let there be two hollow cylindric conductors $A$ and $B$, Fig. 5, of indefinite length, having the axis of $x$ for their common axis, one on the positive and the other on the negative side of the origin, and separated by a short interval near the origin of coordinates.

Let a cylinder $C$ of length $2l$ be placed with its middle point at a distance $x$ on the positive side of the origin, so as to extend into both the hollow cylinders.

Let the potential of the hollow cylinder on the positive side be

$A$, that of the one on the negative side $B$, and that of the internal one $C$, and let us put $\alpha$ for the capacity per unit of length of $C$ with respect to $A$, and $\beta$ for the same quantity with respect to $B$.

The surface-densities of the parts of the cylinders at fixed points near the origin and at points at given small distances from the ends of the inner cylinder will not be affected by the value of $x$ provided a considerable length of the inner cylinder enters each of the hollow cylinders. Near the ends of the hollow cylinders, and near the ends of the inner cylinder, there will be distributions of electricity which we are not yet able to calculate, but the distribution near the origin will not be altered by the motion of the inner cylinder provided neither of its ends comes near the origin, and the distributions at the ends of the inner cylinder will move with it, so that the only effect of the motion will be to increase or diminish the length of those parts of the inner cylinder where the distribution is similar to that on an infinite cylinder.

Hence the whole energy of the system will be, so far as it depends on $x$,

$$Q = \tfrac{1}{2}\alpha(l+x)(C-A)^2 + \tfrac{1}{2}\beta(l-x)(C-B)^2 + \text{quantities independent of } x;$$

and the resultant force parallel to the axis of the cylinders since the energy is expressed in terms of the potentials will by Art. 93 *b* be

$$X = \frac{dQ}{dx} = \tfrac{1}{2}\alpha(C-A)^2 - \tfrac{1}{2}\beta(C-B)^2.$$

If the cylinders $A$ and $B$ are of equal section, $\alpha = \beta$, and

$$X = \alpha(B-A)(C - \tfrac{1}{2}(A+B)).$$

It appears, therefore, that there is a constant force acting on the inner cylinder tending to draw it into that one of the outer cylinders from which its potential differs most.

If $C$ be numerically large and $A+B$ comparatively small, then the force is approximately $X = \alpha(B-A)C$;

so that the difference of the potentials of the two cylinders can be measured if we can measure $X$, and the delicacy of the measurement will be increased by raising $C$, the potential of the inner cylinder.

This principle in a modified form is adopted in Thomson's Quadrant Electrometer, Art. 219.

The same arrangement of three cylinders may be used as a measure of capacity by connecting $B$ and $C$. If the potential of

$A$ is zero, and that of $B$ and $C$ is $V$, then the quantity of electricity on $A$ will be

$$E_3 = (q_{13} + a(l + x))\,V;$$

where $q_{13}$ is a quantity depending on the distribution of electricity on the ends of the cylinder but not upon $x$, so that by moving $C$ to the right till $x$ becomes $x + \xi$ the capacity of the cylinder $C$ becomes increased by the definite quantity $a\xi$, where

$$a = \frac{1}{2\log\frac{b}{a}},$$

$a$ and $b$ being the radii of the opposed cylindric surfaces.

# CHAPTER IX.

## SPHERICAL HARMONICS.

128.] THE mathematical theory of spherical harmonics has been made the subject of several special treatises. The *Handbuch der Kugelfunctionen* of Dr. E. Heine, which is the most elaborate work on the subject, has now (1878) reached a second edition in two volumes, and Dr. F. Neumann has published his *Beiträge zur Theorie der Kugelfunctionen* (Leipzig, Teubner, 1878). The treatment of the subject in Thomson and Tait's *Natural Philosophy* is considerably improved in the second edition (1879), and Mr. Todhunter's *Elementary Treatise on Laplace's Functions, Lamé's Functions, and Bessel's Functions*, together with Mr. Ferrers' *Elementary Treatise on Spherical Harmonics and subjects connected with them*, have rendered it unnecessary to devote much space in a book on electricity to the purely mathematical development of the subject.

I have retained however the specification of a spherical harmonic in terms of its poles.

### *On Singular Points at which the Potential becomes Infinite.*

129 *a*.] If a charge, $A_0$, of electricity is uniformly spread over the surface of a sphere the coordinates of whose centre are $(a, b, c)$, the potential at any point $(x, y, z)$ outside the sphere is, by Art. 125,

$$V = \frac{A_0}{r}, \tag{1}$$

where

$$r^2 = (x-a)^2 + (y-b)^2 + (z-c)^2. \tag{2}$$

As the expression for $V$ is independent of the radius of the sphere, the form of the expression will be the same if we suppose the radius infinitely small. The physical interpretation of the expression would be that the charge $A_0$ is placed on the surface of an infinitely small sphere, which is sensibly the same as a mathematical point. We have already (Arts. 55, 81) shewn that there is a limit to the surface-density of electricity, so that it is physically impossible to place a finite charge of electricity on a sphere of less than a certain radius.

Nevertheless, as the equation (1) represents a possible distribution of potential in the space surrounding a sphere, we may for mathematical purposes treat it as if it arose from a charge $A_0$ condensed at the mathematical point $(a, b, c)$, and we may call the point a singular point of order zero.

There are other kinds of singular points, the properties of which we shall presently investigate, but before doing so we must define certain expressions which we shall find useful in dealing with directions in space, and with the points on a sphere which correspond to them.

129 *b*.] An *axis* is any definite direction in space. We may suppose it defined by a mark made on the surface of a sphere at the point where the radius drawn *from* the centre in the direction of the axis meets the surface. This point is called the Pole of the axis. An axis has therefore one pole only, not two.

If $\mu$ is the cosine of the angle between the axis $h$ and any vector $r$, and if

$$p = \mu r, \tag{3}$$

$p$ is the resolved part of $r$ in the direction of the axis $h$.

Different axes are distinguished by different suffixes, and the cosine of the angle between two axes is denoted by $\lambda_{mn}$, where $m$ and $n$ are the suffixes specifying the axis.

Differentiation with respect to an axis, $h$, whose direction cosines are $L, M, N$, is denoted by

$$\frac{d}{dh} = L\frac{d}{dx} + M\frac{d}{dy} + N\frac{d}{dz}. \tag{4}$$

From these definitions it is evident that

$$\frac{dr}{dh_m} = \frac{p_m}{r} = \mu_m, \tag{5}$$

$$\frac{dp_n}{dh_m} = \lambda_{mn} = \frac{dp_m}{dh_n}, \tag{6}$$

$$\frac{d\mu_m}{dh_n} = \frac{\lambda_{mn} - \mu_m\mu_n}{r}. \tag{7}$$

If we now suppose that the potential at the point $(x, y, z)$ due to a singular point of any order placed at the origin is

$$Af(x, y, z),$$

then if such a point be placed at the extremity of the axis $h$, the potential at $(x, y, z)$ will be

$$Af[(x-Lh),\ (y-Mh),\ (z-Nh)],$$

and if a point in all respects the same, except that the sign of $A$ is reversed, be placed at the origin, the potential due to the pair of points will be

$$V = Af[(x-Lh),\ (y-Mh),\ (z-Nh)] - Af(x, y, z),$$
$$= -Ah\frac{d}{dh}f(x, y, z) + \text{terms containing } h^2.$$

If we now diminish $h$ and increase $A$ without limit, their product continuing finite and equal to $A'$, the ultimate value of the potential of the pair of points will be

$$V' = -A'\frac{d}{dh}f(x, y, z). \tag{8}$$

If $f(x, y, z)$ satisfies Laplace's equation, then, since this equation is linear, $V'$, which is the difference of two functions, each of which separately satisfies the equation, must itself satisfy it.

129 *c*.] Now the potential due to a singular point of order zero,

$$V_0 = A_0\frac{1}{r}, \tag{9}$$

satisfies Laplace's equation, therefore every function formed from this by differentiation with respect to any number of axes in succession must also satisfy that equation.

A point of the first order may be formed by taking two points of order zero, having equal and opposite charges $-A_0$ and $A_0$, and placing the first at the origin and the second at the extremity of the axis $h_1$. The value of $h_1$ is then diminished and that of $A_0$ increased indefinitely, but so that the product $A_0 h_1$ is always equal to $A_1$. The ultimate result of this process, when the two points coincide, is a point of the first order whose moment is $A_1$ and whose axis is $h_1$. A point of the first order is therefore a double point. Its potential is

$$V_1 = -h_1\frac{d}{dh_1}V_0$$
$$= A_1\frac{\mu_1}{r^2}. \tag{10}$$

By placing a point of the first order at the origin, whose moment is $-A_1$, and another at the extremity of the axis $h_2$

whose moment is $A_1$, and then diminishing $h_2$ and increasing $A_1$, so that

$$A_1 h_2 = \tfrac{1}{2} A_2, \tag{11}$$

we obtain a point of the second order, whose potential is

$$\begin{aligned} V_2 &= -h_2 \frac{d}{dh_2} V_1 \\ &= A_2 \tfrac{1}{2} \frac{3\mu_1\mu_2 - \lambda_{12}}{r^3}. \end{aligned} \tag{12}$$

We may call a point of the second order a quadruple point because it is constructed by making four points of order zero approach each other. It has two axes $h_1$ and $h_2$ and a moment $A_2$. The directions of these axes and the magnitude of the moment completely define the nature of the point.

By differentiating with respect to $n$ axes in succession we obtain the potential due to a point of the $n^{\text{th}}$ order. It will be the product of three factors, a constant, a certain combination of cosines, and $r^{-(n+1)}$. It is convenient, for reasons which will appear as we go on, to make the numerical value of the constant such that when all the axes coincide with the vector, the coefficient of the moment is $r^{-(n+1)}$. We therefore divide by $n$ when we differentiate with respect to $h_n$.

In this way we obtain a definite numerical value for a particular potential, to which we restrict the name of The Solid Harmonic of degree $-(n+1)$, namely

$$V_n = (-1)^n \frac{1}{1.2.3\ldots n} \frac{d}{dh_1} \cdot \frac{d}{dh_2} \cdots \frac{d}{dh_n} \cdot \frac{1}{r}. \tag{13}$$

If this quantity is multiplied by a constant it is still the potential due to a certain point of the $n^{\text{th}}$ order.

129 d.] The result of the operation (13) is of the form

$$V_n = Y_n r^{-(n+1)}, \tag{14}$$

where $Y_n$ is a function of the $n$ cosines $\mu_1 \ldots \mu_n$ of the angles between $r$ and the $n$ axes, and of the $\frac{1}{2} n(n-1)$ cosines $\lambda_{12}$, &c. of the angles between pairs of the axes.

If we consider the directions of $r$ and the $n$ axes as determined by points on a spherical surface, we may regard $Y_n$ as a quantity varying from point to point on that surface, being a function of the $\frac{1}{2} n(n+1)$ distances between the $n$ poles of the axes and the pole of the vector. We therefore call $Y_n$ The Surface Harmonic of order $n$.

130 a.] We have next to shew that to every surface-harmonic

of order $n$ there corresponds not only a solid harmonic of degree $-(n+1)$ but another of degree $n$, or that

$$H_n = Y_n r^n = V_n r^{2n+1} \tag{15}$$

satisfies Laplace's equation.

For
$$\frac{dH_n}{dx} = (2n+1)\, r^{2n-1} x V_n + r^{2n+1} \frac{dV_n}{dx},$$

$$\frac{d^2H_n}{dx^2} = (2n+1)\left[(2n-1)x^2 + r^2\right] r^{2n-3} V_n + 2(2n+1) r^{2n-1} x \frac{dV_n}{dx} + r^{2n+1} \frac{d^2V_n}{dx^2}.$$

Hence

$$\frac{d^2H_n}{dx} + \frac{d^2H_n}{dy^2} + \frac{d^2H_n}{dz^2} = (2n+1)(2n+2) r^{2n-1} V_n + 2(2n+1) r^{2n-1}\left(x\frac{dV_n}{dx} + y\frac{dV_n}{dy} + z\frac{dV_n}{dz}\right) + r^{2n+1}\left(\frac{d^2V_n}{dx^2} + \frac{d^2V_n}{dy^2} + \frac{d^2V_n}{dz^2}\right). \tag{16}$$

Now, since $V_n$ is a homogeneous function of $x$, $y$, and $z$, of negative degree $n+1$,

$$x\frac{dV_n}{dx} + y\frac{dV_n}{dy} + z\frac{dV_n}{dz} = -(n+1)V_n. \tag{17}$$

The first two terms therefore of the right-hand member of equation (16) destroy each other, and, since $V_n$ satisfies Laplace's equation, the third term is zero, so that $H_n$ also satisfies Laplace's equation, and is therefore a solid harmonic of degree $n$.

This is a particular case of the more general theorem of electrical inversion, which asserts that if $F(x, y, z)$ is a function of $x$, $y$, and $z$ which satisfies Laplace's equation, then there exists another function,

$$\frac{a}{r} F\left(\frac{a^2x}{r^2}, \frac{a^2y}{r^2}, \frac{a^2z}{r^2}\right),$$

which also satisfies Laplace's equation. See Art. 162.

130 *b*.] The surface harmonic $Y_n$ contains $2n$ arbitrary variables, for it is defined by the positions of its $n$ poles on the sphere, and each of these is defined by two coordinates.

Hence the solid harmonics $V_n$ and $H_n$ also contain $2n$ arbitrary variables. Each of these quantities, however, when multiplied by a constant, will satisfy Laplace's equation.

To prove that $AH_n$ is the most general rational homogeneous function of degree $n$ which can satisfy Laplace's equation, we observe that $K$, the general rational homogeneous function of degree $n$, contains $\frac{1}{2}(n+1)(n+2)$ terms. But $\nabla^2 K$ is a homogeneous function of degree $n-2$, and therefore contains $\frac{1}{2}n(n-1)$ terms, and the condition $\nabla^2 K = 0$ requires that each of these must vanish. There are therefore $\frac{1}{2}n(n-1)$ equations between the coefficients of the $\frac{1}{2}(n+1)(n+2)$ terms of the function $K$, leaving $2n+1$ independent constants in the most general form of the homogeneous function of degree $n$ which satisfies Laplace's equation. But $H_n$, when multiplied by an arbitrary constant, satisfies the required conditions, and has $2n+1$ arbitrary constants. It is therefore of the most general form.

131 *a*.] We are now able to form a distribution of potential such that neither the potential itself nor its first derivatives become infinite at any point.

The function $V_n = Y_n r^{-(n+1)}$ satisfies the condition of vanishing at infinity, but becomes infinite at the origin.

The function $H_n = Y_n r^n$ is finite and continuous at finite distances from the origin, but does not vanish at an infinite distance.

But if we make $a^n Y_n r^{-(n+1)}$ the potential at all points outside a sphere whose centre is the origin, and whose radius is $a$, and $a^{-(n+1)} Y_n r^n$ the potential at all points within the sphere, and if on the sphere itself we suppose electricity spread with a surface density $\sigma$ such that

$$4\pi\sigma a^2 = (2n+1)Y_n, \tag{18}$$

then all the conditions will be satisfied for the potential due to a shell charged in this manner.

For the potential is everywhere finite and continuous, and vanishes at an infinite distance; its first derivatives are everywhere finite and are continuous except at the charged surface, where they satisfy the equation

$$\frac{dV}{d\nu} + \frac{dV'}{d\nu'} + 4\pi\sigma = 0, \tag{19}$$

and Laplace's equation is satisfied at all points both inside and outside of the sphere.

This, therefore, is a distribution of potential which satisfies the conditions, and by Art. 100 *c* it is the only distribution which can satisfy them.

131 *b*.] The potential due to a sphere of radius $a$ whose surface-density is given by the equation

$$4\pi a^2\sigma = (2n+1)Y_n, \tag{20}$$

is, at all points external to the sphere, identical with that due to the corresponding singular point of order $n$.

Let us now suppose that there is an electrical system which we may call $E$, external to the sphere, and that $\Psi$ is the potential due to this system, and let us find the value of $\Sigma(\Psi e)$ for the singular point. This is the part of the electric energy depending on the action of the external system on the singular point.

If $A_0$ is the charge of a singular point of order zero, then the potential energy in question is

$$W_0 = A_0 \Psi. \tag{21}$$

If there are two such points, a negative one at the origin and a positive one of equal numerical value at the extremity of the axis $h_1$, then the potential energy will be

$$-A_0\Psi + A_0\left(\Psi + h_1\frac{d\Psi}{dh_1} + \tfrac{1}{2}h_1{}^2\frac{d^2\Psi}{dh_1{}^2} + \&c.\right),$$

and when $A_0$ increases and $h_1$ diminishes indefinitely, but so that $A_0 h_1 = A_1$, the value of the potential energy for a point of the first order will be

$$W_1 = A_1\frac{d\Psi}{dh_1}. \tag{22}$$

Similarly for a point of order $n$ the potential energy will be

$$W_n = \frac{1}{1\,.\,2\ldots n}A_n\frac{d^n\Psi}{dh_1\ldots dh_n}\text{*}. \tag{23}$$

131 *c*.] If we suppose the charge of the external system to be made up of parts, any one of which is denoted by $dE$, and that of the singular point of order $n$ to be made up of parts any one of which is $de$, then

$$\Psi = \Sigma\left(\frac{1}{r}dE\right). \tag{24}$$

But if $V_n$ is the potential due to the singular point,

$$V_n = \Sigma\left(\frac{1}{r}de\right), \tag{25}$$

and the potential energy due to the action of $E$ on $e$ is

$$W_n = \Sigma(\Psi de) = \Sigma\Sigma\left(\frac{1}{r}dEde\right) = \Sigma(V_n dE), \tag{26}$$

the last expression being the potential energy due to the action of $e$ on $E$.

* We shall find it convenient, in what follows, to denote the product of the positive integral numbers $1\,.\,2\,.\,3\ldots n$ by $n\,!$

Similarly, if $\sigma ds$ is the charge on an element $ds$ of the shell, since the potential due to the shell at the external system $E$ is $V_n$, we have

$$W_n = \Sigma\,(V_n dE) = \Sigma\Sigma\left(\frac{1}{r} dE\,\sigma ds\right) = \Sigma\,(\Psi\sigma ds). \tag{27}$$

The last term contains a summation to be extended over the surface of the sphere. Equating it to the first expression for $W_n$, we have

$$\iint \Psi\sigma ds = \Sigma(\Psi de)$$

$$= \frac{1}{n!} A_n \frac{d^n\Psi}{dh_1 \ldots dh_n}. \tag{28}$$

If we remember that $4\pi\sigma a^2 = (2n+1)Y_n$, and that $A_n = a^n$, this becomes

$$\iint \Psi Y_n\,ds = \frac{4\pi}{n!\,(2n+1)} a^{n+2} \frac{d^n\Psi}{dh_1 \ldots dh_n}. \tag{29}$$

This equation reduces the operation of taking the surface integral of $\Psi Y_n ds$ over every element of the surface of a sphere of radius $a$, to that of differentiating $\Psi$ with respect to the $n$ axes of the harmonic and taking the value of the differential coefficient at the centre of the sphere, provided that $\Psi$ satisfies Laplace's equation at all points within the sphere, and $Y_n$ is a surface harmonic of order $n$.

132.] Let us now suppose that $\Psi$ is a solid harmonic of positive degree $m$ of the form

$$\Psi = a^{-m} Y_m r^m. \tag{30}$$

At the spherical surface, $r = a$, and $\Psi = Y_m$, so that equation (29) becomes in this case

$$\iint Y_m Y_n\,ds = \frac{4\pi}{n!\,(2n+1)} a^{n-m+2} \frac{d^n(Y_m r^m)}{dh_1 \ldots dh_n}, \tag{31}$$

where the value of the differential coefficient is to be taken at the centre of the sphere.

When $n$ is less than $m$, the result of the differentiation is a homogeneous function of $x$, $y$, and $z$ of degree $m-n$, the value of which at the centre of the sphere is zero. If $n$ is equal to $m$ the result of the differentiation is a constant, the value of which we shall determine in Art. 134. If the differentiation is carried further, the result is zero. Hence the surface-integral $\iint Y_m Y_n ds$ vanishes whenever $m$ and $n$ are different.

The steps by which we have arrived at this result are all of them purely mathematical, for though we have made use of terms having a physical meaning, such as electrical energy, each of these terms is regarded not as a physical phenomenon to be investigated, but as a definite mathematical expression. A mathematician has as much right to make use of these as of any other mathematical functions which he may find useful, and a physicist, when he has to follow a mathematical calculation, will understand it all the better if each of the steps of the calculation admits of a physical interpretation.

133.] We shall now determine the form of the surface harmonic $Y_n$ as a function of the position of a point $P$ on the sphere with respect to the $n$ poles of the harmonic.

We have

$$\left.\begin{aligned} Y_0 = 1, \qquad Y_1 = \mu_1, \qquad Y_2 = \tfrac{3}{2}\mu_1\mu_2 - \tfrac{1}{2}\lambda_{12}, \\ Y_3 = \tfrac{5}{2}\mu_1\mu_2\mu_3 - \tfrac{1}{2}(\mu_1\lambda_{23} + \mu_2\lambda_{31} + \mu_3\lambda_{12}), \end{aligned}\right\} \tag{32}$$

and so on.

Every term of $Y_n$ therefore consists of products of cosines, those of the form $\mu$, with a single suffix, being cosines of the angles between $P$ and the different poles, and those of the form $\lambda$, with double suffixes, being cosines of the angles between the poles.

Since each axis is introduced by one of the $n$ differentiations, the symbol of that axis must occur once and only once among the suffixes of the cosines of each term.

Hence if in any term there are $s$ cosines with double suffixes, there must be $n-2s$ cosines with single suffixes.

Let the sum of all products of cosines in which $s$ of them have double suffixes be written in the abbreviated form

$$\Sigma\,(\mu^{n-2s}\lambda^s).$$

In every one of the products all the suffixes occur once, and none is repeated.

If we wish to express that a particular suffix, $m$, occurs among the $\mu$'s only or among the $\lambda$'s only, we write it as a suffix to the $\mu$ or the $\lambda$. Thus the equation

$$\Sigma\,(\mu^{n-2s}\lambda^s) = \Sigma\,(\mu_m^{n-2s}\lambda^s) + \Sigma\,(\mu^{n-2s}\lambda_m^s) \tag{33}$$

expresses that the whole set of products may be divided into two parts, in one of which the suffix $m$ occurs among the direction cosines of the variable point $P$, and in the other among the cosines of the angles between the poles.

Let us now assume that for a particular value of $n$

$$Y_n = A_{n.0}\Sigma(\mu^n) + A_{n.1}\Sigma(\mu^{n-2}\lambda^1) + \&c. + A_{n.s}\Sigma(\mu^{n-2s}\lambda^s) + \&c., \tag{34}$$

where the $A$'s are numerical coefficients. We may write the series in the abbreviated form

$$Y_n = S\left[A_{n.s}\Sigma(\mu^{n-2s}\lambda^s)\right], \tag{35}$$

where $S$ indicates a summation in which all values of $s$, including zero, not greater than $\frac{1}{2}n$, are to be taken.

To obtain the corresponding solid harmonic of negative degree $(n+1)$ and order $n$, we multiply by $r^{-(n+1)}$, and obtain

$$V_n = S\left[A_{n.s}r^{2s-2n-1}\Sigma(p^{n-2s}\lambda^s)\right], \tag{36}$$

putting $r\mu = p$, as in equation (3).

If we differentiate $V_n$ with respect to a new axis $h_m$ we obtain $-(n+1)V_{n+1}$, and therefore

$$(n+1)V_{n+1} = S\left[A_{n.s}(2n+1-2s)r^{2s-2n-3}\Sigma(p_m^{n-2s+1}\lambda^s) - A_{n.s}r^{2s-2n-1}\Sigma(p^{n-2s-1}\lambda_m^{s+1})\right]. \tag{37}$$

If we wish to obtain the terms containing $s$ cosines with double suffixes, we must diminish $s$ by unity in the last term, and we find

$$(n+1)V_{n+1} = S\left[r^{2s-2n-3}\left\{A_{n.s}(2n-2s+1)\Sigma(p_m^{n-2s+1}\lambda^s) - A_{n.s-1}\Sigma(p^{n-2s+1}\lambda_m^s)\right\}\right]. \tag{38}$$

Now the two classes of products are not distinguished from each other in any way except that the suffix $m$ occurs among the $p$'s in one and among the $\lambda$'s in the other. Hence their coefficients must be the same, and since we ought to be able to obtain the same result by putting $n+1$ for $n$ in the expression for $V_n$ and multiplying by $n+1$, we obtain the following equations,

$$(n+1)A_{n+1.s} = (2n-2s+1)A_{n.s} = -A_{n.s-1}. \tag{39}$$

If we put $s = 0$, we obtain

$$(n+1)A_{n+1.0} = (2n+1)A_{n.0}; \tag{40}$$

and therefore, since $A_{1.0} = 1$,

$$A_{n.0} = \frac{2n!}{2^n(n!)^2}; \tag{41}$$

and from this we obtain the general value of the coefficient

$$A_{n.s} = (-1)^s\frac{(2n-2s)!}{2^{n-s}n!(n-s)}; \tag{42}$$

and finally the trigonometrical expression for the surface har-

monic, as
$$Y_n = S\left[(-1)^s \frac{(2n-2s)!}{2^{n-s}\, n!\,(n-s)!} \Sigma\left(\mu^{n-2s}\lambda^s\right)\right] * . \tag{43}$$

This expression gives the value of the surface harmonic at any point $P$ of the spherical surface in terms of the cosines of the distances of $P$ from the different poles and of the distances of the poles from each other.

It is easy to see that if any one of the poles be removed to the opposite point of the spherical surface, the value of the harmonic will have its sign reversed. For any cosine involving the index of this pole will have its sign reversed, and in each term of the harmonic the index of the pole occurs once and only once.

Hence if two or any even number of poles are removed to the points respectively opposite to them, the value of the harmonic will be unaltered.

Professor Sylvester has shewn (*Phil. Mag.*, Oct. 1876) that, when the harmonic is given, the problem of finding the $n$ lines which coincide with the axes has one and only one solution, though, as we have just seen, the directions to be reckoned positive along these axes may be reversed in pairs.

134.] We are now able to determine the value of the surface integral $\iint Y_m Y_n\, ds$ when the order of the two surface harmonics is the same, though the directions of their axes may be in general different.

For this purpose we have to form the solid harmonic $Y_m r^m$ and to differentiate it with respect to each of the $n$ axes of $Y_n$.

Any term of $Y_m r^m$ of the form $r^m \mu^{m-2s}\lambda^s$ may be written $r^{2s} p_m^{m-2s} \lambda_{mn}^s$. Differentiating this $n$ times in succession with respect to the $n$ axes of $Y_n$, we find that in differentiating $r^{2s}$

* {We may deduce from this that

$$\frac{d^p}{dx^p}\frac{d^q}{dy^q}\frac{d^r}{dz^r}\frac{1}{R} = \frac{(-1)^n . 2n!}{2^n\, n!} R^{-(2n+1)}\Big\{x^p y^q z^r - \frac{R^2}{2n-1}\left({}_pc_2 x^{p-2}y^q z^r + {}_qc_2 x^p y^{q-2} z^r + {}_rc_2 x^p y^q z^{r-2}\right)$$
$$+ \frac{1}{(2n-1)}\frac{1}{(2n-3)} R^4 \big({}_pc_4 . x^{p-4}y^q z^r + {}_qc_4 . x^p y^{q-4} z^r + {}_rc_4 . x^p y^q z^{r-4} + {}_pc_2\, {}_qc_2 x^{p-2} y^{q-2} z^r$$
$$+ {}_pc_2\, {}_rc_2 x^{p-2} y^q z^{r-2} + {}_qc_2\, {}_rc_2 x^p y^{q-2} z^{r-2}\big)$$
$$- \frac{1}{(2n-1)(2n-3)(2n-5)} R^6 \big({}_pc_6 . x^{p-6} y^q z^r + {}_qc_6 . x^p y^{q-6} z^r + {}_rc_6 . x^p y^q z^{r-6}$$
$$+ {}_pc_4\, {}_qc_2 x^{p-4} y^{q-2} z^r + {}_pc_4\, {}_rc_2 x^{p-4} y^q z^{r-2} + {}_pc_2\, {}_qc_4 . x^{p-2} y^{q-4} z^r + {}_qc_4\, {}_rc_2 x^p y^{q-4} z^{r-2}$$
$$+ {}_pc_2\, {}_rc_4 . x^{p-2} y^q z^{r-4} + {}_qc_2\, {}_rc_4 . x^p y^{q-2} z^{r-4} + {}_pc_2\, {}_qc_2\, {}_rc_2\, x^{p-2} y^{q-2} z^{r-2}\big) + \dots\Big\},$$

where $n = p+q+r$ and $R^2 = x^2+y^2+z^2$, and ${}_mc_n$ denotes the number of permutations of $m$ things $n$ at a time divided by $2^{\frac{n}{2}}\left(\frac{n}{2}\right)!$.}

with respect to $s$ of these axes we introduce $s$ of the $p_n$'s, and the numerical factor

$$2s(2s-2)\ldots 2, \text{ or } 2^s s!.$$

In continuing the differentiation with respect to the next $s$ axes, the $p_n$'s become converted into $\lambda_{nn}$'s, but no numerical factor is introduced, and in differentiating with respect to the remaining $n-2s$ axes, the $p_m$'s become converted into $\lambda_{mn}$'s, so that the result is $2^s s!\,\lambda_{nn}^s \lambda_{mm}^s \lambda_{mn}^{m-2s}$.

We have therefore, by equation (31),

$$\iint Y_m Y_n \, ds = \frac{4\pi}{n!\,(2n+1)} a^{n-m+2} \frac{d^n (Y_m r^m)}{dh_1 \ldots dh_n}, \tag{44}$$

and by equation (43),

$$Y_m r^m = S\left[(-1)^s \frac{(2m-2s)!}{2^{m-s} m!\,(m-s)!} \Sigma\left(r^{2s} p_m^{m-2s} \lambda_{mm}^s\right)\right]. \tag{45}$$

Hence, performing the differentiations and remembering that $m=n$, we find

$$\iint Y_m Y_n \, ds = \frac{4\pi a^2}{(2n+1)\,(n!)^2} S\left[(-1)^s \frac{(2n-2s)!\,s!}{2^{n-2s}(n-s)!} \Sigma\left(\lambda_{mm}^s \lambda_{nn}^s \lambda_{mn}^{n-2s}\right)\right]. \tag{46}$$

135 $a$.] The expression (46) for the surface-integral of the product of two surface-harmonics assumes a remarkable form if we suppose all the axes of one of the harmonics, $Y_m$, to coincide with each other, so that $Y_m$ becomes what we shall afterwards define as the zonal harmonic of order $m$, denoted by the symbol $P_m$.

In this case all the cosines of the form $\lambda_{nm}$ may be written $\mu_n$, where $\mu_n$ denotes the cosine of the angle between the common axis of $P_m$ and one of the axes of $Y_n$. The cosines of the form $\lambda_{mm}$ will all become equal to unity, so that for $\Sigma\lambda_{mm}^s$ we must put the number of combinations of $s$ symbols, each of which is distinguished by two suffixes out of $n$, no suffix being repeated. Hence

$$\Sigma \lambda_{mm}^s = \frac{n!}{2^s s!\,(n-2s)!} *. \tag{47}$$

* {We can see this if we consider how many permutations of the suffixes of one term in the expression $\Sigma\lambda_{mm}^s$ we can form. The suffixes consist of $s$ groups of two numbers each, by altering the order of the groups we can form $s!$ arrangements, and by interchanging the order of the numbers inside the groups we can form from any one of these arrangements $2^s$ other arrangements, so that from each of the groups of suffixes we can get $2^s s!$ arrangements; thus, if $N$ be the number of terms in the series $\Sigma\lambda_{mm}^s$, $N 2^s s!$ arrangements of the $n$ numbers taken $2s$ at a time may be made, but the whole number of arrangements thus made is evidently the number of permutations of $n$ things taken $2s$ at a time, or $\frac{n!}{(n-2s)!}$; thus $N 2^s s! = \frac{n!}{(n-2s)!}$, or $N = \frac{n!}{2^s s!\,(n-2s)!}$}.

The number of permutations of the remaining $n-2s$ indices of the axes of $P_m$ is $(n-2s)!$ Hence

$$\Sigma(\lambda_{mn}^{n-2s}) = (n-2s)!\,\mu^{n-2s}. \tag{48}$$

Equation (46) therefore becomes, when all the axes of $Y_m$ coincide with each other,

$$\iint Y_n P_m\, ds = \frac{4\pi a^2}{(2n+1)\,n!} S\left[(-1)^s \frac{(2n-2s)!}{2^{n-s}(n-s)!}\Sigma(\mu^{n-2s}\lambda^s)\right] \tag{49}$$

$$= \frac{4\pi a^2}{2n+1} Y_{n(m)}, \text{ by equation (43)}, \tag{50}$$

where $Y_{n(m)}$ denotes the value of $Y_n$ at the pole of $P_m$.

We may obtain the same result by the following shorter process:—

Let a system of rectangular coordinates be taken so that the axis of $z$ coincides with the axis of $P_m$, and let $Y_n r^n$ be expanded as a homogeneous function of $x$, $y$, $z$ of degree $n$.

At the pole of $P_m$, $x = y = 0$ and $z = r$, so that if $Cz^n$ is the term not involving $x$ or $y$, $C$ is the value of $Y_n$ at the pole of $P_m$.

Equation (31) becomes in this case

$$\iint Y_n P_m\, ds = \frac{4\pi a^2}{2n+1}\frac{1}{n!}\frac{d^m}{dz^m}(Y_n r^n).$$

As $m$ is equal to $n$, the result of differentiating $Cz^n$ is $n!\,C$, and is zero for the other terms. Hence

$$\iint Y_n P_m\, ds = \frac{4\pi a^2}{2n+1} C,$$

$C$ being the value of $Y_n$ at the pole of $P_m$.

135 *b*.] This result is a very important one in the theory of spherical harmonics, as it shews how to determine a series of spherical harmonics which expresses the value of a quantity having any arbitrarily assigned finite and continuous value at each point of a spherical surface.

For let $F$ be the value of the quantity and $ds$ the element of surface at a point $Q$ of the spherical surface, then if we multiply $F ds$ by $P_n$, the zonal harmonic whose pole is the point $P$ of the same surface, and integrate over the surface, the result, since it depends on the position of the point $P$, may be considered as a function of the position of $P$.

But since the value at $P$ of the zonal harmonic whose pole is $Q$ is equal to the value at $Q$ of the zonal harmonic of the same order whose pole is $P$, we may suppose that for every element

$ds$ of the surface a zonal harmonic is constructed having its pole at $Q$ and having a coefficient $Fds$.

We shall thus have a system of zonal harmonics superposed on each other with their poles at every point of the sphere where $F$ has a value. Since each of these is a multiple of a surface harmonic of order $n$, their sum is a multiple of a surface harmonic (not necessarily zonal) of order $n$.

The surface integral $\iint FP_n ds$ considered as a function of the point $P$ is therefore a multiple of a surface harmonic $Y_n$; so that

$$\frac{2n+1}{4\pi a^2}\iint FP_n ds$$

is also that particular surface harmonic of the $n^{\text{th}}$ order which belongs to the series of harmonics which expresses $F$, provided $F$ can be so expressed.

For if $F$ can be expressed in the form

$$F = A_0 Y_0 + A_1 Y_1 + \&c. + A_n Y_n + \&c.,$$

then if we multiply by $P_n ds$ and take the surface integral over the whole sphere, all terms involving products of harmonics of different orders will vanish, leaving

$$\iint FP_n ds = \frac{4\pi a^2}{2n+1} A_n Y_n.$$

Hence the only possible expansion of $F$ in spherical harmonics is

$$F = \frac{1}{4\pi a^2}\left[\iint FP_0 ds + \&c. + (2n+1)\iint FP_n ds + \&c.\right]. \tag{51}$$

## *Conjugate Harmonics.*

136.] We have seen that the surface integral of the product of two harmonics of different orders is always zero. But even when the two harmonics are of the same order, the surface integral of their product may be zero. The two harmonics are then said to be conjugate to each other. The condition of two harmonics of the same order being conjugate to each other is expressed in terms of equation (46) by making its members equal to zero.

If one of the harmonics is zonal, the condition of conjugacy is that the value of the other harmonic at the pole of the zonal harmonic must be zero.

If we begin with a given harmonic of the $n^{\text{th}}$ order, then, in order that a second harmonic may be conjugate to it, its $2n$ variables must satisfy one condition.

If a third harmonic is to be conjugate to both, its $2n$ variables must satisfy two conditions. If we go on constructing harmonics, each of which is conjugate to all those before it, the number of conditions for each will be equal to the number of harmonics already in existence, so that the $(2n+1)^{\text{th}}$ harmonic will have $2n$ conditions to satisfy by means of its $2n$ variables, and will therefore be completely determined.

Any multiple $AY_n$ of a surface harmonic of the $n^{\text{th}}$ order can be expressed as the sum of multiples of any set of $2n+1$ conjugate harmonics of the same order, for the coefficients of the $2n+1$ conjugate harmonics are a set of disposable quantities equal in number to the $2n$ variables of $Y_n$ and the coefficient $A$.

In order to find the coefficient of any one of the conjugate harmonics, say $Y_n^\sigma$ suppose that

$$AY_n = A_0 Y_n^\sigma + \&\text{c.} + A_\sigma Y_n^\sigma + \&\text{c.}$$

Multiply by $Y_n^\sigma\, ds$ and find the surface integral over the sphere. All the terms involving products of harmonics conjugate to each other will vanish, leaving

$$A\iint Y_n Y_n^\sigma ds = A_\sigma \iint (Y_n^\sigma)^2 ds, \tag{52}$$

an equation which determines $A_\sigma$.

Hence if we suppose a set of $2n+1$ conjugate harmonics given, any other harmonic of the $n^{\text{th}}$ order can be expressed in terms of them, and this only in one way. Hence no other harmonic can be conjugate to all of them.

137.] We have seen that if a complete system of $2n+1$ harmonics of the $n^{\text{th}}$ order, all conjugate to each other, be given, any other harmonic of that order can be expressed in terms of these. In such a system of $2n+1$ harmonics there are $2n(2n+1)$ variables connected by $n(2n+1)$ equations, $n(2n+1)$ of the variables may therefore be regarded as arbitrary.

We might, as Thomson and Tait have suggested, select as a system of conjugate harmonics one in which each harmonic has its $n$ poles distributed so that $j$ of them coincide at the pole of the axis of $x$, $k$ at the pole of $y$, and $l$ $(= n-j-k)$ at the pole of $z$. The $n+1$ distributions for which $l = 0$ and the $n$ distributions for which $l = 1$ being given, all the others may be expressed in terms of these.

The system which has been actually adopted by all mathematicians (including Thomson and Tait) is that in which $n-\sigma$ of the poles are made to coincide at a point which we may call the Positive Pole of the sphere, and the remaining $\sigma$ poles are placed at equal distances round the equator when their number is odd, or at equal distances round one half of the equator when their number is even.

In this case $\mu_1, \mu_2, \ldots \mu_{n-\sigma}$ are each of them equal to $\cos\theta$, which we shall denote by $\mu$. If we also write $\nu$ for $\sin\theta$, $\mu_{n-\sigma+1}, \ldots \mu_n$ are of the form $\nu\cos(\phi-\beta)$, where $\beta$ is the azimuth of one of the poles on the equator.

Also the value of $\lambda_{pq}$ is unity if $p$ and $q$ are both less than $n-\sigma$, zero when one is greater and the other less than this number, and $\cos s\pi/\sigma$ when both are greater, $s$ being an integral number less than $\sigma$.

138.] When all the poles coincide at the pole of the sphere, $\sigma = 0$, and the harmonic is called a Zonal harmonic. As the zonal harmonic is of great importance we shall reserve for it the symbol $P_n$.

We may obtain its value either from the trigonometrical expression (43) or more directly by differentiation, thus

$$P_n = (-1)^n \frac{r^{n+1}}{n!} \frac{d^n}{dz^n}\left(\frac{1}{r}\right), \tag{53}$$

$$\begin{aligned} P_n = \frac{1.3.5\ldots(2n-1)}{1.2.3\ldots n}\Big[\mu^n - \frac{n(n-1)}{2.(2n-1)}\mu^{n-2} \\ + \frac{n(n-1)(n-2)(n-3)}{2.4.(2n-1)(2n-3)}\mu^{n-4} - \&c.\Big] \\ = \Sigma\left[(-1)^p \frac{(2n-2p)!}{2^n p!(n-p)!(n-2p)!}\mu^{n-2p}\right], \end{aligned} \tag{54}$$

where we must give to $p$ every integral value from zero to the greatest integer which does not exceed $\frac{1}{2}n$.

It is sometimes convenient to express $P_n$ as a homogeneous function of $\cos\theta$ and $\sin\theta$, or, as we write them, $\mu$ and $\nu$,

$$\begin{aligned} P_n = \mu^n - \frac{n(n-1)}{2.2}\mu^{n-2}\nu^2 + \frac{n(n-1)(n-2)(n-3)}{2.2.4.4}\mu^{n-4}\nu^4 - \&c., \\ = \Sigma\left[(-1)^p \frac{n!}{2^{2p}(p!)^2(n-2p)!}\mu^{n-2p}\nu^{2p}\right]. \end{aligned} \tag{55}$$

It is shewn in the mathematical treatises on this subject that

$P_n(\mu)$ is the coefficient of $h^n$ in the expansion of $(1-2\mu h+h^2)^{-\frac{1}{2}}$ {and that it is also equal to $\frac{1}{2^n n!}\frac{d^n}{d\mu^n}(\mu^2-1)^n$}.

The surface integral of the square of the zonal harmonic, or

$$\iint (P_n)^2 ds = 2\pi a^2 \int_{-1}^{+1} (P_n(\mu))^2 d\mu = \frac{4\pi a^2}{2n+1}. \tag{56}$$

Hence
$$\int_{-1}^{+1} (P_n(\mu))^2 d\mu = \frac{2}{2n+1}. \tag{57}$$

139.] If we consider a zonal harmonic simply as a function of $\mu$, and without any explicit reference to a spherical surface, it may be called a Legendre's Coefficient.

If we consider the zonal harmonic as existing on a spherical surface the points of which are defined by the coordinates $\theta$ and $\phi$, and if we suppose the pole of the zonal harmonic to be at the point $(\theta', \phi')$, then the value of the zonal harmonic at the point $(\theta, \phi)$ is a function of the four angles $\theta'$, $\phi'$, $\theta$, $\phi$, and because it is a function of $\mu$, the cosine of the arc joining the points $(\theta, \phi)$ and $(\theta', \phi')$, it will be unchanged in value if $\theta$ and $\theta'$, and also $\phi$ and $\phi'$, are made to change places. The zonal harmonic so expressed has been called Laplace's Coefficient. Thomson and Tait call it the Biaxal Harmonic.

Any homogeneous function of $x$, $y$, $z$ which satisfies Laplace's equation may be called a Solid harmonic, and the value of a solid harmonic at the surface of a sphere whose centre is the origin may be called a Surface harmonic. In this book we have defined a surface harmonic by means of its $n$ poles, so that it has only $2n$ variables. The more general surface harmonic, which has $2n+1$ variables, is the more restricted surface harmonic multiplied by an arbitrary constant. The more general surface harmonic, when expressed in terms of $\theta$ and $\phi$, is called a Laplace's Function.

140 $a$.] To obtain the other harmonics of the symmetrical system, we have to differentiate with respect to $\sigma$ axes in the plane of $xy$ inclined to each other at angles equal to $\pi/\sigma$. This may be most conveniently done by means of the system of imaginary coordinates given in Thomson and Tait's *Natural Philosophy*, vol. I, p. 148 (or p. 185 of 2nd edition).

If we write $\qquad \xi = x+iy, \qquad \eta = x-iy,$

where $i$ denotes $\sqrt{-1}$, the operation of differentiating with respect to the $\sigma$ axes if one of these axes makes an angle $a$ with $x$ may be written when $\sigma$ is odd in the form

$$\left(e^{i\alpha}\frac{d}{d\xi}+e^{-i\alpha}\frac{d}{d\eta}\right)\left(e^{i\left(\alpha+\frac{2\pi}{\sigma}\right)}\frac{d}{d\xi}+e^{-i\left(\alpha+\frac{2\pi}{\sigma}\right)}\frac{d}{d\eta}\right)\left(e^{i\left(\alpha+\frac{4\pi}{\sigma}\right)}\frac{d}{d\xi}+e^{-i\left(\alpha+\frac{4\pi}{\sigma}\right)}\frac{d}{d\eta}\right)\ldots$$

This equals

$$\cos\sigma\alpha\left\{\frac{d^\sigma}{d\xi^\sigma}+\frac{d^\sigma}{d\eta^\sigma}\right\}+\sin\sigma\alpha.i\left\{\frac{d^\sigma}{d\xi^\sigma}-\frac{d^\sigma}{d\eta^\sigma}\right\}. \tag{58}$$

If $\sigma$ is even we may prove that the operation of differentiating may be written

$$(-1)^{\frac{\sigma+2}{2}}\left\{\cos\sigma\alpha.i\left(\frac{d^\sigma}{d\xi^\sigma}-\frac{d^\sigma}{d\eta^\sigma}\right)-\sin\sigma\alpha\left(\frac{d^\sigma}{d\xi^\sigma}+\frac{d^\sigma}{d\eta^\sigma}\right)\right\}. \tag{59}$$

Thus, if $i\left(\frac{d^\sigma}{d\xi^\sigma}-\frac{d^\sigma}{d\eta^\sigma}\right)=\overset{(\sigma)}{Ds}$, $\quad \frac{d^\sigma}{d\xi^\sigma}+\frac{d^\sigma}{d\eta^\sigma}=\overset{(\sigma)}{Dc}$,

we may express the operation of differentiating with respect to the $\sigma$ axes in terms of $\overset{(\sigma)}{Ds}$, $\overset{(\sigma)}{Dc}$. These are, of course, real operations, and may be expressed without the use of imaginary symbols, thus:

$$2^{\sigma-1}\overset{(\sigma)}{Ds}=\sigma\frac{d^{\sigma-1}}{dx^{\sigma-1}}\frac{d}{dy}-\frac{\sigma(\sigma-1)(\sigma-2)}{1.2.3}\frac{d^{\sigma-3}}{dx^{\sigma-3}}\frac{d^3}{dy^3}+\&\text{c.}, \tag{60}$$

$$2^{\sigma-1}\overset{(\sigma)}{Dc}=\frac{d^\sigma}{dx^\sigma}-\frac{\sigma(\sigma-1)}{1.2}\frac{d^{\sigma-2}}{dx^{\sigma-2}}\frac{d^2}{dy^2}+\&\text{c.} \tag{61}$$

We shall also write

$$\frac{d^{n-\sigma}}{dz^{n-\sigma}}\overset{(\sigma)}{Ds}=\underset{n}{\overset{(\sigma)}{Ds}},\quad\text{and}\quad\frac{d^{n-\sigma}}{dz^{n-\sigma}}\overset{(\sigma)}{Dc}=\underset{n}{\overset{(\sigma)}{Dc}}; \tag{62}$$

so that $\underset{n}{\overset{(\sigma)}{Ds}}$ and $\underset{n}{\overset{(\sigma)}{Dc}}$ denote the operations of differentiating with respect to $n$ axes, $n-\sigma$ of which coincide with the axis of $z$, while the remaining $\sigma$ make equal angles with each other in the plane of $xy$, $\underset{n}{\overset{(\sigma)}{Ds}}$ being used when the axis of $y$ coincides with one of the axes, and $\underset{n}{\overset{(\sigma)}{Dc}}$ when the axis of $y$ bisects the angle between two of the axes.

The two tesseral surface harmonics of order $n$ and type $\sigma$ may now be written

$$\underset{n}{\overset{(\sigma)}{Ys}}=(-1)^n\frac{1}{n!}r^{n+1}\underset{n}{\overset{(\sigma)}{Ds}}\frac{1}{r}, \tag{63}$$

$$\underset{n}{\overset{(\sigma)}{Yc}}=(-1)^n\frac{1}{n!}r^{n+1}\underset{n}{\overset{(\sigma)}{Dc}}\frac{1}{r}. \tag{64}$$

Writing $\mu=\cos\theta$, $\nu=\sin\theta$, $\rho^2=x^2+y^2$, $r^2=\xi\eta+z^2$,

so that $z=\mu r$, $\rho=\nu r$, $x=\rho\cos\phi$, $y=\rho\sin\phi$,

we have

$$\overset{(\sigma)}{Ds}\frac{1}{r}=(-1)^\sigma\frac{(2\sigma)!}{2^{2\sigma}\sigma!}i(\eta^\sigma-\xi^\sigma)\frac{1}{r^{2\sigma+1}}, \tag{65}$$

$$\overset{(\sigma)}{D}c\,\frac{1}{r} = (-1)^{\sigma}\frac{(2\sigma)!}{2^{2\sigma}\sigma!}(\xi^{\sigma}+\eta^{\sigma})\frac{1}{r^{2\sigma+1}}, \tag{66}$$

in which we may write

$$\frac{i}{2}(\eta^{\sigma}-\xi^{\sigma}) = \rho^{\sigma}\sin\sigma\phi, \qquad \frac{1}{2}(\xi^{\sigma}+\eta^{\sigma}) = \rho^{\sigma}\cos\sigma\phi. \tag{67}$$

We have now only to differentiate with respect to $z$, which we may do so as to obtain the result either in terms of $r$ and $z$, or as a homogeneous function of $z$ and $\rho$ divided by a power of $r$,

$$\frac{d^{n-\sigma}}{dz^{n-\sigma}}\frac{1}{r^{2\sigma+1}} = (-1)^{n-\sigma}\frac{(2n)!}{2^n n!}\frac{2^{\sigma}\sigma!}{(2\sigma)!}\frac{1}{r^{2n+1}}\times$$
$$\left[z^{n-\sigma} - \frac{(n-\sigma)(n-\sigma-1)}{2(2n-1)}z^{n-\sigma-2}r^2 + \&\text{c.}\right]^{*}, \tag{68}$$

or
$$\frac{d^{n-\sigma}}{dz^{n-\sigma}}\frac{1}{r^{2\sigma+1}} = (-1)^{n-\sigma}\frac{(n+\sigma)!}{(2\sigma)!}\frac{1}{r^{2n+1}}\times$$
$$\left[z^{n-\sigma} - \frac{(n-\sigma)(n-\sigma-1)}{4(\sigma+1)}z^{n-\sigma-2}\rho^2 + \&\text{c.}\right]. \tag{69}$$

If we write

$$\Theta_n^{(\sigma)} = \nu^{\sigma}\left[\mu^{n-\sigma} - \frac{(n-\sigma)(n-\sigma-1)}{2(2n-1)}\mu^{n-\sigma-2}\right.$$
$$\left. + \frac{(n-\sigma)(n-\sigma-1)(n-\sigma-2)(n-\sigma-3)}{2\,.\,4\,(2n-1)(2n-3)}\mu^{n-\sigma-4} - \&\text{c.}\right], \tag{70}$$

and

$$\vartheta_n^{(\sigma)} = \nu^{\sigma}\left[\mu^{n-\sigma} - \frac{(n-\sigma)(n-\sigma-1)}{4(\sigma+1)}\mu^{n-\sigma-2}\nu^2\right.$$
$$\left. + \frac{(n-\sigma)(n-\sigma-1)(n-\sigma-2)(n-\sigma-3)}{4\,.\,8\,(\sigma+1)(\sigma+2)}\mu^{n-\sigma-4}\nu^4 - \&\text{c.}\right], \tag{71}$$

then
$$\Theta_n^{(\sigma)} = \frac{2^{n-\sigma}n!\,(n+\sigma)!}{(2n)!\,\sigma!}\vartheta_n^{(\sigma)}, \tag{72}$$

so that these two functions differ only by a constant factor.

We may now write the expressions for the two tesseral harmonics of order $n$ and type $\sigma$ in terms either of $\Theta$ or $\vartheta$,

$$\overset{(\sigma)}{\underset{n}{Y}}s = \frac{(2n)!}{2^{n+\sigma}n!\,n!}\Theta_n^{(\sigma)}\,2\sin\sigma\phi = \frac{(n+\sigma)!}{2^{2\sigma}n!\,\sigma!}\vartheta_n^{(\sigma)}\,2\sin\sigma\phi, \tag{73}$$

$$\overset{(\sigma)}{\underset{n}{Y}}c = \frac{(2n)!}{2^{n+\sigma}n!\,n!}\Theta_n^{(\sigma)}\,2\cos\sigma\phi = \frac{(n+\sigma)!}{2^{2\sigma}n!\,\sigma!}\vartheta_n^{(\sigma)}\,2\cos\sigma\phi\,\dagger. \tag{74}$$

* {Equation (68) may easily be proved by noticing that the left-hand side is $(n-\sigma)!$ times the coefficient of $h^{n-\sigma}$ in $\left\{\frac{1}{\xi\eta+(z+h)^2}\right\}^{\frac{2\sigma+1}{2}}$, or $\frac{1}{r^{2\sigma+1}}\left\{1+\frac{2hz+h^2}{r^2}\right\}^{-\frac{(2\sigma+1)}{2}}$; if we write this as $\frac{1}{r^{2\sigma+1}}\left\{\left(1+\mu\frac{h}{r}\right)^2+\nu^2\frac{h^2}{r^2}\right\}^{-\frac{(2\sigma+1)}{2}}$ and pick out the coefficient of $h^{n-\sigma}$, we get equation (69).}

† {This value must be halved when $\sigma = 0$.}

We must remember that when $\sigma = 0$, $\sin\sigma\phi = 0$ and $\cos\sigma\phi = 1$.

For every value of $\sigma$ from 1 to $n$ inclusive there is a pair of harmonics, but when $\sigma = 0$, $\underset{n}{\overset{(0)}{Ys}} = 0$, and $\underset{n}{\overset{(0)}{Yc}} = P_n$, the zonal harmonic. The whole number of harmonics of order $n$ is therefore $2n+1$, as it ought to be.

140 *b*.] The numerical value of $Y$ adopted in this treatise is that which we find by differentiating $r^{-1}$ with respect to the $n$ axes and dividing by $n!$ It is the product of four factors, the sine or cosine of $\sigma\phi$, $\nu^\sigma$, a function of $\mu$ (or of $\mu$ and $\nu$), and a numerical coefficient.

The product of the second and third factors, that is to say, the part depending on $\theta$, has been expressed in terms of three different symbols which differ from each other only by their numerical factors. When it is expressed as the product of $\nu^\sigma$ into a series of descending powers of $\mu$, the first term being $\mu^{n-\sigma}$, it is the function which we, following Thomson and Tait, denote by $\Theta$.

The function which Heine (*Handbuch der Kugelfunctionen*, § 47) denotes by $P_\sigma^{(n)}$, and calls *eine zugeordnete Function erster Art*, or, as Todhunter translates it, an 'Associated Function of the First Kind,' is related to $\Theta_n^{(\sigma)}$ by the equation

$$\Theta_n^{(\sigma)} = (-1)^{\frac{\sigma}{2}} P_\sigma^{(n)}. \tag{75}$$

The series of descending powers of $\mu$, beginning with $\mu^{n-\sigma}$, is expressed by Heine by the symbol $\mathfrak{P}_\sigma^{(n)}$, and by Todhunter by the symbol $\varpi(\sigma, n)$.

This series may also be expressed in two other forms,

$$\begin{aligned}\mathfrak{P}_\sigma^{(n)} = \varpi(\sigma, n) &= \frac{(n-\sigma)!}{(2n)!}\frac{d^{n+\sigma}}{d\mu^{n+\sigma}}(\mu^2-1)^n \\ &= \frac{2^n(n-\sigma)!\,n!}{(2n)!}\frac{d^\sigma}{d\mu^\sigma}P_n.\end{aligned} \tag{76}$$

The last of these, in which the series is obtained by differentiating the zonal harmonic with respect to $\mu$, seems to have suggested the symbol $T_n^{(\sigma)}$ adopted by Ferrers, who defines it thus

$$T_n^{(\sigma)} = \nu^\sigma \frac{d^\sigma}{d\mu^\sigma} P_n = \frac{(2n)!}{2^n(n-\sigma)!\,n!}\Theta_n^{(\sigma)}. \tag{77}$$

When the same quantity is expressed as a homogeneous function of $\mu$ and $\nu$, and divided by the coefficient of $\mu^{n-\sigma}\nu^\sigma$, it is what we have already denoted by $\mathfrak{I}_n^{(\sigma)}$.

140 *c*.] The harmonics of the symmetrical system have been classified by Thomson and Tait with reference to the form of the spherical curves at which they become zero.

The value of the zonal harmonic at any point of the sphere is a function of the cosine of the polar distance, which if equated to zero gives an equation of the $n^{\text{th}}$ degree, all whose roots lie between $-1$ and $+1$, and therefore correspond to $n$ parallels of latitude on the sphere.

The zones included between these parallels are alternately positive and negative, the circle surrounding the pole being always positive.

The zonal harmonic is therefore suitable for expressing a function which becomes zero at certain parallels of latitude on the sphere, or at certain conical surfaces in space.

The other harmonics of the symmetrical system occur in pairs, one involving the cosine and the other the sine of $\sigma\phi$. They therefore become zero at $\sigma$ meridian circles on the sphere and also at $n-\sigma$ parallels of latitude, so that the spherical surface is divided into $2\sigma(n-\sigma-1)$ quadrilaterals or tesserae, together with $4\sigma$ triangles at the poles. They are therefore useful in investigations relating to quadrilaterals or tesserae on the sphere bounded by meridian circles and parallels of latitude.

They are all called Tesseral harmonics except the last pair, which becomes zero at $n$ meridian circles only, which divide the spherical surface into $2n$ sectors. This pair are therefore called Sectorial harmonics.

141.] We have next to find the surface integral of the square of any tesseral harmonic taken over the sphere. This we may do by the method of Art. 134. We convert the surface harmonic $Y_n^{(\sigma)}$ into a solid harmonic of positive degree by multiplying it by $r^n$, we differentiate this solid harmonic with respect to the $n$ axes of the harmonic itself, and then make $x=y=z=0$, and we multiply the result by $\dfrac{4\pi a^2}{n!(2n+1)}$.

These operations are indicated in our notation by

$$\iint \left(Y_n^{(\sigma)}\right)^2 ds = \frac{4\pi a^2}{n!(2n+1)} D_n^{(\sigma)}\left(r^n\, Y_n^{(\sigma)}\right). \tag{78}$$

Writing the solid harmonic in the form of a homogeneous function of $z$ and $\xi$ and $\eta$, viz.,

$$r^n \overset{(\sigma)}{\underset{n}{Ys}} = \frac{(n+\sigma)!}{2^{2\sigma} n!\,\sigma!} i\,(\eta^\sigma - \xi^\sigma)\left[z^{n-\sigma} - \frac{(n-\sigma)(n-\sigma-1)}{4(\sigma+1)} z^{n-\sigma-2}\xi\eta + \&c.\right], \quad (79)$$

we find that on performing the differentiations with respect to $z$, all the terms of the series except the first disappear, and the factor $(n-\sigma)!$ is introduced.

Continuing the differentiations with respect to $\xi$ and $\eta$ we get rid also of these variables and introduce the factor $-2i\sigma!$, so that the final result is

$$\iint \Big(\overset{(\sigma)}{\underset{n}{Ys}}\Big)^2 ds = \frac{8\pi a^2}{2n+1} \frac{(n+\sigma)!\,(n-\sigma)!}{2^{2\sigma} n!\,n!}. \quad (80)$$

We shall denote the second member of this equation by the abbreviated symbol $[n, \sigma]$.

This expression is correct for all values of $\sigma$ from 1 to $n$ inclusive, but there is no harmonic in $\sin \sigma\phi$ corresponding to $\sigma = 0$.

In the same way we can shew that

$$\iint \Big(\overset{(\sigma)}{\underset{n}{Yc}}\Big)^2 ds = \frac{8\pi a^2}{2n+1} \frac{(n+\sigma)!\,(n-\sigma)!}{2^{2\sigma} n!\,n!} \quad (81)$$

for all values of $\sigma$ from 1 to $n$ inclusive.

When $\sigma = 0$, the harmonic becomes the zonal harmonic, and

$$\iint \Big(\overset{(0)}{\underset{n}{Yc}}\Big)^2 ds = \iint (P_n)^2 ds = \frac{4\pi a^2}{2n+1}; \quad (82)$$

a result which may be obtained directly from equation (50) by putting $Y_n = P_m$ and remembering that the value of the zonal harmonic at its pole is unity.

142 *a.*] We can now apply the method of Art. 136 to determine the coefficient of any given tesseral surface harmonic in the expansion of any arbitrary function of the position of a point on a sphere. For let $F$ be the arbitrary function, and let $A_n^\sigma$ be the coefficient of $Y_n^{(\sigma)}$ in the expansion of this function in surface harmonics of the symmetrical system, then

$$\iint F Y_n^{(\sigma)} ds = A_n^{(\sigma)} \iint \big(Y_n^{(\sigma)}\big)^2 ds = A_n^{(\sigma)} [n, \sigma], \quad (83)$$

where $[n, \sigma]$ is the abbreviation for the value of the surface integral given in equation (80).

142 *b.*] Let $\Psi$ be any function which satisfies Laplace's equation, and which has no singular values within a distance $a$ of a point $O$, which we may take as the origin of coordinates. It is always possible to expand such a function in a series of solid harmonics of positive degree, having their origin at $O$.

One way of doing this is to describe a sphere about $O$ as centre with a radius less than $a$, and to expand the value of the potential at the surface of the sphere in a series of surface harmonics. Multiplying each of these harmonics by $r/a$ raised to a power equal to the order of the surface harmonic, we obtain the solid harmonics of which the given function is the sum.

But a more convenient method, and one which does not involve integration, is by differentiation with respect to the axes of the harmonics of the symmetrical system.

For instance, let us suppose that in the expansion of $\Psi$, there is a term of the form $\overset{(\sigma)}{\underset{n}{Ac}} \overset{(\sigma)}{\underset{n}{Yc}} r^n$.

If we perform on $\Psi$ and on its expansion the operation

$$\frac{d^{n-\sigma}}{dz^{n-\sigma}}\left(\frac{d^\sigma}{d\xi^\sigma} + \frac{d^\sigma}{d\eta^\sigma}\right),$$

and put $x, y, z$ equal to zero after differentiating, all the terms of the expansion vanish except that containing $\overset{(\sigma)}{\underset{n}{Ac}}$.

Expressing the operator on $\Psi$ in terms of differentiations with respect to the real axes, we obtain the equation

$$\frac{d^{n-\sigma}}{dz^{n-\sigma}}\left[\frac{d^\sigma}{dx^\sigma} - \frac{\sigma(\sigma-1)}{1\,.\,2}\frac{d^{\sigma-2}}{dx^{\sigma-2}}\frac{d^2}{dy^2} + \&c.\right]\Psi$$
$$= \overset{(\sigma)}{\underset{n}{Ac}}\frac{(n+\sigma)!\,(n-\sigma)!}{2^\sigma n!}, \tag{84}$$

from which we can determine the coefficient of any harmonic of the series in terms of the differential coefficients of $\Psi$ with respect to $x, y, z$ at the origin.

143.] It appears from equation (50) that it is always possible to express a harmonic as the sum of a system of zonal harmonics of the same order, having their poles distributed over the surface of the sphere. The simplification of this system, however, does not appear easy. I have, however, for the sake of exhibiting to the eye some of the features of spherical harmonics, calculated the zonal harmonics of the third and fourth orders, and drawn, by the method already described for the addition of functions, the equipotential lines on the sphere for harmonics which are the sums of two zonal harmonics. See Figures VI to IX at the end of this volume.

Fig. VI represents the difference of two zonal harmonics of the third order whose axes are inclined at 120° in the plane of the paper, and this difference is the harmonic of the second type in which $\sigma = 1$, the axis being perpendicular to the paper.

In Fig. VII the harmonic is also of the third order, but the axes of the zonal harmonics of which it is the sum are inclined at 90°, and the result is not of any type of the symmetrical system. One of the nodal lines is a great circle, but the other two which are intersected by it are not circles.

Fig. VIII represents the difference of two zonal harmonics of the fourth order whose axes are at right angles. The result is a tesseral harmonic for which $n = 4$, $\sigma = 2$.

Fig. IX represents the sum of the same zonal harmonics. The result gives some notion of one type of the more general harmonic of the fourth order. In this type the nodal line on the sphere consists of six ovals not intersecting each other. Within these ovals the harmonic is positive, and in the sextuply connected part of the spherical surface which lies outside the ovals, the harmonic is negative.

All these figures are orthogonal projections of the spherical surface.

I have also drawn in Fig. V a plane section through the axis of a sphere, to shew the equipotential surfaces and lines of force due to a spherical surface electrified according to the values of a spherical harmonic of the first order.

Within the sphere the equipotential surfaces are equidistant planes, and the lines of force are straight lines parallel to the axis, their distances from the axis being as the square roots of the natural numbers. The lines outside the sphere may be taken as a representation of those which would be due to the earth's magnetism if it were distributed according to the most simple type.

144 $a$.] We are now able to determine the distribution of electricity on a spherical conductor under the action of electric forces whose potential is given.

By the methods already given we expand $\Psi$, the potential due to the given forces, in a series of solid harmonics of positive degree having their origin at the centre of the sphere.

Let $A_n r^n Y_n$ be one of these, then since within the conducting sphere the potential is uniform, there must be a term $-A_n r^n Y_n$ arising from the distribution of electricity on the surface of the sphere, and therefore in the expansion of $4\pi\sigma$ there must be a term

$$4\pi\sigma_n = (2n+1)a^{n-1}A_n Y_n.$$

In this way we can determine the coefficients of the harmonics of all orders except zero in the expression for the surface density. The coefficient corresponding to order zero depends on the charge, $e$, of the sphere, and is given by $4\pi\sigma_0 = a^{-2}e$.

The potential of the sphere is

$$V = \Psi_0 + \frac{e}{a}.$$

144 *b*.] Let us next suppose that the sphere is placed in the neighbourhood of conductors connected with the earth, and that Green's Function, $G$, has been determined in terms of $x, y, z$ and $x', y', z'$, the coordinates of any two points in the region in which the sphere is placed.

If the surface density on the sphere is expressed in a series of spherical harmonics, then the electrical phenomena outside the sphere, arising from this charge on the sphere, are identical with those arising from an imaginary series of singular points all at the centre of the sphere, the first of which is a single point having a charge equal to that of the sphere and the others are multiple points of different orders corresponding to the harmonics which express the surface density.

Let Green's function be denoted by $G_{pp'}$, where $p$ indicates the point whose coordinates are $x, y, z$, and $p'$ the point whose coordinates are $x', y', z'$.

If a charge $A_0$ is placed at the point $p'$, then, considering $x', y', z'$ as constants, $G_{pp'}$ becomes a function of $x, y, z$; and the potential arising from the electricity induced on surrounding bodies by $A_0$ is

$$\Psi = A_0 G_{pp'}. \tag{1}$$

If, instead of placing the charge $A_0$ at the point $p'$, it were distributed uniformly over a sphere of radius $a$ having its centre at $p'$, the value of $\Psi$ at points outside the sphere would be the same.

If the charge on the sphere is not uniformly distributed, let its surface density be expressed, as it always can, in a series of spherical harmonics, thus

$$4\pi a^2 \sigma = A_0 + 3A_1 Y_1 + \&c. + (2n+1) A_n Y_n + \dots. \tag{2}$$

The potential arising from any term of this distribution, say

$$4\pi a^2 \sigma_n = (2n+1) A_n Y_n, \tag{3}$$

will be $\frac{r^n}{a^{n+1}} A_n Y_n$ for points inside the sphere, and $\frac{a^n}{r^{n+1}} A_n Y_n$ for points outside the sphere.

Now the latter expression, by equations (13), (14), Arts. 129 $c$ and 129 $d$ is equal to
$$(-1)^n A_n \frac{a^n}{n!} \frac{d^n}{dh_1 \ldots dh_n} \frac{1}{r};$$
or the potential outside the sphere, due to the charge on the surface of the sphere, is equivalent to that due to a certain multiple point whose axes are $h_1 \ldots h_n$ and whose moment is $A_n a^n$.

Hence the distribution of electricity on the surrounding conductors and the potential due to this distribution is the same as that which would be due to such a multiple point.

The potential, therefore, at the point $p$, or $(x, y, z)$, due to the induced electrification of surrounding bodies, is
$$\Psi_n = (-1)^n A_n \frac{a^n}{n!} \frac{d'^n}{d'h_1 \ldots d'h_n} G, \tag{4}$$
where the accent over the $d$'s indicates that the differentiations are to be performed with respect to $x', y', z'$. These coordinates are afterwards to be made equal to those of the centre of the sphere.

It is convenient to suppose $Y_n$ broken up into its $2n+1$ constituents of the symmetrical system. Let $A_n^{(\sigma)}\, Y_n^{(\sigma)}$ be one of these, then
$$\frac{d'^n}{d'h_1 \ldots d'h_n} = D'^{(\sigma)}_n. \tag{5}$$
It is unnecessary here to supply the affix $s$ or $c$, which indicates whether $\sin \sigma\phi$ or $\cos \sigma\phi$ occurs in the harmonic.

We may now write the complete expression for $\Psi$, the potential arising from induced electrification,
$$\Psi = A_0 G + \Sigma\Sigma\left[(-1)^n A_n^{(\sigma)} \frac{a^n}{n!} D'^{(\sigma)}_n G\right]. \tag{6}$$
But within the sphere the potential is constant, or
$$\Psi + \frac{1}{a} A_0 + \Sigma\Sigma\left[\frac{r^{n_1}}{a^{n_1+1}} A_{n_1}^{(\sigma_1)}\, Y_{n_1}^{(\sigma_1)}\right] = \text{constant}. \tag{7}$$

Now perform on this expression the operation $D_{n_1}^{(\sigma_1)}$, where the differentiations are to be with respect to $x, y, z$, and the values of $n_1$ and $\sigma_1$ are independent of those of $n$ and $\sigma$. All the terms of (7) will disappear except that in $Y_{n_1}^{(\sigma_1)}$, and we find
$$-2\frac{(n_1+\sigma_1)!\,(n_1-\sigma_1)!}{2^{2\sigma_1} n_1!} \frac{1}{a^{n_1+1}} A_{n_1}^{(\sigma_1)}$$
$$= A_0 D_{n_1}^{(\sigma_1)} G + \Sigma\Sigma\left[(-1)^n A_n^{\sigma} \frac{a^n}{n!} D_{n_1}^{(\sigma_1)} D'^{(\sigma)}_n G\right]. \tag{8}$$

We thus obtain a set of equations, the first member of each of which contains one of the coefficients which we wish to determine. The first term of the second member contains $A_0$, the charge of the sphere, and we may regard this as the principal term.

Neglecting, for the present, the other terms, we obtain as a first approximation

$$A_{n_1}^{(\sigma_1)} = -\frac{1}{2}\frac{2^{2\sigma_1} n_1!}{(n_1+\sigma_1)!(n_1-\sigma_1)!} A_0 a^{n_1+1} D_{n_1}^{(\sigma_1)} G. \tag{9}$$

If the shortest distance from the centre of the sphere to the nearest of the surrounding conductors is denoted by $b$,

$$a^{n_1+1} D_{n_1}^{(\sigma_1)} G < n_1! \left(\frac{a}{b}\right)^{n_1+1}.$$

If, therefore, $b$ is large compared with $a$, the radius of the sphere, the coefficients of the other spherical harmonics are very small compared with $A_0$. The ratio of a term after the first on the right-hand side of equation (8) to the first term will therefore be of an order of magnitude similar to $\left(\frac{a}{b}\right)^{2n+n_1+1}$

We may therefore neglect them in a first approximation, and in a second approximation we may insert in these terms the values of the coefficients obtained by the first approximation, and so on till we arrive at the degree of approximation required.

*Distribution of electricity on a nearly spherical conductor.*

145 *a*.] Let the equation of the surface of the conductor be

$$r = a(1+F), \tag{1}$$

where $F$ is a function of the direction of $r$, that is to say of $\theta$ and $\phi$, and is a quantity the square of which may be neglected in this investigation.

Let $F$ be expanded in the form of a series of surface harmonics

$$F = f_0 + f_1 Y_1 + f_2 Y_2 + \&c. + f_n Y_n. \tag{2}$$

Of these terms, the first depends on the excess of the mean radius above $a$. If therefore we assume that $a$ is the mean radius, that is to say approximately the radius of a sphere whose volume is equal to that of the given conductor, the coefficient $f_0$ will disappear.

The second term, that in $f_1$, depends on the distance of the centre of mass of the conductor, supposed of uniform density, from the origin. If therefore we take that centre for origin, the coefficient $f_1$ will also disappear.

We shall begin by supposing that the conductor has a charge $A_0$, and that no external electrical force acts on it. The potential outside the conductor must therefore be of the form

$$V = A_0 \frac{1}{r} + A_1 Y_1' \frac{1}{r^2} + \&\text{c.} + A_n Y_n' \frac{1}{r^{n+1}} + \dots, \tag{3}$$

where the surface harmonics are not assumed to be of the same types as in the expansion of $F$.

At the surface of the conductor the potential is that of the conductor, namely, the constant quantity $\alpha$.

Hence, expanding the powers of $r$ in terms of $a$ and $F$, and neglecting the square and higher powers of $F$, we have

$$\alpha = A_0 \frac{1}{a}(1-F) + A_1 \frac{1}{a^2} Y_1'(1-2F) + \&\text{c.}$$

$$+ A_n \frac{1}{a^{n+1}} Y_n'(1-(n+1)F) + \dots. \tag{4}$$

Since the coefficients $A_1$, &c. are evidently small compared with $A_0$, we may begin by neglecting products of these coefficients into $F$.

If we then write for $F$ in its first term its expansion in spherical harmonics, and equate to zero the terms involving harmonics of the same order, we find

$$\alpha = A_0 \frac{1}{a}, \tag{5}$$

$$A_1 Y_1' = A_0 a f_1 Y_1 = 0, \tag{6}$$

$$\dots\dots\dots$$

$$A_n Y_n' = A_0 a^n f_n Y_n. \tag{7}$$

It follows from these equations that the $Y'$'s must be of the same type as the $Y$'s, and therefore identical with them, and that $A_1 = 0$ and $A_n = A_0 a^n f_n$.

To determine the density at any point of the surface, we have the equation
$$4\pi\sigma = -\frac{dV}{d\nu} = -\frac{dV}{dr}\cos\epsilon, \text{ approximately}; \tag{8}$$

where $\nu$ is the normal and $\epsilon$ is the angle which the normal makes with the radius. Since in this investigation we suppose $F$ and its first differential coefficients with respect to $\theta$ and $\phi$ to be small, we may put $\cos\epsilon = 1$, so that

$$4\pi\sigma = -\frac{dV}{dr} = A_0 \frac{1}{r^2} + \&\text{c.} + (n+1) A_n Y_n \frac{1}{r^{n+2}} + \dots. \tag{9}$$

Expanding the powers of $r$ in terms of $a$ and $F$, and neglecting products of $F$ into $A_n$, we find

$$4\pi\sigma = A_0\frac{1}{a^2}(1-2F)+\text{\&c.}+(n+1)A_n\frac{1}{a^{n+2}}Y_n. \qquad (10)$$

Expanding $F$ in spherical harmonics and giving $A_n$ its value as already found, we obtain

$$4\pi\sigma = A_0\frac{1}{a^2}[1+f_2Y_2+2f_3Y_3+\text{\&c.}+(n-1)f_nY_n]. \qquad (11)$$

Hence, if the surface differs from that of a sphere by a thin stratum whose depth varies according to the values of a spherical harmonic of order $n$, the ratio of the difference of the surface densities at any two points to their sum will be $n-1$ times the ratio of the difference of the radii at the same two points to their sum.

145 *b*.] If the nearly spherical conductor (1) is acted on by external electric forces, let the potential, $U$, arising from these forces be expanded in a series of spherical harmonics of positive degree, having their origin at the centre of volume of the conductor

$$U = B_0+B_1rY_1'+B_2r^2Y_2'+\text{\&c.}+B_nr^nY_n'+\dots, \qquad (12)$$

where the accent over $Y$ indicates that this harmonic is not necessarily of the same type as the harmonic of the same order in the expansion of $F$.

If the conductor had been accurately spherical, the potential arising from its surface charge at a point outside the conductor would have been

$$V = A_0\frac{1}{r}-B_1\frac{a^3}{r^2}Y_1'-\text{\&c.}-B_n\frac{a^{2n+1}}{r^{n+1}}Y_n'-\dots. \qquad (13)$$

Let the actual potential arising from the surface charge be $V+W$, where

$$W = C_1\frac{1}{r^2}Y_1''+\text{\&c.}+C_m\frac{1}{r^{m+1}}Y_m''+\dots; \qquad (14)$$

the harmonics with a double accent being different from those occurring either in $F$ or in $U$, and the coefficients $C$ being small because $F$ is small.

The condition to be fulfilled is that, when $r = a\,(1+F)$,

$$U+V+W = \text{constant} = A_0\frac{1}{a}+B_0,$$

the potential of the conductor.

Expanding the powers of $r$ in terms of $a$ and $F$, and retaining the first power of $F$ when it is multiplied by $A$ or $B$, but neglecting it when it is multiplied by the small quantities $C$, we find

$$F\left[-A_0\frac{1}{a}+3B_1aY_1'+5B_2a^2Y_2'+\&c.+(2n+1)B_na^nY_n'+\dots\right]$$
$$+C_1\frac{1}{a^2}Y_1''+\&c.+C_m\frac{1}{a^{m+1}}Y_m''+\dots=0. \quad (15)$$

To determine the coefficients $C$, we must perform the multiplication indicated in the first line, and express the result in a series of spherical harmonics. This series, with the signs reversed, will be the series for $W$ at the surface of the conductor.

The product of two surface spherical harmonics of orders $n$ and $m$, is a rational function of degree $n+m$ in $x/r$, $y/r$, and $z/r$, and can therefore be expanded in a series of spherical harmonics of orders not exceeding $m+n$. If, therefore, $F$ can be expanded in spherical harmonics of orders not exceeding $m$, and if the potential due to external forces can be expanded in spherical harmonics of orders not exceeding $n$, the potential arising from the surface charge will involve spherical harmonics of orders not exceeding $m+n$.

This surface density can then be found from the potential by the approximate equation

$$4\pi\sigma+\frac{d}{dr}(U+V+W)=0. \quad (16)$$

145 c.] *A nearly spherical conductor enclosed in a nearly spherical and nearly concentric conducting vessel.*

Let the equation of the surface of the conductor be

$$r=a(1+F), \quad (17)$$

where
$$F=f_1Y_1+\&c.+f_n^{(\sigma)}Y_n^{(\sigma)}. \quad (18)$$

Let the equation of the inner surface of the vessel be

$$r=b(1+G), \quad (19)$$

where
$$G=g_1Y_1+\&c.+g_n^{(\sigma)}Y_n^{(\sigma)}, \quad (20)$$

the $f$'s and $g$'s being small compared with unity, and $Y_n^{(\sigma)}$ being the surface harmonic of order $n$ and type $\sigma$.

Let the potential of the conductor be $\alpha$, and that of the vessel $\beta$. Let the potential at any point between the conductor and the vessel be expanded in spherical harmonics, thus

$$\Psi = h_0 + h_1 Y_1 r + \&c. + h_n^{(\sigma)} Y_n^{(\sigma)} r^n + \dots$$
$$+ k_0 \frac{1}{r} + k_1 Y_1 \frac{1}{r^2} + \&c. + k_n^{(\sigma)} Y_n^{(\sigma)} \frac{1}{r^{n+1}} + \dots, \quad (21)$$

then we have to determine the constants of the forms $h$ and $k$ so that when $r = a(1+F)$, $\Psi = \alpha$, and when $r = b(1+G)$, $\Psi = \beta$.

It is manifest, from our former investigation, that all the $h$'s and $k$'s except $h_0$ and $k_0$ will be small quantities, the products of which into $F$ may be neglected. We may, therefore, write

$$\alpha = h_0 + k_0 \frac{1}{a}(1-F) + \&c. + \left(h_n^{(\sigma)} a^n + k_n^{(\sigma)} \frac{1}{a^{n+1}}\right) Y_n^{(\sigma)} + \dots, \quad (22)$$

$$\beta = h_0 + k_0 \frac{1}{b}(1-G) + \&c. + \left(h_n^{(\sigma)} b^n + k_n^{(\sigma)} \frac{1}{b^{n+1}}\right) Y_n^{(\sigma)} + \dots. \quad (23)$$

We have therefore
$$\alpha = h_0 + k_0 \frac{1}{a}, \quad (24)$$

$$\beta = h_0 + k_0 \frac{1}{b}, \quad (25)$$

$$k_0 \frac{1}{a} f_n^{(\sigma)} = h_n^{(\sigma)} a^n + k_n^{(\sigma)} \frac{1}{a^{n+1}}, \quad (26)$$

$$k_0 \frac{1}{b} g_n^{(\sigma)} = h_n^{(\sigma)} b^n + k_n^{(\sigma)} \frac{1}{b^{n+1}}, \quad (27)$$

whence we find for $k_0$, the charge of the inner conductor,

$$k_0 = (\alpha - \beta) \frac{ab}{b-a}, \quad (28)$$

and for the coefficients of the harmonics of order $n$

$$h_n^{(\sigma)} = k_0 \frac{b^n g_n^{(\sigma)} - a^n f_n^{(\sigma)}}{b^{2n+1} - a^{2n+1}}, \quad (29)$$

$$k_n^{(\sigma)} = k_0 a^n b^n \frac{b^{n+1} f_n^{(\sigma)} - a^{n+1} g_n^{(\sigma)}}{b^{2n+1} - a^{2n+1}}, \quad (30)$$

where we must remember that the coefficients $f_n^{(\sigma)}$, $g_n^{(\sigma)}$, $h_n^{(\sigma)}$, $k_n^{(\sigma)}$ are those belonging to the same type as well as order.

The surface density on the inner conductor is given by the equation

$$4\pi\sigma a^2 = k_0 (1 + \dots + A_n Y_n^{(\sigma)} + \dots)$$

where
$$A_n = \frac{f_n^{(\sigma)}\{(n+2)a^{2n+1} + (n-1)b^{2n+1}\} - g_n^{(\sigma)}(2n+1)a^{n+1}b^n}{b^{2n+1} - a^{2n+1}}. \quad (31)$$

146.] As an example of the application of zonal harmonics, let us investigate the equilibrium of electricity on two spherical conductors.

Let $a$ and $b$ be the radii of the spheres, and $c$ the distance between their centres. We shall also, for the sake of brevity, write $a = cx$, and $b = cy$, so that $x$ and $y$ are numerical quantities less than unity.

Let the line joining the centres of the spheres be taken as the axis of the zonal harmonics, and let the pole of the zonal harmonics belonging to either sphere be the point of that sphere nearest to the other.

Let $r$ be the distance of any point from the centre of the first sphere, and $s$ the distance of the same point from that of the second sphere.

Let the surface density, $\sigma_1$, of the first sphere be given by the equation

$$4\pi\sigma_1 a^2 = A + A_1 P_1 + 3 A_2 P_2 + \&c. + (2m+1) A_m P_m, \qquad (1)$$

so that $A$ is the total charge of the sphere, and $A_1$, &c. are the coefficients of the zonal harmonics $P_1$, &c.

The potential due to this distribution of charge may be represented by

$$U' = \frac{1}{a}\left[A + A_1 P_1 \frac{r}{a} + A_2 P_2 \frac{r^2}{a^2} + \&c. + A_m P_m \frac{r^m}{a^m}\right] \qquad (2)$$

for points inside the sphere, and by

$$U = \frac{1}{r}\left[A + A_1 P_1 \frac{a}{r} + A_2 P_2 \frac{a^2}{r^2} + \&c. + A_m P_m \frac{a^m}{r^m}\right] \qquad (3)$$

for points outside.

Similarly, if the surface density on the second sphere is given by the equation

$$4\pi\sigma_2 b^2 = B + B_1 P_1 + \&c. + (2n+1) B_n P_n, \qquad (4)$$

the potential inside and outside this sphere due to this charge may be represented by equations of the form

$$V' = \frac{1}{b}\left[B + B_1 P_1 \frac{s}{b} + \&c. + B_n P_n \frac{s^n}{b^n}\right], \qquad (5)$$

$$V = \frac{1}{s}\left[B + B_1 P_1 \frac{b}{s} + \&c. + B_n P_n \frac{b^n}{s^n}\right], \qquad (6)$$

where the several harmonics are related to the second sphere.

The charges of the spheres are $A$ and $B$ respectively.

The potential at every point within the first sphere is constant and equal to $a$, the potential of that sphere, so that within the first sphere

$$U' + V = a. \qquad (7)$$

Similarly, if the potential of the second sphere is $\beta$, for points within that sphere,
$$U + V' = \beta. \tag{8}$$
For points outside both spheres the potential is $\Psi$, where
$$U + V = \Psi. \tag{9}$$
On the axis, between the centres of the spheres,
$$r + s = c. \tag{10}$$
Hence, differentiating with respect to $r$, and after differentiation making $r = 0$, and remembering that at the pole each of the zonal harmonics is unity, we find
$$\left.\begin{aligned} &A_1 \frac{1}{a^2} - \frac{dV}{ds} = 0, \\ &A_2 \frac{2\,!}{a^3} + \frac{d^2V}{ds^2} = 0, \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ &A_m \frac{m\,!}{a^{m+1}} + (-1)^m \frac{d^mV}{ds^m} = 0, \end{aligned}\right\} \tag{11}$$
where, after differentiation, $s$ is to be made equal to $c$.

If we perform the differentiations, and write $a/c = x$ and $b/c = y$, these equations become
$$\left.\begin{aligned} 0 &= A_1 + Bx^2 + 2B_1x^2y + 3B_2x^2y^2 + \&\text{c.} + (n+1)B_nx^2y^n, \\ 0 &= A_2 + Bx^3 + 3B_1x^3y + 6B_2x^3y^2 + \&\text{c.} + \tfrac{1}{2}(n+1)(n+2)B_nx^3y^n, \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ 0 &= A_m + Bx^{m+1} + (m+1)B_1x^{m+1}y + \tfrac{1}{2}(m+1)(m+2)B_2x^{m+1}y^2 \\ &\qquad + \&\text{c.} + \frac{(m+n)\,!}{m\,!\,n\,!}B_nx^{m+1}y^n. \end{aligned}\right\} \tag{12}$$
By the corresponding operations for the second sphere we find,
$$\left.\begin{aligned} 0 &= B_1 + Ay^2 + 2A_1xy^2 + 3A_2x^2y^2 + \&\text{c.} + (m+1)A_mx^my^2, \\ 0 &= B_2 + Ay^3 + 3A_1xy^3 + 6A_2x^2y^3 + \&\text{c.} + \tfrac{1}{2}(m+1)(m+2)A_mx^my^3, \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ 0 &= B_n + Ay^{n+1} + (n+1)A_1xy^{n+1} + \tfrac{1}{2}(n+1)(n+2)A_2x^2y^{n+1} + \&\text{c.} \\ &\qquad + \frac{(m+n)\,!}{m\,!\,n\,!}A_mx^my^{n+1}. \end{aligned}\right\} \tag{13}$$
To determine the potentials, $a$ and $\beta$, of the two spheres we have the equations (7) and (8), which we may now write
$$ca = A\frac{1}{x} + B + B_1y + B_2y^2 + \&\text{c.} + B_ny^n, \tag{14}$$
$$c\beta = B\frac{1}{y} + A + A_1x + A_2x^2 + \&\text{c.} + A_mx^m. \tag{15}$$

If, therefore, we confine our attention to the coefficients $A_1$ to $A_m$ and $B_1$ to $B_n$, we have $m+n$ equations from which to determine these quantities in terms of $A$ and $B$, the charges of the two spheres, and by inserting the values of these coefficients in (14) and (15) we may express the potentials of the spheres in terms of their charges.

These operations may be expressed in the form of determinants, but for purposes of calculation it is more convenient to proceed as follows.

Inserting in equations (12) the values of $B_1 \dots B_n$ from equations (13), we find

$$\begin{aligned} A_1 = -Bx^2 &+ A\ x^2y^3[2.1+3.1y^2+4.1y^4+5.1y^6+6.1y^8+\dots] \\ &+A_1x^3y^3[2.2+3.3y^2+4.4y^4+5.5y^6+\dots] \\ &+A_2x^4y^3[2.3+3.6y^2+4.10y^4+\dots] \\ &+A_3x^5y^3[2.4+3.10y^2+\dots] \\ &+A_4x^6y^3[2.5+\dots] \\ &+\dots\dots\dots \end{aligned} \tag{16}$$

$$\begin{aligned} A_2 = -Bx^3 &+ A\ x^3y^3[3.1+6.1y^2+10.1y^4+15.1y^6+\dots] \\ &+A_1x^4y^3[3.2+6.3y^2+10.4y^4+\dots] \\ &+A_2x^5y^3[3.3+6.6y^2+\dots] \\ &+A_3x^6y^3[3.4+\dots] \\ &+\dots\dots\dots \end{aligned} \tag{17}$$

$$\begin{aligned} A_3 = -Bx^4 &+ A\ x^4y^3[4.1+10.1y^2+20.1y^4+\dots] \\ &+A_1x^5y^3[4.2+10.3y^2+\dots] \\ &+A_2x^6y^3[4.3+\dots] \\ &+\dots\dots\dots \end{aligned} \tag{18}$$

$$\begin{aligned} A_4 = -Bx^5 &+ A\ x^5y^3[5.1+15.1y^2+\dots] \\ &+A_1x^6y^3[5.2+\dots] \\ &+\dots\dots\dots \end{aligned} \tag{19}$$

By substituting in the second members of these equations the approximate values of $A_1$ &c., and repeating the process for further approximations, we may carry the approximation to the coefficient to any extent in ascending powers and products of $x$ and $y$. If we write

$$A_n = p_nA - q_nB,$$
$$B_n = -r_nA + s_nB,$$

we find

$$
\begin{aligned}
p_1 = {} & x^2y^3\,[\,2+3y^2+4y^4+5y^6+6y^8+7y^{10}+8y^{12}+9y^{14}+\&c.]\\
&+x^5y^6\,[\,8+30y^2+75y^4+154y^6+280y^8+\&c.]\\
&+x^7y^6\,[18+90y^2+288y^4+735y^6+\&c.]\\
&+x^9y^6\,[32+200y^2+780y^4+\&c.]\\
&+x^{11}y^6[50+375y^2+\&c.]\\
&+x^{13}y^6[72+\&c.]\\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\
&+x^8y^9\,[32+192y^2+\&c.]\\
&+x^{10}y^9[144+\&c.]\\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot
\end{aligned}
\tag{20}
$$

$$
\begin{aligned}
q_1 = {} & x^2\\
&+x^5y^3\,[\,4+9y^2+16y^4+25y^6+36y^8+49y^{10}+64y^{12}+\&c.]\\
&+x^7y^3\,[\,6+18y^2+40y^4+75y^6+126y^8+196y^{10}+\&c.]\\
&+x^9y^3\,[\,8+30y^2+80y^4+175y^6+336y^8+\&c.]\\
&+x^{11}y^3[10+45y^2+140y^4+350y^6+\&c.]\\
&+x^{13}y^3[12+63y^2+224y^4+\&c.]\\
&+x^{15}y^3[14+84y^2+\&c.]\\
&+x^{17}y^3[16+\&c.]\\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\
&+x^8y^6\,[\,16+72y^2+209y^4+488y^6+\&c.]\\
&+x^{10}y^6[\,60+342y^2+1222y^4+\&c.]\\
&+x^{12}y^6[150+1050y^2+\&c.]\\
&+x^{14}y^6[308+\&c.]\\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\
&+x^{11}y^9[\,64+\&c.]\\
&+\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot
\end{aligned}
\tag{21}
$$

It will be more convenient in subsequent operations to write these coefficients in terms of $a$, $b$, and $c$, and to arrange the terms according to their dimensions in $c$. This will make it easier to differentiate with respect to $c$. We thus find

$$
\begin{aligned}
p_1 = {} & 2a^2b^3c^{-5}+3a^2b^5c^{-7}+4a^2b^7c^{-9}+(5a^2b^9+8a^5b^6)c^{-11}\\
&+(6a^2b^{11}+39a^5b^8+18a^7b^6)c^{-13}\\
&+(7a^2b^{13}+75a^5b^{10}+90a^7b^8+32a^9b^6)c^{-15}\\
&+(8a^2b^{15}+154a^5b^{12}+288a^7b^{10}+32a^8b^9+200a^9b^8+50a^{11}b^6)c^{-17}
\end{aligned}
$$

$$+(9a^2b^{17}+280a^5b^{14}+735a^7b^{12}+192a^8b^{11}+780a^9b^{10}$$
$$+144a^{10}b^9+375a^{11}b^8+72a^{13}b^6)c^{-19}+\dots \quad (22)$$

$$q_1 = a^2c^{-2}+4a^5b^3c^{-8}+(6a^7b^3+9a^5b^5)c^{-10}$$
$$+(8a^9b^3+18a^7b^5+16a^5b^7)c^{-12}$$
$$+(10a^{11}b^3+30a^9b^5+16a^8b^6+40a^7b^7+25a^5b^9)c^{-14}$$
$$+(12a^{13}b^3+45a^{11}b^5+60a^{10}b^6+80a^9b^7$$
$$+72a^8b^8+75a^7b^9+36a^5b^{11})c^{-16}$$
$$+(14a^{15}b^3+63a^{13}b^5+150a^{12}b^6+140a^{11}b^7+342a^{10}b^8$$
$$+175a^9b^9+209a^8b^{10}+126a^7b^{11}+49a^5b^{13})c^{-18}$$
$$+(16a^{17}b^3+84a^{15}b^5+308a^{14}b^6+224a^{13}b^7+1050a^{12}b^8$$
$$+414a^{11}b^9+1222a^{10}b^{10}+336a^9b^{11}+488a^8b^{12}+196a^7b^{13}$$
$$+64a^5b^{15})c^{-20}+\dots \quad (23)$$

$$p_2 = 3a^3b^3c^{-6}+6a^3b^5c^{-8}+10a^3b^7c^{-10}+(12a^6b^6+15a^3b^9)c^{-12}$$
$$+(27a^8b^6+54a^6b^8+21a^3b^{11})c^{-14}$$
$$+(48a^{10}b^6+162a^8b^8+158a^6b^{10}+28a^3b^{13})c^{-16}$$
$$+(75a^{12}b^6+360a^{10}b^8+48a^9b^9+606a^8b^{10}$$
$$+372a^6b^{12}+36a^3b^{15})c^{-18}+\dots \quad (24)$$

$$q_2 = a^3c^{-3}+6a^6b^3c^{-9}+(9a^8b^3+18a^6b^5)c^{-11}$$
$$+(12a^{10}b^3+36a^8b^5+40a^6b^7)c^{-13}$$
$$+(15a^{12}b^3+60a^{10}b^5+24a^9b^6+100a^8b^7+75a^6b^9)c^{-15}$$
$$+(18a^{14}b^3+90a^{12}b^5+90a^{11}b^6+200a^{10}b^7$$
$$+126a^9b^8+225a^8b^9+126a^6b^{11})c^{-17}$$
$$+(21a^{16}b^3+126a^{14}b^5+225a^{13}b^6+350a^{12}b^7+594a^{11}b^8$$
$$+525a^{10}b^9+418a^9b^{10}+441a^8b^{11}+196a^6b^{13})c^{-19}+\dots \quad (25)$$

$$p_3 = 4a^4b^3c^{-7}+10a^4b^5c^{-9}+20a^4b^7c^{-11}+(16a^7b^6+35a^4b^9)c^{-13}$$
$$+(36a^9b^6+84a^7b^8+56a^4b^{11})c^{-15}$$
$$+(64a^{11}b^6+252a^9b^8+282a^7b^{10}+84a^4b^{13})c^{-17}+\dots \quad (26)$$

$$q_3 = a^4c^{-4}+8a^7b^3c^{-10}+(12a^9b^3+30a^7b^5)c^{-12}$$
$$+(16a^{11}b^3+60a^9b^5+80a^7b^7)c^{-14}$$
$$+(20a^{13}b^3+100a^{11}b^5+32a^{10}b^6+200a^9b^7+175a^7b^9)c^{-16}$$
$$+(24a^{15}b^3+150a^{13}b^5+120a^{12}b^6+400a^{11}b^7+192a^{10}b^8$$
$$+525a^9b^9+336a^7b^{11})c^{-18}+\dots \quad (27)$$

$$p_4 = 5a^5b^3c^{-8} + 15a^5b^5c^{-10} + 35a^5b^7c^{-12} + (20a^8b^6 + 70a^5b^9)c^{-14}$$
$$+ (45a^{10}b^6 + 120a^8b^8 + 126a^5b^{11})c^{-16} + \dots \quad (28)$$

$$q_4 = a^5c^{-5} + 10a^8b^3c^{-11} + (15a^{10}b^3 + 45a^8b^5)c^{-13}$$
$$+ (20a^{12}b^3 + 90a^{10}b^5 + 140a^8b^7)c^{-15}$$
$$+ (25a^{14}b^3 + 150a^{12}b^5 + 40a^{11}b^6 + 350a^{10}b^7 + 350a^8b^9)c^{-17} + \dots \quad (29)$$

$$p_5 = 6a^6b^3c^{-9} + 21a^6b^5c^{-11} + 56a^6b^7c^{-13}$$
$$+ (24a^9b^6 + 126a^6b^9)c^{-15} + \dots \quad (30)$$

$$q_5 = a^6c^{-6} + 12a^9b^3c^{-12} + (18a^{11}b^3 + 63a^9b^5)c^{-14}$$
$$+ (24a^{13}b^3 + 126a^{11}b^5 + 224a^9b^7)c^{-16} + \dots \quad (31)$$

$$p_6 = 7a^7b^3c^{-10} + 28a^7b^5c^{-12} + 84a^7b^7c^{-14} + \dots \quad (32)$$

$$q_6 = a^7c^{-7} + 14a^{10}b^3c^{-13} + (21a^{12}b^3 + 84a^{10}b^5)c^{-15} + \dots \quad (33)$$

$$p_7 = 8a^8b^3c^{-11} + 36a^8b^5c^{-13} + \dots \quad (34)$$

$$q_7 = a^8c^{-8} + 16a^{11}b^3c^{-14} + \dots \quad (35)$$

$$p_8 = 9a^9b^3c^{-12} + \dots \quad (36)$$

$$q_8 = a^9c^{-9} + \dots \quad (37)$$

The values of the $r$'s and $s$'s may be written down by interchanging $a$ and $b$ in the $q$'s and $p$'s respectively.

If we now calculate the potentials of the two spheres in terms of these coefficients in the form

$$\alpha = lA \;\; + mB, \quad (38)$$

$$\beta = mA + nB, \quad (39)$$

then $l, m, n$ are the coefficients of potential (Art. 87), and of these

$$m = c^{-1} + p_1ac^{-2} + p_2a^2c^{-3} + \&c., \quad (40)$$

$$n = b^{-1} - q_1ac^{-2} - q_2a^2c^{-3} - \&c., \quad (41)$$

or, expanding in terms of $a$, $b$, $c$,

$$m = c^{-1} + 2a^3b^3c^{-7} + 3a^3b^3(a^2+b^2)c^{-9} + a^3b^3(4a^4 + 6a^2b^2 + 4b^4)c^{-11}$$
$$+ a^3b^3[5a^6 + 10a^4b^2 + 8a^3b^3 + 10a^2b^4 + 5b^6]c^{-13}$$
$$+ a^3b^3[6a^8 + 15a^6b^2 + 30a^5b^3 + 20a^4b^4$$
$$+ 30a^3b^5 + 15a^2b^6 + 6b^8]c^{-15}$$
$$+ a^3b^3[7a^{10} + 21a^8b^2 + 75a^7b^3 + 35a^6b^4 + 144a^5b^5$$
$$+ 35a^4b^6 + 75a^3b^7 + 21a^2b^8 + 7b^{10}]c^{-17}$$
$$+ a^3b^3[8a^{12} + 28a^{10}b^2 + 154a^9b^3 + 56a^8b^4 + 446a^7b^5 + 102a^6b^6$$
$$+ 446a^5b^7 + 56a^4b^8 + 154a^3b^9 + 28a^2b^{10} + 8b^{12}]c^{-19}$$

$$+a^3b^3[9a^{14}+36a^{12}b^2+280a^{11}b^3+84a^{10}b^4+1107a^9b^5+318a^8b^6$$
$$+1668a^7b^7+318a^6b^8+1107a^5b^9+84a^4b^{10}+280a^3b^{11}$$
$$+36a^2b^{12}+9b^{14}]c^{-21}+\dots. \tag{42}$$

$$n = b^{-1}-a^3c^{-4}-a^5c^{-6}-a^7c^{-8}-(a^3+4b^3)a^6c^{-10}$$
$$-(a^5+12a^2b^3+9b^5)a^6c^{-12}-(a^7+25a^4b^3+36a^2b^5+16b^7)a^6c^{-14}$$
$$-(a^9+44a^6b^3+96a^4b^5+16a^3b^6+80a^2b^7+25b^9)a^6c^{-16}$$
$$-(a^{11}+70a^8b^3+210a^6b^5+84a^5b^6+260a^4b^7$$
$$+72a^3b^8+150a^2b^9+36b^{11})a^6c^{-18}$$
$$-(a^{13}+104a^{10}b^3+406a^8b^5+272a^7b^6+680a^6b^7+468a^5b^8$$
$$+575a^4b^9+209a^3b^{10}+252a^2b^{11}+49b^{13})a^6c^{-20}$$
$$-(a^{15}+147a^{12}b^3+720a^{10}b^5+693a^9b^6+1548a^8b^7+1836a^7b^8$$
$$+1814a^6b^9+1640a^5b^{10}+1113a^4b^{11}+488a^3b^{12}$$
$$+392a^2b^{13}+64b^{15})a^6c^{-22}+\dots. \tag{43}$$

The value of $l$ can be obtained from that of $n$ by interchanging $a$ and $b$.

The potential energy of the system is, by Art. 87,

$$W = \tfrac{1}{2}lA^2+mAB+\tfrac{1}{2}nB^2, \tag{44}$$

and the repulsion between the two spheres is, by Art. 93$a$,

$$-\frac{dW}{dc} = \tfrac{1}{2}A^2\frac{dl}{dc}+AB\frac{dm}{dc}+\tfrac{1}{2}B^2\frac{dn}{dc}. \tag{45}$$

The surface density at any point of either sphere is given by equations (1) and (4) in terms of the coefficients $A_n$ and $B_n$.

# CHAPTER X.

## CONFOCAL QUADRIC SURFACES *

147.] Let the general equation of a confocal system be

$$\frac{x^2}{\lambda^2-a^2}+\frac{y^2}{\lambda^2-b^2}+\frac{z^2}{\lambda^2-c^2}=1, \tag{1}$$

where $\lambda$ is a variable parameter, which we shall distinguish by a suffix for the species of quadric, viz. we shall take $\lambda_1$ for the hyperboloids of two sheets, $\lambda_2$ for the hyperboloids of one sheet, and $\lambda_3$ for the ellipsoids. The quantities

$$a,\ \lambda_1,\ b,\ \lambda_2,\ c,\ \lambda_3$$

are in ascending order of magnitude. The quantity $a$ is introduced for the sake of symmetry, but in our results we shall always suppose $a = 0$.

If we consider the three surfaces whose parameters are $\lambda_1, \lambda_2, \lambda_3$, we find, by elimination between their equations, that the value of $x^2$ at their point of intersection satisfies the equation

$$x^2(b^2-a^2)(c^2-a^2) = (\lambda_1{}^2-a^2)(\lambda_2{}^2-a^2)(\lambda_3{}^2-a^2). \tag{2}$$

The values of $y^2$ and $z^2$ may be found by transposing $a, b, c$ symmetrically.

Differentiating this equation with respect to $\lambda_1$, we find

$$\frac{dx}{d\lambda_1}=\frac{\lambda_1}{\lambda_1{}^2-a^2}x. \tag{3}$$

If $ds_1$ is the length of the intercept of the curve of intersection of $\lambda_2$ and $\lambda_3$ cut off between the surfaces $\lambda_1$ and $\lambda_1+d\lambda_1$, then

$$\overline{\frac{ds_1}{d\lambda_1}}\Big|^2=\overline{\frac{dx}{d\lambda_1}}\Big|^2+\overline{\frac{dy}{d\lambda_1}}\Big|^2+\overline{\frac{dz}{d\lambda_1}}\Big|^2=\frac{\lambda_1{}^2(\lambda_2{}^2-\lambda_1{}^2)(\lambda_3{}^2-\lambda_1{}^2)}{(\lambda_1{}^2-a^2)(\lambda_1{}^2-b^2)(\lambda_1{}^2-c^2)}. \tag{4}$$

* This investigation is chiefly borrowed from a very interesting work,—*Leçons sur les Fonctions Inverses des Transcendantes et les Surfaces Isothermes.* Par G. Lamé, Paris, 1857.

The denominator of this fraction is the product of the squares of the semi-axes of the surface $\lambda_1$.

If we put

$$D_1^2 = \lambda_3^2-\lambda_2^2,\ D_2^2 = \lambda_3^2-\lambda_1^2,\ \text{and}\ D_3^2 = \lambda_2^2-\lambda_1^2, \tag{5}$$

and if we make $a = 0$, then

$$\frac{ds_1}{d\lambda_1} = \frac{D_2 D_3}{\sqrt{b^2-\lambda_1^2}\sqrt{c^2-\lambda_1^2}}. \tag{6}$$

It is easy to see that $D_2$ and $D_3$ are the semi-axes of the central section of $\lambda_1$ which is conjugate to the diameter passing through the given point, and that $D_3$ is parallel to $ds_2$, and $D_2$ to $ds_3$.

If we also substitute for the three parameters $\lambda_1$, $\lambda_2$, $\lambda_3$ their values in terms of three functions $\alpha$, $\beta$, $\gamma$, defined by the equations

$$\left.\begin{aligned} \alpha &= \int_0^{\lambda_1} \frac{c\,d\lambda_1}{\sqrt{(b^2-\lambda_1^2)(c^2-\lambda_1^2)}}, \\ \beta &= \int_b^{\lambda_2} \frac{c\,d\lambda_2}{\sqrt{(\lambda_2^2-b^2)(c^2-\lambda_2^2)}}, \\ \gamma &= \int_c^{\lambda_3} \frac{c\,d\lambda_3}{\sqrt{(\lambda_3^2-b^2)(\lambda_3^2-c^2)}}; \end{aligned}\right\} \tag{7}$$

then $ds_1 = \dfrac{1}{c} D_2 D_3 d\alpha,\ \ ds_2 = \dfrac{1}{c} D_3 D_1 d\beta,\ \ ds_3 = \dfrac{1}{c} D_1 D_2 d\gamma.$ (8)

148.] Now let $V$ be the potential at any point, $\alpha$, $\beta$, $\gamma$, then the resultant force in the direction of $ds_1$ is

$$R_1 = -\frac{dV}{ds_1} = -\frac{dV}{d\alpha}\frac{d\alpha}{ds_1} = -\frac{dV}{d\alpha}\frac{c}{D_2 D_3}. \tag{9}$$

Since $ds_1$, $ds_2$, and $ds_3$ are at right angles to each other, the surface-integral over the element of area $ds_2 ds_3$ is

$$\begin{aligned} R_1 ds_2 ds_3 &= -\frac{dV}{d\alpha}\frac{c}{D_2 D_3}\cdot\frac{D_3 D_1}{c}\cdot\frac{D_1 D_2}{c}\cdot d\beta\, d\gamma \\ &= -\frac{dV}{d\alpha}\frac{D_1^2}{c}\, d\beta\, d\gamma. \end{aligned} \tag{10}$$

Now consider the element of volume intercepted between the surface $\alpha$, $\beta$, $\gamma$, and $\alpha+d\alpha$, $\beta+d\beta$, $\gamma+d\gamma$. There will be eight such elements, one in each octant of space.

We have found the surface-integral of the normal component of the force (measured inwards) for the element of surface intercepted from the surface $\alpha$ by the surfaces $\beta$ and $\beta+d\beta$, $\gamma$ and $\gamma+d\gamma$.

The surface-integral for the corresponding element of the surface $a+da$ will be

$$+\frac{dV}{da}\frac{D_1^2}{c}d\beta\,d\gamma+\frac{d^2V}{da^2}\frac{D_1^2}{c}da\,d\beta\,d\gamma$$

since $D_1$ is independent of $a$. The surface-integral for the two opposite faces of the element of volume will be the sum of these quantities, or

$$\frac{d^2V}{da^2}\frac{D_1^2}{c}\,da\,d\beta\,d\gamma.$$

Similarly the surface-integrals for the other two pairs of faces will be

$$\frac{d^2V}{d\beta^2}\frac{D_2^2}{c}\,da\,d\beta\,d\gamma \text{ and } \frac{d^2V}{d\gamma^2}\frac{D_3^2}{c}\,da\,d\beta\,d\gamma.$$

These six faces enclose an element whose volume is

$$ds_1\,ds_2\,ds_3=\frac{D_1^2D_2^2D_3^2}{c^3}\,da\,d\beta\,d\gamma,$$

and if $\rho$ is the volume-density within that element, we find by Art. 77 that the total surface-integral of the element, together with the quantity of electricity within it multiplied by $4\pi$, is zero, or, dividing by $da\,d\beta\,d\gamma$,

$$\frac{d^2V}{da^2}D_1^2+\frac{d^2V}{d\beta^2}D_2^2+\frac{d^2V}{d\gamma^2}D_3^2+4\pi\rho\frac{D_1^2D_2^2D_3^2}{c^2}=0, \qquad (11)$$

which is the form of Poisson's extension of Laplace's equation referred to ellipsoidal coordinates.

If $\rho=0$ the fourth term vanishes, and the equation is equivalent to that of Laplace.

For the general discussion of this equation the reader is referred to the work of Lamé already mentioned.

149.] To determine the quantities $a$, $\beta$, $\gamma$, we may put them in the form of ordinary elliptic integrals by introducing the auxiliary angles $\theta$, $\phi$, and $\psi$, where

$$\lambda_1=b\sin\theta, \qquad (12)$$

$$\lambda_2=\sqrt{c^2\sin^2\phi+b^2\cos^2\phi}, \qquad (13)$$

$$\lambda_3=c\sec\psi. \qquad (14)$$

If we put $b=kc$, and $k^2+k'^2=1$, we may call $k$ and $k'$ the two complementary moduli of the confocal system, and we find

$$a=\int_0^\theta\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}, \qquad (15)$$

an elliptic integral of the first kind, which we may write according to the usual notation $F(k,\theta)$.

In the same way we find

$$\beta = \int_0^{\phi} \frac{d\phi}{\sqrt{1-k'^2\cos^2\phi}} = F(k') - F(k', \phi), \tag{16}$$

where $F(k')$ is the complete function for modulus $k'$,

$$\gamma = \int_0^{\psi} \frac{d\psi}{\sqrt{1-k^2\cos^2\psi}} = F(k) - F(k, \psi). \tag{17}$$

Here $\alpha$ is represented as a function of the angle $\theta$, which is accordingly a function of the parameter $\lambda_1$, $\beta$ as a function of $\phi$ and thence of $\lambda_2$, and $\gamma$ as a function of $\psi$ and thence of $\lambda_3$.

But these angles and parameters may be considered as functions of $\alpha$, $\beta$, $\gamma$. The properties of such inverse functions, and of those connected with them, are explained in the treatise of M. Lamé on this subject.

It is easy to see that since the parameters are periodic functions of the auxiliary angles, they will be periodic functions of the quantities $\alpha$, $\beta$, $\gamma$: the periods of $\lambda_1$ and $\lambda_3$ are $4F(k)$, and that of $\lambda_2$ is $2F(k')$.

*Particular Solutions.*

150.] If $V$ is a linear function of $\alpha$, $\beta$, or $\gamma$, the equation is satisfied. Hence we may deduce from the equation the distribution of electricity on any two confocal surfaces of the same family maintained at given potentials, and the potential at any point between them.

*The Hyperboloids of Two Sheets.*

When $\alpha$ is constant the corresponding surface is a hyperboloid of two sheets. Let us make the sign of $\alpha$ the same as that of $x$ in the sheet under consideration. We shall thus be able to study one of these sheets at a time.

Let $\alpha_1$, $\alpha_2$ be the values of $\alpha$ corresponding to two single sheets, whether of different hyperboloids or of the same one, and let $V_1$, $V_2$, be the potentials at which they are maintained. Then, if we make

$$V = \frac{\alpha_1 V_2 - \alpha_2 V_1 + \alpha(V_1 - V_2)}{\alpha_1 - \alpha_2} \tag{18}$$

the conditions will be satisfied at the two surfaces and throughout the space between them. If we make $V$ constant and equal to $V_1$ in the space beyond the surface $\alpha_1$, and constant and equal to $V_2$ in the space beyond the surface $\alpha_2$, we shall have obtained the complete solution of this particular case.

The resultant force at any point of either sheet is

$$\pm R_1 = -\frac{dV}{ds_1} = -\frac{dV}{da}\frac{da}{ds_1}, \tag{19}$$

$$\text{or} \quad R_1 = \frac{V_1 - V_2}{a_2 - a_1}\frac{c}{D_2 D_3}. \tag{20}$$

If $p_1$ be the perpendicular from the centre on the tangent plane at any point, and $P_1$ the product of the semi-axes of the surface, then $p_1 D_2 D_3 = P_1$.

Hence we find
$$R_1 = \frac{V_1 - V_2}{a_2 - a_1}\frac{cp_1}{P_1}, \tag{21}$$

or the force at any point of the surface is proportional to the perpendicular from the centre on the tangent plane.

The surface-density $\sigma$ may be found from the equation

$$4\pi\sigma = R_1. \tag{22}$$

The total quantity of electricity on a segment cut off by a plane whose equation is $x = d$ from one sheet of the hyperboloid is

$$Q = \frac{c}{2}\frac{V_1 - V_2}{a_2 - a_1}\left(\frac{d}{\lambda_1} - 1\right). \tag{23}$$

The quantity on the whole infinite sheet is therefore infinite.

The limiting forms of the surface are:—

(1) When $a = F(k)$ the surface is the part of the plane of $xz$ on the positive side of the positive branch of the hyperbola whose equation is
$$\frac{x^2}{b^2} - \frac{z^2}{c^2 - b^2} = 1. \tag{24}$$

(2) When $a = 0$ the surface is the plane of $yz$.

(3) When $a = -F(k)$ the surface is the part of the plane of $xz$ on the negative side of the negative branch of the same hyperbola.

### *The Hyperboloid of One Sheet.*

By making $\beta$ constant we obtain the equation of the hyperboloid of one sheet. The two surfaces which form the boundaries of the electric field must therefore belong to two different hyperboloids. The investigation will in other respects be the same as for the hyperboloids of two sheets, and when the difference of potentials is given the density at any point of the surface will be proportional to the perpendicular from the centre on the tangent plane, and the whole quantity on the infinite sheet will be infinite.

*Limiting Forms.*

(1) When $\beta = 0$ the surface is the part of the plane of $xz$ between the two branches of the hyperbola whose equation is written above, (24).

(2) When $\beta = F(k')$ the surface is the part of the plane of $xy$ which is on the outside of the focal ellipse whose equation is

$$\frac{x^2}{c^2} + \frac{y^2}{c^2 - b^2} = 1. \tag{25}$$

*The Ellipsoids.*

For any given ellipsoid $\gamma$ is constant. If two ellipsoids, $\gamma_1$ and $\gamma_2$, be maintained at potentials $V_1$ and $V_2$, then, for any point $\gamma$ in the space between them, we have

$$V = \frac{\gamma_1 V_2 - \gamma_2 V_1 + \gamma (V_1 - V_2)}{\gamma_1 - \gamma_2}. \tag{26}$$

The surface-density at any point is

$$\sigma = -\frac{1}{4\pi} \frac{V_1 - V_2}{\gamma_1 - \gamma_2} \frac{c p_3}{P_3}, \tag{27}$$

where $p_3$ is the perpendicular from the centre on the tangent plane, and $P_3$ is the product of the semi-axes.

The whole charge of electricity on either surface is given by

$$Q_2 = c \frac{V_1 - V_2}{\gamma_1 - \gamma_2} = -Q_1, \tag{28}$$

and is finite.

When $\gamma = F(k)$ the surface of the ellipsoid is at an infinite distance in all directions.

If we make $V_2 = 0$ and $\gamma_2 = F(k)$, we find for the quantity of electricity on an ellipsoid $\gamma$ maintained at potential $V$ in an infinitely extended field,

$$Q = c \frac{V}{F(k) - \gamma}. \tag{29}$$

The limiting form of the ellipsoids occurs when $\gamma = 0$, in which case the surface is the part of the plane of $xy$ within the focal ellipse, whose equation is written above, (25).

The surface-density on either side of the elliptic plate whose equation is (25), and whose eccentricity is $k$, is

$$\sigma = \frac{V}{4\pi\sqrt{c^2 - b^2}} \frac{1}{F(k)} \frac{1}{\sqrt{1 - \dfrac{x^2}{c^2} - \dfrac{y^2}{c^2 - b^2}}}, \tag{30}$$

and its charge is

$$Q = c \frac{V}{F(k)}. \tag{31}$$

*Particular Cases.*

151.] If $c$ remains finite, while $b$ and therefore $k$ is diminished till it becomes ultimately zero, the system of surfaces becomes transformed in the following manner:—

The real axis and one of the imaginary axes of each of the hyperboloids of two sheets are indefinitely diminished, and the surface ultimately coincides with two planes intersecting in the axis of $z$.

The quantity $a$ becomes identical with $\theta$, and the equation of the system of meridional planes to which the first system is reduced is

$$\frac{x^2}{(\sin a)^2} - \frac{y^2}{(\cos a)^2} = 0. \tag{32}$$

As regards the quantity $\beta$, if we take the definition given in page 233, (7), we shall be led to an infinite value of the integral at the lower limit. In order to avoid this we define $\beta$ in this particular case as the value of the integral

$$\int_{\lambda_2}^{c} \frac{c\,d\lambda_2}{\lambda_2\sqrt{c^2-\lambda_2^2}}.$$

If we now put $\lambda_2 = c\sin\phi$, $\beta$ becomes

$$\int_{\phi}^{\frac{\pi}{2}} \frac{d\phi}{\sin\phi}, \qquad \text{i. e. } \log\cot\tfrac{1}{2}\phi;$$

whence
$$\cos\phi = \frac{e^{\beta}-e^{-\beta}}{e^{\beta}+e^{-\beta}}, \tag{33}$$

and therefore
$$\sin\phi = \frac{2}{e^{\beta}+e^{-\beta}}. \tag{34}$$

If we call the exponential quantity $\frac{1}{2}(e^{\beta}+e^{-\beta})$ the hyperbolic cosine of $\beta$, or more concisely the hypocosine of $\beta$, or cosh $\beta$, and if we call $\frac{1}{2}(e^{\beta}-e^{-\beta})$ the hyposine of $\beta$, or sinh $\beta$, and if in the same way we employ functions of a similar character analogous to the other simple trigonometrical ratios, then $\lambda_2 = c\,\mathrm{sech}\,\beta$, and the equation of the system of hyperboloids of one sheet is

$$\frac{x^2+y^2}{(\mathrm{sech}\,\beta)^2} - \frac{z^2}{(\tanh\beta)^2} = c^2. \tag{35}$$

The quantity $\gamma$ is reduced to $\psi$, so that $\lambda_3 = c\sec\gamma$, and the equation of the system of ellipsoids is

$$\frac{x^2+y^2}{(\sec\gamma)^2} + \frac{z^2}{(\tan\gamma)^2} = c^2. \tag{36}$$

Ellipsoids of this kind, which are figures of revolution about their conjugate axes, are called planetary ellipsoids.

The quantity of electricity on a planetary ellipsoid maintained at potential $V$ in an infinite field, is

$$Q = c\frac{V}{\frac{1}{2}\pi - \gamma}, \tag{37}$$

where $c \sec\gamma$ is the equatorial radius, and $c \tan\gamma$ is the polar radius.

If $\gamma = 0$, the figure is a circular disk of radius $c$, and

$$\sigma = \frac{V}{2\pi^2\sqrt{c^2 - r^2}}, \tag{38}$$

$$Q = c\frac{V}{\frac{1}{2}\pi}. \tag{39}$$

152.] *Second Case.* Let $b = c$, then $k = 1$ and $k' = 0$,

$$a = \log\tan\frac{\pi + 2\theta}{4}, \text{ whence } \lambda_1 = c\tanh a, \tag{40}$$

and the equation of the hyperboloids of revolution of two sheets becomes

$$\frac{x^2}{(\tanh a)^2} - \frac{y^2 + z^2}{(\operatorname{sech} a)^2} = c^2. \tag{41}$$

The quantity $\beta$ becomes reduced to $\phi$, and each of the hyperboloids of one sheet is reduced to a pair of planes intersecting in the axis of $x$ whose equation is

$$\frac{y^2}{(\sin\beta)^2} - \frac{z^2}{(\cos\beta)^2} = 0. \tag{42}$$

This is a system of meridional planes in which $\beta$ is the longitude.

The quantity $\gamma$ as defined in page 233, (7), becomes in this case infinite at the lower limit. To avoid this let us define it as the value of the integral $\displaystyle\int_{\lambda_3}^{\infty}\frac{c\,d\lambda_3}{\lambda_3^2 - c^2}$.

If we then put $\lambda_3 = c\sec\psi$, we find $\gamma = \displaystyle\int_{\psi}^{\frac{\pi}{2}}\frac{d\psi}{\sin\psi}$, whence $\lambda_3 = c\coth\gamma$, and the equation of the family of ellipsoids is

$$\frac{x^2}{(\coth\gamma)^2} + \frac{y^2 + z^2}{(\operatorname{cosech}\gamma)^2} = c^2. \tag{43}$$

These ellipsoids, in which the transverse axis is the axis of revolution, are called ovary ellipsoids.

The quantity of electricity on an ovary ellipsoid maintained at potential $V$ in an infinite field becomes in this case, by (29),

$$cV \div \int_{\psi_0}^{\frac{\pi}{2}}\frac{d\psi}{\sin\psi}, \tag{44}$$

where $c\sec\psi_0$ is the polar radius.

If we denote the polar radius by $A$ and the equatorial by $B$, the result just found becomes

$$V\frac{\sqrt{A^2-B^2}}{\log\dfrac{A+\sqrt{A^2-B^2}}{B}}. \tag{45}$$

If the equatorial radius is very small compared to the polar radius, as in a wire with rounded ends,

$$Q=\frac{AV}{\log 2A-\log B}. \tag{46}$$

When both $b$ and $c$ become zero, their ratio remaining finite, the system of surfaces becomes two systems of confocal cones, and a system of spherical surfaces of which the radii are inversely proportional to $\gamma$.

If the ratio of $b$ to $c$ is zero or unity, the system of surfaces becomes one system of meridian planes, one system of right cones having a common axis, and a system of concentric spherical surfaces of which the radii are inversely proportional to $\gamma$. This is the ordinary system of spherical polar coordinates.

### *Cylindric Surfaces.*

153.] When $c$ is infinite the surfaces are cylindric, the generating lines being parallel to the axes of $z$. One system of cylinders is hyperbolic, viz. that into which the hyperboloids of two sheets degenerate. Since, when $c$ is infinite, $k$ is zero, and therefore $\theta=a$, it follows that the equation of this system is

$$\frac{x^2}{\sin^2 a}-\frac{y^2}{\cos^2 a}=b^2. \tag{47}$$

The other system is elliptic, and since when $k=0$, $\beta$ becomes

$$\int_b^{\lambda_2}\frac{d\lambda_2}{\sqrt{\lambda_2{}^2-b^2}},\quad\text{or}\quad \lambda_2=b\cosh\beta,$$

the equation of this system is

$$\frac{x^2}{(\cosh\beta)^2}+\frac{y^2}{(\sinh\beta)^2}=b^2. \tag{48}$$

These two systems are represented in Fig. X at the end of this volume.

### *Confocal Paraboloids.*

154.] If in the general equations we transfer the origin of coordinates to a point on the axis of $x$ distant $t$ from the centre of the system, and if for $x$, $\lambda$, $b$, and $c$ we substitute $t+x$, $t+\lambda$, $t+b$, and $t+c$ respectively, and then make $t$ increase indefinitely, we obtain, in the limit, the equation of a system of paraboloids whose foci are at the points $x=b$ and $x=c$, viz. the equation is

$$4(x-\lambda)+\frac{y^2}{\lambda-b}+\frac{z^2}{\lambda-c}=0. \tag{49}$$

If the variable parameter is $\lambda$ for the first system of elliptic paraboloids, $\mu$ for the hyperbolic paraboloids, and $\nu$ for the second system of elliptic paraboloids, we have $\lambda$, $b$, $\mu$, $c$, $\nu$ in ascending order of magnitude, and

$$\left.\begin{aligned} x &= \lambda+\mu+\nu-c-b, \\ y^2 &= 4\frac{(b-\lambda)(\mu-b)(\nu-b)}{c-b}, \\ z^2 &= 4\frac{(c-\lambda)(c-\mu)(\nu-c)}{c-b}. \end{aligned}\right\} \tag{50}$$

In order to avoid infinite values in the integrals (7) the corresponding integrals in the paraboloidal system are taken between different limits.

We write in this case

$$\alpha=\int_\lambda^b \frac{d\lambda}{\sqrt{(b-\lambda)(c-\lambda)}},$$

$$\beta=\int_b^\mu \frac{d\mu}{\sqrt{(\mu-b)(c-\mu)}},$$

$$\gamma=\int_c^\nu \frac{d\nu}{\sqrt{(\nu-b)(\nu-c)}}.$$

From these we find

$$\left.\begin{aligned} \lambda &= \tfrac{1}{2}(c+b)-\tfrac{1}{2}(c-b)\cosh\alpha, \\ \mu &= \tfrac{1}{2}(c+b)-\tfrac{1}{2}(c-b)\cos\beta, \\ \nu &= \tfrac{1}{2}(c+b)+\tfrac{1}{2}(c-b)\cosh\gamma; \end{aligned}\right\} \tag{51}$$

$$\left.\begin{aligned} x &= \tfrac{1}{2}(c+b)+\tfrac{1}{2}(c-b)(\cosh\gamma-\cos\beta-\cosh\alpha), \\ y &= 2(c-b)\sinh\frac{\alpha}{2}\sin\frac{\beta}{2}\cosh\frac{\gamma}{2}, \\ z &= 2(c-b)\cosh\frac{\alpha}{2}\cos\frac{\beta}{2}\sinh\frac{\gamma}{2}. \end{aligned}\right\} \tag{52}$$

When $b=c$ we have the case of paraboloids of revolution about the axis of $x$, and {see foot note}

$$\begin{aligned} x &= a(e^{2\alpha}-e^{2\gamma}), \\ y &= 2ae^{\alpha+\gamma}\cos\beta, \\ z &= 2ae^{\alpha+\gamma}\sin\beta. \end{aligned} \tag{53}$$

The surfaces for which $\beta$ is constant are planes through the axis, $\beta$ being the angle which such a plane makes with a fixed plane through the axis.

The surfaces for which $a$ is constant are confocal paraboloids. When $a = -\infty$ the paraboloid is reduced to a straight line terminating at the origin.

We may also find the values of $a$, $\beta$, $\gamma$ in terms of $r$, $\theta$, and $\phi$, the spherical polar coordinates referred to the focus as origin, and the axis of the paraboloids as the axis of $\theta$,

$$\left.\begin{aligned} a &= \log\left(r^{\frac{1}{2}} \cos \tfrac{1}{2}\theta\right), \\ \beta &= \phi, \\ \gamma &= \log\left(r^{\frac{1}{2}} \sin \tfrac{1}{2}\theta\right). \end{aligned}\right\} \qquad (54)$$

We may compare the case in which the potential is equal to $a$, with the zonal solid harmonic $r^i\, Q_i$. Both satisfy Laplace's equation, and are homogeneous functions of $x$, $y$, $z$, but in the case derived from the paraboloid there is a discontinuity at the axis {since $a$ is altered by writing $\theta + 2\pi$ for $\theta$}.

The surface-density on an electrified paraboloid in an infinite field (including the case of a straight line infinite in one direction) is inversely as the square root of the distance from the focus, or, in the case of the line, from the extremity of the line *.

* {The results of Art. 154 can be deduced as follows. Changing the variables from $x, y, z$ to $\lambda, \mu, \nu$, Laplace's equation becomes

$$\frac{d}{d\lambda}\left\{\frac{(\mu-\nu)(b-\lambda)^{\frac{1}{2}}(c-\lambda)^{\frac{1}{2}}}{(\mu-b)^{\frac{1}{2}}(c-\mu)^{\frac{1}{2}}(\nu-b)^{\frac{1}{2}}(\nu-c)^{\frac{1}{2}}}\frac{d\phi}{d\lambda}\right\} + \ldots = 0,$$

or $$(\nu-\mu)\{b-\lambda\}^{\frac{1}{2}}\{c-\lambda\}^{\frac{1}{2}}\frac{d}{d\lambda}\left\{(b-\lambda)^{\frac{1}{2}}(c-\lambda)^{\frac{1}{2}}\frac{d\phi}{d\lambda}\right\}$$
$$+(\nu-\lambda)\{\mu-b\}^{\frac{1}{2}}\{c-\mu\}^{\frac{1}{2}}\frac{d}{d\mu}\left\{(\mu-b)^{\frac{1}{2}}(c-\mu)^{\frac{1}{2}}\frac{d\phi}{d\mu}\right\}$$
$$+(\mu-\lambda)\{\nu-b\}^{\frac{1}{2}}\{\nu-c\}^{\frac{1}{2}}\frac{d}{d\nu}\left\{(\nu-b)^{\frac{1}{2}}(\nu-c)^{\frac{1}{2}}\frac{d\phi}{d\nu}\right\} = 0$$

or if
$$\frac{da}{d\lambda} = \frac{1}{(b-\lambda)^{\frac{1}{2}}(c-\lambda)^{\frac{1}{2}}},$$
$$\frac{d\beta}{d\mu} = \frac{1}{(\mu-b)^{\frac{1}{2}}(c-\mu)^{\frac{1}{2}}},$$
$$\frac{d\gamma}{d\nu} = \frac{1}{(\nu-b)^{\frac{1}{2}}(\nu-c)^{\frac{1}{2}}},$$

Laplace's equation becomes

$$(\nu-\mu)\frac{d^2\phi}{da^2} + (\nu-\lambda)\frac{d^2\phi}{d\beta^2} + (\mu-\lambda)\frac{d^2\phi}{d\gamma^2} = 0.$$

So that a linear function of $a$, $\beta$, $\gamma$ satisfies Laplace's equation.

When $b = c$, we may take

$$a = -\int_0^\lambda \frac{d\lambda}{b-\lambda},$$

$$\gamma = \int_{2b}^\nu \frac{d\nu}{\nu - b},$$

$$\lambda = \{b1 - e^a\},$$

$$\nu = b\{1 + e^\gamma\}.$$

From (51)

$$(\mu - b) = \tfrac{1}{2}(c-b)\{1 - \cos\beta\},$$

$$(c-\mu) = \tfrac{1}{2}(c-b)\{1+\cos\beta\};$$

hence from (50),

$$x = b + b(e^\gamma - e^a),$$

$$y^2 = 4b^2 e^{\gamma+a}\sin^2\frac{\beta}{2},$$

$$z^2 = 4b^2 e^{\gamma+a}\cos^2\frac{\beta}{2}.$$

If we take the origin at the focus $x = b$, and write $2\beta'$ for $\beta$, $ae^{2\gamma'}$ for $b\epsilon^\gamma$, $a\epsilon^{2\beta}$ for $b\epsilon^a$, we get

$$x = e^{2\gamma'} - e2^{a'},$$

$$y = 2ae^{a'+\gamma'}\sin\beta'$$

$$z = 2a\epsilon^{a'+\gamma'}\cos\beta'.$$

From which equations of the form (54) may easily be deduced.

Since from these equations the force along the radius varies as $1/r$, the normal force, and therefore the surface-density, will vary as $\frac{1}{r}\cdot\frac{r}{p}$ where $p$ is the perpendicular from the focus on the tangent plane, thus the surface-density varies as $1/p$, and therefore inversely as the square root of $r$.}

# CHAPTER XI.

## THEORY OF ELECTRIC IMAGES AND ELECTRIC INVERSION.

155.] We have already shewn that when a conducting sphere is under the influence of a known distribution of electricity, the distribution of electricity on the surface of the sphere can be determined by the method of spherical harmonics.

For this purpose we require to expand the potential of the influenced system in a series of solid harmonics of positive degree, having the centre of the sphere as origin, and we then find a corresponding series of solid harmonics of negative degree, which express the potential due to the electrification of the sphere.

By the use of this very powerful method of analysis, Poisson determined the electrification of a sphere under the influence of a given electrical system, and he also solved the more difficult problem to determine the distribution of electricity on two conducting spheres in presence of each other. These investigations have been pursued at great length by Plana and others, who have confirmed the accuracy of Poisson.

In applying this method to the most elementary case of a sphere under the influence of a single electrified point, we require to expand the potential due to the electrified point in a series of solid harmonics, and to determine a second series of solid harmonics which express the potential, due to the electrification of the sphere, in the space outside.

It does not appear that any of these mathematicians observed that this second series expresses the potential due to an imaginary electrified point, which has no physical existence as an electrified point, but which may be called an electrical image, because the action of the surface on external points is the same as that which would be produced by the imaginary electrified point if the spherical surface was removed.

This discovery seems to have been reserved for Sir W. Thomson, who has developed it into a method of great power for the solution of electrical problems, and at the same time capable of being presented in an elementary geometrical form.

His original investigations, which are contained in the *Cambridge and Dublin Mathematical Journal*, 1848, are expressed in terms of the ordinary theory of attraction at a distance, and make no use of the method of potentials and of the general theorems of Chapter IV, though they were probably discovered by these methods. Instead, however, of following the method of the author, I shall make free use of the idea of the potential and of equipotential surfaces, whenever the investigation can be rendered more intelligible by such means.

*Theory of Electric Images.*

156.] Let $A$ and $B$, Figure 7, represent two points in a uniform dielectric medium of infinite extent. Let the charges of $A$ and $B$ be $e_1$ and $e_2$ respectively. Let $P$ be any point in space whose distances from $A$ and $B$ are $r_1$ and $r_2$ respectively. Then the value of the potential at $P$ will be

$$V = \frac{e_1}{r_1} + \frac{e_2}{r_2}. \qquad (1)$$

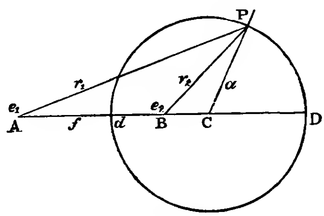


Fig. 7.

The equipotential surfaces due to this distribution of electricity are represented in Fig. I (at the end of this volume) when $e_1$ and $e_2$ are of the same sign, and in Fig. II when they are of opposite signs. We have now to consider that surface for which $V = 0$, which is the only spherical surface in the system. When $e_1$ and $e_2$ are of the same sign, this surface is entirely at an infinite distance, but when they are of opposite signs there is a plane or spherical surface at a finite distance over which the potential is zero.

The equation of this surface is

$$\frac{e_1}{r_1} + \frac{e_2}{r_2} = 0. \qquad (2)$$

Its centre is at a point $C$ in $AB$, produced, such that

$$AC : BC :: {e_1}^2 : {e_2}^2,$$

and the radius of the sphere is

$$AB \frac{e_1 e_2}{{e_1}^2 - {e_2}^2}.$$

The two points $A$ and $B$ are inverse points with respect to this sphere, that is to say, they lie in the same radius, and the radius is a mean proportional between their distances from the centre.

Since this spherical surface is at potential zero, if we suppose it constructed of thin metal and connected with the earth, there will be no alteration of the potential at any point either outside or inside, but the electrical action everywhere will remain that due to the two electrified points $A$ and $B$.

If we now keep the metallic shell in connection with the earth and remove the point $B$, the potential within the sphere will become everywhere zero, but outside it will remain the same as before. For the surface of the sphere still remains at the same potential, and no change has been made in the exterior electrification.

Hence, if an electrified point $A$ be placed outside a spherical conductor which is at potential zero, the electrical action at all points outside the sphere will be that due to the point $A$ together with another point $B$ within the sphere, which we may call the electrical image of $A$.

In the same way we may shew that if $B$ is a point placed inside the spherical shell, the electrical action within the sphere is that due to $B$, together with its image $A$.

157.] *Definition of an Electrical Image.* An electrical image is an electrified point or system of points on one side of a surface which would produce on the other side of that surface the same electrical action which the actual electrification of that surface really does produce.

In Optics a point or system of points on one side of a mirror or lens which if it existed would emit the system of rays which actually exists on the other side of the mirror or lens, is called a *virtual* image.

Electrical images correspond to virtual images in Optics in being related to the space on the other side of the surface. They do not correspond to them in actual position, or in the merely approximate character of optical foci.

There are no *real* electrical images, that is, imaginary electrified points which would produce, in the region on the same side of the electrified surface, an effect equivalent to that of the electrified surface.

For if the potential in any region of space is equal to that due to a certain electrification in the same region it must be actually produced by that electrification. In fact, the electrification at any point may be found from the potential near that point by the application of Poisson's equation.

Let $a$ be the radius of the sphere.

Let $f$ be the distance of the electrified point $A$ from the centre $C$.

Let $e$ be the charge of this point.

Then the image of the point is at $B$, on the same radius of the sphere at a distance $\frac{a^2}{f}$, and the charge of the image is $-e\frac{a}{f}$.

We have shewn that this image will produce the same effect on the opposite side of the surface as the actual electrification of the surface does. We shall next determine the surface-density of this electrification at any point $P$ of the spherical surface, and for this purpose we shall make use of the theorem of Coulomb, Art. 80, that if $R$ is the resultant force at the surface of a conductor, and $\sigma$ the superficial density,

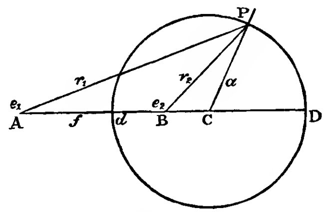



Fig. 7.

$$R = 4\pi\sigma,$$

$R$ being measured away from the surface.

We may consider $R$ as the resultant of two forces, a repulsion $\frac{e}{AP^2}$ acting along $AP$, and an attraction $e\frac{a}{f}\frac{1}{PB^2}$ acting along $PB$.

Resolving these forces in the directions of $AC$ and $CP$, we find that the components of the repulsion are

$$\frac{ef}{AP^3} \text{ along } AC, \text{ and } \frac{ea}{AP^3} \text{ along } CP.$$

Those of the attraction are

$$-e\frac{a}{f}\frac{1}{BP^3}BC \text{ along } AC, \text{ and } -e\frac{a^2}{f}\frac{1}{BP^3} \text{ along } CP.$$

Now $BP = \frac{a}{f}AP$, and $BC = \frac{a^2}{f}$, so that the components of the attraction may be written

$$-ef\frac{1}{AP^3} \text{ along } AC, \text{ and } -e\frac{f^2}{a}\frac{1}{AP^3} \text{ along } CP.$$

The components of the attraction and the repulsion in the direction of $AC$ are equal and opposite, and therefore the resultant force is entirely in the direction of the radius $CP$. This only confirms what we have already proved, that the sphere is an equipotential surface, and therefore a surface to which the resultant force is everywhere perpendicular.

The resultant force measured along $CP$, the normal to the surface in the direction towards the side on which $A$ is placed, is

$$R = -e\frac{f^2-a^2}{a}\frac{1}{AP^3}. \tag{3}$$

If $A$ is taken inside the sphere $f$ is less than $a$, and we must measure $R$ inwards. For this case therefore

$$R = -e\frac{a^2-f^2}{a}\frac{1}{AP^3}. \tag{4}$$

In all cases we may write

$$R = -e\frac{AD.Ad}{CP}\frac{1}{AP^3}, \tag{5}$$

where $AD$, $Ad$ are the segments of any line through $A$ cutting the sphere, and their product is to be taken positive in all cases.

158.] From this it follows, by Coulomb's theorem, Art. 80, that the surface-density at $P$ is

$$\sigma = -e\frac{AD.Ad}{4\pi.CP}\frac{1}{AP^3}. \tag{6}$$

The density of the electricity at any point of the sphere varies inversely as the cube of its distance from the point $A$.

The effect of this superficial distribution, together with that of the point $A$, is to produce on the same side of the surface as the point $A$ a potential equivalent to that due to $e$ at $A$, and its image $-e\frac{a}{f}$ at $B$, and on the other side of the surface the potential is everywhere zero. Hence the effect of the superficial distribution by itself is to produce a potential on the side of $A$ equivalent to that due to the image $-e\frac{a}{f}$ at $B$, and on the opposite side a potential equal and opposite to that of $e$ at $A$.

The whole charge on the surface of the sphere is evidently $-e\frac{a}{f}$ since it is equivalent to the image at $B$.

We have therefore arrived at the following theorems on the action of a distribution of electricity on a spherical surface, the surface-density being inversely as the cube of the distance from a point $A$ either without or within the sphere.

Let the density be given by the equation

$$\sigma = \frac{C}{AP^3}, \tag{7}$$

where $C$ is some constant quantity, then by equation (6)

$$C = -e\frac{AD.Ad}{4\pi a}. \tag{8}$$

The action of this superficial distribution on any point separated from $A$ by the surface is equal to that of a quantity of electricity $-e$, or

$$\frac{4\pi aC}{AD.Ad}$$

concentrated at $A$.

Its action on any point on the same side of the surface with $A$ is equal to that of a quantity of electricity

$$\frac{4\pi Ca^2}{f.AD.Ad}$$

concentrated at $B$ the image of $A$.

The whole quantity of electricity on the sphere is equal to the first of these quantities if $A$ is within the sphere, and to the second if $A$ is without the sphere.

These propositions were established by Sir W. Thomson in his original geometrical investigations with reference to the distribution of electricity on spherical conductors, to which the student ought to refer.

159.] If a system in which the distribution of electricity is known is placed in the neighbourhood of a conducting sphere of radius $a$, which is maintained at potential zero by connection with the earth, then the electrifications due to the several parts of the system will be superposed.

Let $A_1$, $A_2$, &c. be the electrified points of the system, $f_1$, $f_2$, &c. their distances from the centre of the sphere, $e_1$, $e_2$, &c. their charges, then the images $B_1$, $B_2$, &c. of these points will be in the same radii as the points themselves, and at distances $\frac{a^2}{f_1}$, $\frac{a^2}{f_2}$, &c. from the centre of the sphere, and their charges will be

$$-e_1\frac{a}{f_1}, \quad -e_2\frac{a}{f_2}, \text{ \&c.}$$

The potential on the outside of the sphere due to the superficial electrification will be the same as that which would be produced by the system of images $B_1, B_2$, &c. This system is therefore called the electrical image of the system $A_1, A_2$, &c.

If the sphere instead of being at potential zero is at potential $V$, we must superpose a distribution of electricity on its outer surface having the uniform surface-density

$$\sigma = \frac{V}{4\pi a}.$$

The effect of this at all points outside the sphere will be equal to that of a quantity $Va$ of electricity placed at its centre, and at all points inside the sphere the potential will be simply increased by $V$.

The whole charge on the sphere due to an external system of influencing points, $A_1, A_2$, &c. is

$$E = Va - e_1\frac{a}{f_1} - e_2\frac{a}{f_3} - \&c., \tag{9}$$

from which either the charge $E$ or the potential $V$ may be calculated when the other is given.

When the electrified system is within the spherical surface the induced charge on the surface is equal and of opposite sign to the inducing charge, as we have before proved it to be for every closed surface, with respect to points within it.

*160.] The energy due to the mutual action between an electrified point $e$, at a distance $f$ from the centre of the sphere

* [The discussion in the text will perhaps be more easily understood if the problem be regarded as an example of Art. 86. Let us then suppose that what is described as an electrified point is really a small spherical conductor, the radius of which is $b$ and the potential $v$. We have thus a particular case of the problem of two spheres of which one solution has already been given in Art. 146, and another will be given in Art. 173. In the case before us however the radius $b$ is so small that we may consider the electricity of the small conductor to be uniformly distributed over its surface and all the electric images except the first image of the small conductor to be disregarded. Since the charge $E$ on the sphere is given, we must in addition to the charge $-ea/f$ at the image have a charge $ea/f$ at the centre of the sphere.

We thus have $$V = \frac{E}{a} + \frac{e}{f},$$

$$v = \frac{E + e\frac{a}{f}}{f} - \frac{ea}{f^2 - a^2} + \frac{e}{b}.$$

The energy of the system is therefore, Art. 85,

$$\frac{E^2}{2a} + \frac{Ee}{f} + \frac{e^2}{2}\left(\frac{1}{b} - \frac{a^3}{f^2(f^2 - a^2)}\right).$$

By means of the above equations we may also express the energy in terms of the potentials: to the same order of approximation it is

$$\frac{aV^2}{2} - \frac{ab}{f}Vv + \frac{1}{2}\left(b + \frac{ab^2}{f^2 - a^2}\right)v^2.]$$

greater than $a$ the radius, and the electrification of the spherical surface due to the influence of the electrified point and the charge of the sphere, is

$$M = \frac{Ee}{f} - \frac{1}{2}\frac{e^2 a^3}{f^2(f^2-a^2)}, \tag{10}$$

$V$ is the potential, and $E$ the charge of the sphere.

The repulsion between the electrified point and the sphere is therefore, by Art. 92,

$$\begin{aligned} F &= ea\left(\frac{V}{f^2} - \frac{ef}{(f^2-a^2)^2}\right) \\ &= \frac{e}{f^2}\left(E - e\frac{a^3(2f^2-a^2)}{f(f^2-a^2)^2}\right). \end{aligned} \tag{11}$$

Hence the force between the point and the sphere is always an attraction in the following cases—

(1) When the sphere is uninsulated.

(2) When the sphere has no charge.

(3) When the electrified point is very near the surface.

In order that the force may be repulsive, the potential of the sphere must be positive and greater than $e\dfrac{f^3}{(f^2-a^2)^2}$, and the charge of the sphere must be of the same sign as $e$ and greater than $e\dfrac{a^3(2f^2-a^2)}{f(f^2-a^2)^2}$.

At the point of equilibrium the equilibrium is unstable, the force being an attraction when the bodies are nearer and a repulsion when they are farther off.

When the electrified point is within the spherical surface the force on the electrified point is always away from the centre of the sphere, and is equal to

$$\frac{e^2 af}{(a^2-f^2)^2}.$$

The surface-density at the point of the sphere nearest to the electrified point when it lies outside the sphere is

$$\begin{aligned} \sigma_1 &= \frac{1}{4\pi a^2}\left\{Va - e\frac{a(f+a)}{(f-a)^2}\right\} \\ &= \frac{1}{4\pi a^2}\left\{E - e\frac{a^2(3f-a)}{f(f-a)^2}\right\}. \end{aligned} \tag{12}$$

The surface-density at the point of the sphere farthest from the electrified point is

$$\sigma = \frac{1}{4\pi a^2}\left\{Va - e\frac{a(f-a)}{(f+a)^2}\right\}$$
$$= \frac{1}{4\pi a^2}\left\{E + e\frac{a^2(3f+a)}{f(f+a)^2}\right\}. \tag{13}$$

When $E$, the charge of the sphere, lies between

$$e\frac{a^2(3f-a)}{f(f-a)^2} \quad\text{and}\quad -e\frac{a^2(3f+a)}{f(f+a)^2}$$

the electrification will be negative next the electrified point and positive on the opposite side. There will be a circular line of division between the positively and the negatively electrified parts of the surface, and this line will be a line of equilibrium.

If
$$E = ea\left(\frac{1}{\sqrt{f^2-a^2}} - \frac{1}{f}\right), \tag{14}$$

the equipotential surface which cuts the sphere in the line of equilibrium is a sphere whose centre is the electrified point and whose radius is $\sqrt{f^2-a^2}$.

The lines of force and equipotential surfaces belonging to a case of this kind are given in Figure IV at the end of this volume.

*Images in an Infinite Plane Conducting Surface.*

161.] If the two electrified points $A$ and $B$ in Art. 156 are electrified with equal charges of electricity of opposite signs, the surface of zero potential will be the plane, every point of which is equidistant from $A$ and $B$.

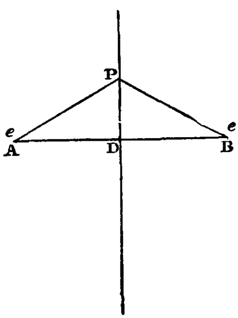


Fig. 8.

Hence, if $A$ be an electrified point whose charge is $e$, and $AD$ a perpendicular on the plane, produce $AD$ to $B$ so that $DB = AD$, and place at $B$ a charge equal to $-e$, then this charge at $B$ will be the image of $A$, and will produce at all points on the same side of the plane as $A$, an effect equal to that of the actual electrification of the plane. For the potential on the side of $A$ due to $A$ and $B$ fulfils the conditions that $\nabla^2 V = 0$ everywhere except at $A$, and that $V = 0$ at the plane, and there is only one form of $V$ which can fulfil these conditions.

To determine the resultant force at the point $P$ of the plane, we observe that it is compounded of two forces each equal to $\frac{e}{AP^2}$, one acting along $AP$ and the other along $PB$. Hence the resultant of these forces is in a direction parallel to $AB$ and equal to

$$\frac{e}{AP^2} \cdot \frac{AB}{AP}.$$

Hence $R$, the resultant force measured from the surface towards the space in which $A$ lies, is

$$R = -\frac{2eAD}{AP^3}, \tag{15}$$

and the density at the point $P$ is

$$\sigma = -\frac{eAD}{2\pi AP^3}. \tag{16}$$

*On Electrical Inversion.*

162.] The method of electrical images leads directly to a method of transformation by which we may derive from any electrical problem of which we know the solution any number of other problems with their solutions.

We have seen that the image of a point at a distance $r$ from the centre of a sphere of radius $R$ is in the same radius and at a distance $r'$ such that $rr' = R^2$. Hence the image of a system of points, lines, or surfaces is obtained from the original system by the method known in pure geometry as the method of inversion, and described by Chasles, Salmon, and other mathematicians.

If $A$ and $B$ are two points, $A'$ and $B'$ their images, $O$ being the centre of inversion, and $R$ the radius of the sphere of inversion,

$$OA.OA' = R^2 = OB.OB'.$$

Hence the triangles $OAB$, $OB'A'$ are similar, and $AB : A'B' :: OA : OB' :: OA.OB : R^2$.

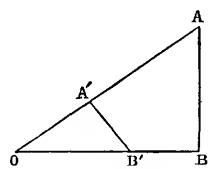


Fig. 9.

If a quantity of electricity $e$ be placed at $A$, its potential at $B$ will be $V = \frac{e}{AB}$.

If $e'$ be placed at $A'$, its potential at $B'$ will be

$$V' = \frac{e'}{A'B'}.$$

In the theory of electrical images

$$e : e' :: OA : R :: R : OA'.$$

Hence $$V : V' :: R : OB, \tag{17}$$

or the potential at $B$ due to the electricity at $A$ is to the potential at the image of $B$ due to the electrical image of $A$ as $R$ is to $OB$.

Since this ratio depends only on $OB$ and not on $OA$, the potential at $B$ due to any system of electrified bodies is to that at $B'$ due to the image of the system as $R$ is to $OB$.

If $r$ be the distance of any point $A$ from the centre, and $r'$ that of its image $A'$, and if $e$ be the electrification of $A$, and that of $A'$, also if $L$, $S$, $K$ be linear, superficial, and solid element at $A$, and $L'$, $S'$, $K'$ their images at $A'$, and $\lambda$, $\sigma$, $\rho$, $\lambda'$, $\sigma'$, $\rho'$ the corresponding line-surface and volume-densities of electricity at the two points, $V$ the potential at $A$ due to the original system, and $V'$ the potential at $A'$ due to the inverse system, then

$$\left.\begin{aligned} \frac{r'}{r} = \frac{L'}{L} = \frac{R^2}{r^2} = \frac{r'^2}{R^2}, \quad \frac{S'}{S} = \frac{R^4}{r^4} = \frac{r'^4}{R^4}, \quad \frac{K'}{K} = \frac{R^6}{r^6} = \frac{r'^6}{R^6}, \\ \frac{e'}{e} = \frac{R}{r} = \frac{r'}{R}, \quad \frac{\lambda'}{\lambda} = \frac{r}{R} = \frac{R}{r'}, \\ \frac{\sigma'}{\sigma} = \frac{r^3}{R^3} = \frac{R^3}{r'^3}, \quad \frac{\rho'}{\rho} = \frac{r^5}{R^5} = \frac{R^5}{r'^5}, \\ \frac{V'}{V} = \frac{r}{R} = \frac{R}{r'}. \end{aligned}\right\} \quad *(18)$$

If in the original system a certain surface is that of a conductor, and has therefore a constant potential $P$, then in the transformed system the image of the surface will have a potential $P\dfrac{R}{r'}$. But by placing at $O$, the centre of inversion, a quantity of electricity equal to $-PR$, the potential of the transformed surface is reduced to zero.

Hence, if we know the distribution of electricity on a conductor when insulated in open space and charged to the potential $P$, we can find by inversion the distribution on a conductor, whose form is the image of the first, under the influence of an electrified point with a charge $-PR$ placed at the centre of inversion, the conductor being in connexion with the earth.

163.] The following geometrical theorems are useful in studying cases of inversion.

Every sphere becomes, when inverted, another sphere, unless it passes through the centre of inversion, in which case it becomes a plane.

If the distances of the centres of the spheres from the centre

* See Thomson and Tait's *Natural Philosophy*, § 515.

of inversion are $a$ and $a'$, and if their radii are $\alpha$ and $\alpha'$, and if we define the *power* of a sphere with respect to the centre of inversion to be the product of the segments cut off by the sphere from a line through the centre of inversion, then the power of the first sphere is $a^2-\alpha^2$, and that of the second is $a'^2-\alpha'^2$. We have in this case

$$\frac{a'}{a}=\frac{\alpha'}{\alpha}=\frac{R^2}{a^2-\alpha^2}=\frac{a'^2-\alpha'^2}{R^2}, \tag{19}$$

or the ratio of the distances of the centres of the first and second spheres is equal to the ratio of their radii, and to the ratio of the power of the sphere of inversion to the power of the first sphere, or of the power of the second sphere to the power of the sphere of inversion.

The image of the centre of inversion with regard to one sphere is the inverse point of the centre of the other sphere.

In the case in which the inverse surfaces are a plane and a sphere, the perpendicular from the centre of inversion on the plane is to the radius of inversion as this radius is to the diameter of the sphere, and the sphere has its centre on this perpendicular and passes through the centre of inversion.

Every circle is inverted into another circle unless it passes through the centre of inversion, in which case it becomes a straight line.

The angle between two surfaces, or two lines at their intersection, is not changed by inversion.

Every circle which passes through a point and the image of that point with respect to a sphere, cuts the sphere at right angles.

Hence, any circle which passes through a point and cuts the sphere at right angles passes through the image of the point.

164.] We may apply the method of inversion to deduce the distribution of electricity on an uninsulated sphere under the influence of an electrified point from the uniform distribution on an insulated sphere not influenced by any other body.

If the electrified point be at $A$, take it for the centre of inversion, and if $A$ is at a distance $f$ from the centre of the sphere whose radius is $a$, the inverted figure will be a sphere whose radius is $a'$ and whose centre is distant $f'$, where

$$\frac{a'}{a}=\frac{f'}{f}=\frac{R^2}{f^2-a^2}. \tag{20}$$

The centre of either of these spheres corresponds to the inverse point of the other with respect to $A$, or if $C$ is the centre and $B$ the inverse point of the first sphere, $C'$ will be the inverse point, and $B'$ the centre of the second.

Now let a quantity $e'$ of electricity be communicated to the second sphere, and let it be uninfluenced by external forces. It will become uniformly distributed over the sphere with a surface-density
$$\sigma' = \frac{e'}{4\pi a'^2}. \tag{21}$$

Its action at any point outside the sphere will be the same as that of a charge $e'$ placed at $B'$ the centre of the sphere.

At the spherical surface and within it the potential is
$$P' = \frac{e'}{a'}, \tag{22}$$
a constant quantity.

Now let us invert this system. The centre $B'$ becomes in the inverted system the inverse point $B$, and the charge $e'$ at $B'$ becomes $e'\frac{R}{f'}$ at $B$, and at any point separated from $B$ by the surface the potential is that due to this charge at $B$.

The potential at any point $P$ on the spherical surface, or on the same side as $B$, is in the inverted system
$$\frac{e'}{a'}\frac{R}{AP}.$$

If we now superpose on this system a charge $e$ at $A$, where
$$e = -\frac{e'}{a'}R, \tag{23}$$
the potential on the spherical surface, and at all points on the same side as $B$, will be reduced to zero. At all points on the same side as $A$ the potential will be that due to a charge $e$ at $A$, and a charge $e'\frac{R}{f'}$ at $B$.

But
$$e'\frac{R}{f'} = -e\frac{a'}{f'} = -e\frac{a}{f}, \tag{24}$$
as we found before for the charge of the image at $B$.

To find the density at any point of the first sphere we have
$$\sigma = \sigma'\frac{R^3}{AP^3}. \tag{25}$$

Substituting for the value of $\sigma'$ in terms of the quantities belonging to the first sphere, we find the same value as in Art 158,
$$\sigma = -\frac{e(f^2-a^2)}{4\pi aAP^3}. \tag{26}$$

*On Finite Systems of Successive Images.*

165.] If two conducting planes intersect at an angle which is a submultiple of two right angles, there will be a finite system of images which will completely determine the electrification.

For let $AOB$ be a section of the two conducting planes perpendicular to their line of intersection, and let the angle of intersection $AOB = \frac{\pi}{n}$, let $P$ be an electrified point. Then, if we draw a circle with centre $O$ and radius $OP$, and find points which are the successive images of $P$ in the two planes beginning with $OB$, we shall find $Q_1$ for the image of $P$ in $OB$, $P_2$ for the image of $Q_1$ in $OA$, $Q_3$ for that of $P_2$ in $OB$, $P_3$ for that of $Q_3$ in $OA$, $Q_2$ for that of $P_3$ in $OB$, and so on.

If we had begun with the image of $P$ in $AO$ we should have found the same points in the reverse order $Q_2$, $P_3$, $Q_3$, $P_2$, $Q_1$, provided $AOB$ is a submultiple of two right angles.

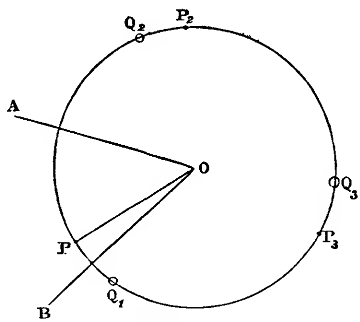


Fig. 10.

For the electrified point and the alternate images $P_2$, $P_3$ are ranged round the circle at angular intervals equal to $2AOB$, and the intermediate images $Q_1$, $Q_2$, $Q_3$ are at intervals of the same magnitude. Hence, if $2AOB$ is a submultiple of $2\pi$, there will be a finite number of images, and none of these will fall within the angle $AOB$. If, however, $AOB$ is not a submultiple of $\pi$, it will be impossible to represent the actual electrification as the result of a finite series of electrified points.

If $AOB = \frac{\pi}{n}$, there will be $n$ negative images $Q_1$, $Q_2$, &c., each equal and of opposite sign to $P$, and $n-1$ positive images $P_2$, $P_3$, &c., each equal to $P$, and of the same sign.

The angle between successive images of the same sign is $\frac{2\pi}{n}$. If we consider either of the conducting planes as a plane of symmetry, we shall find the electrified point and the positive and negative images placed symmetrically with regard to that plane, so that for every positive image there is a negative image in the same normal, and at an equal distance on the opposite side of the plane.

If we now invert this system with respect to any point, the two planes become two spheres, or a sphere and a plane intersecting at an angle $\frac{\pi}{n}$, the influencing point P, the inverse point of $P$, being within this angle.

The successive images lie on the circle which passes through P and intersects both spheres at right angles.

To find the position of the images we may make use of the principle that a point and its image in a sphere are in the same radius of the sphere, and draw successive chords of the circle on which the images lie beginning at P and passing through the centres of the two spheres alternately.

To find the charge which must be attributed to each image, take any point in the circle of intersection, then the charge of each image is proportional to its distance from this point, and its sign is positive or negative according as it belongs to the first or the second system.

166.] We have thus found the distribution of the images when any space bounded by a conductor consisting of two spherical surfaces meeting at an angle $\frac{\pi}{n}$, and kept at potential zero, is influenced by an electrified point.

We may by inversion deduce the case of a conductor consisting of two spherical segments meeting at a re-entering angle $\frac{\pi}{n}$, charged to potential unity and placed in free space.

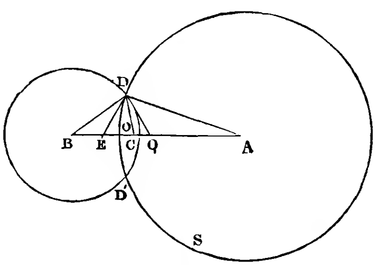


Fig. 11.

For this purpose we invert the system of planes with respect to $P$ and change the signs of the charges. The circle on which the images formerly lay now becomes a straight line through the centres of the spheres.

If the figure (11) represents a section through the line of centres $AB$, and if $D$, $D'$ are the points where the circle of intersection cuts the plane of the paper, then, to find the successive images, draw $DA$ a radius of the first circle, and draw $DC$, $DE$, &c., making angles $\frac{\pi}{n}$, $\frac{2\pi}{n}$, &c. with $DA$. The points $A$, $C$, $E$, &c. at which they cut the line of centres will be the positions of the positive images, and the charge of each will be represented by its distance from $D$. The last of these images will be at the centre of the second circle.

To find the negative images draw $DQ$, $DR$, &c., making angles $\frac{\pi}{n}$, $\frac{2\pi}{n}$, &c. with the line of centres. The intersections of these lines with the line of centres will give the positions of the negative images, and the charge of each will be represented by its distance from $D$ {for if $E$ and $Q$ are inverse points in the sphere $A$ the angles $ADE$, $AQD$ are equal}.

The surface-density at any point of either sphere is the sum of the surface-densities due to the system of images. For instance, the surface-density at any point $S$ of the sphere whose centre is $A$, is

$$\sigma = \frac{1}{4\pi DA}\left\{1 + (AD^2 - AB^2)\frac{DB}{BS^3} + (AD^2 - AC^2)\frac{DC}{CS^3} + \&c.\right\},$$

where $A$, $B$, $C$, &c. are the positive series of images.

When $S$ is on the circle of intersection the density is zero.

To find the total charge on one of the spherical segments, we may find the surface-integral of the induction through that segment due to each of the images.

The total charge on the segment whose centre is $A$ due to the image at $A$ whose charge is $DA$ is

$$DA\frac{DA + OA}{2DA} = \tfrac{1}{2}(DA + OA),$$

where $O$ is the centre of the circle of intersection.

In the same way the charge on the same segment due to the image at $B$ is $\frac{1}{2}(DB + OB)$, and so on, lines such as $OB$ measured from $O$ to the left being reckoned negative.

Hence the total charge on the segment whose centre is $A$ is

$$\tfrac{1}{2}(DA + DB + DC + \&c.) + \tfrac{1}{2}(OA + OB + OC + \&c.)$$
$$- \tfrac{1}{2}(DP + DQ + \&c.) - \tfrac{1}{2}(OP + OQ + \&c.).$$

167.] The method of electrical images may be applied to any space bounded by plane or spherical surfaces all of which cut one another in angles which are submultiples of two right angles.

In order that such a system of spherical surfaces may exist, every solid angle of the figure must be trihedral, and two of its angles must be right angles, and the third either a right angle or a submultiple of two right angles.

Hence the cases in which the number of images is finite are—

(1) A single spherical surface or a plane.

(2) Two planes, a sphere and a plane, or two spheres intersecting at an angle $\frac{\pi}{n}$.

(3) These two surfaces with a third, which may be either plane or spherical, cutting both orthogonally.

(4) These three surfaces with a fourth, plane or spherical, cutting the first two orthogonally and the third at an angle $\frac{\pi}{n'}$. Of these four surfaces one at least must be spherical.

We have already examined the first and second cases. In the first case we have a single image. In the second case we have $2n-1$ images arranged in two series on a circle which passes through the influencing point and is orthogonal to both surfaces. In the third case we have, besides these images and the influencing point, their images with respect to the third surface, that is, $4n-1$ images in all besides the influencing point.

In the fourth case we first draw through the influencing point a circle orthogonal to the first two surfaces, and determine on it the positions and magnitudes of the $n$ negative images and the $n-1$ positive images. Then through each of these $2n$ points, including the influencing point, we draw a circle orthogonal to the third and fourth surfaces, and determine on it two series of images, $n'$ in each series. We shall obtain in this way, besides the influencing point, $2nn'-1$ positive and $2nn'$ negative images. These $4nn'$ points are the intersections of circles belonging to the two systems of lines of curvature of a cyclide.

If each of these points is charged with the proper quantity of electricity, the surface whose potential is zero will consist of $n+n'$ spheres, forming two series of which the successive spheres of the first set intersect at angles $\frac{\pi}{n}$, and those of the second set at angles $\frac{\pi}{n'}$, while every sphere of the first set is orthogonal to every sphere of the second set.

*Case of Two Spheres cutting Orthogonally.* See Fig. IV at the end of this volume.

168.] Let $A$ and $B$, Fig. 12, be the centres of two spheres cutting each other orthogonally in a circle through $D$ and $D'$, and let the straight line $DD'$ cut the line of centres in $C$. Then $C$ is the image of $A$ with respect to the sphere $B$, and also the image of $B$ with respect to the sphere $A$. If $AD = \alpha$, $BD = \beta$, then $AB = \sqrt{\alpha^2+\beta^2}$, and if we place

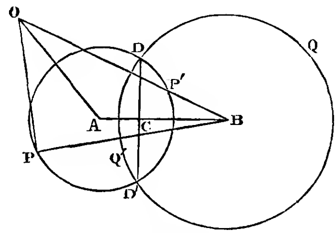


Fig. 12.

at $A$, $B$, $C$ quantities of electricity equal to $\alpha$, $\beta$, and $-\dfrac{\alpha\beta}{\sqrt{\alpha^2+\beta^2}}$ respectively, then both spheres will be equipotential surfaces whose potential is unity.

We may therefore determine from this system the distribution of electricity in the following cases:

(1) On the conductor $PDQD'$ formed of the larger segments of both spheres. Its potential is unity, and its charge is

$$\alpha+\beta-\frac{\alpha\beta}{\sqrt{\alpha^2+\beta^2}} = AD+BD-CD.$$

This quantity therefore measures the capacity of such a figure when free from the inductive action of other bodies.

The density at any point $P$ of the sphere whose centre is $A$, and the density at any point $Q$ of the sphere whose centre is $B$, are respectively

$$\frac{1}{4\pi\alpha}\left(1-\left(\frac{\beta}{BP}\right)^3\right) \quad \text{and} \quad \frac{1}{4\pi\beta}\left(1-\left(\frac{\alpha}{AQ}\right)^3\right).$$

On the circle of intersection the density is zero.

If one of the spheres is very much larger than the other, the density at the vertex of the smaller sphere is ultimately three times that at the vertex of the larger sphere.

(2) On the lens $P'DQ'D'$ formed by the two smaller segments of the spheres, charged with a quantity of electricity $= -\dfrac{\alpha\beta}{\sqrt{\alpha^2+\beta^2}}$, and acted on by points $A$ and $B$, charged with quantities $\alpha$ and $\beta$

at potential unity, and the density at any point is expressed by the same formula.

(3) On the meniscus $DPD'Q'$ charged with a quantity $a$, and acted on by points $B$ and $C$ charged respectively with quantities $\beta$ and $\frac{-a\beta}{\sqrt{a^2+\beta^2}}$, which is also in equilibrium at potential unity.

(4) On the other meniscus $QDP'D'$ charged with a quantity $\beta$ under the action of $A$ and $C$.

We may also deduce the distribution of electricity on the following internal surfaces—

The hollow lens $P'DQ'D'$ under the influence of the internal electrified point $C$ at the centre of the circle $DD'$.

The hollow meniscus under the influence of a point at the centre of the concave surface.

The hollow formed of the two larger segments of both spheres under the influence of the three points $A$, $B$, $C$.

But, instead of working out the solutions of these cases, we shall apply the principle of electrical images to determine the density of the electricity induced at the point $P$ of the external surface of the conductor $PDQD'$ by the action of a point at $O$ charged with unit of electricity.

Let $\quad OA = a, \quad OB = b, \quad OP = r, \quad BP = p,$

$\quad AD = a, \quad BD = \beta, \quad AB = \sqrt{a^2+\beta^2}.$

Invert the system with respect to a sphere of radius unity and centre $O$.

The two spheres will remain spheres, cutting each other orthogonally, and having their centres in the same radii with $A$ and $B$. If we indicate by accented letters the quantities corresponding to the inverted system,

$$a' = \frac{a}{a^2-a^2}, \quad b' = \frac{b}{b^2-\beta^2}, \quad a' = \frac{a}{a^2-a^2}, \quad \beta' = \frac{\beta}{b^2-\beta^2},$$

$$r' = \frac{1}{r}, \quad p'^2 = \frac{\beta^2 r^2 + (b^2-\beta^2)(p^2-\beta^2)}{r^2(b^2-\beta^2)^2}.$$

If, in the inverted system, the potential of the surface is unity, then the density at the point $P'$ is

$$\sigma' = \frac{1}{4\pi a'}\left(1-\left(\frac{\beta'}{p'}\right)^3\right).$$

If, in the original system, the density at $P$ is $\sigma$, then

$$\frac{\sigma}{\sigma'} = \frac{1}{r^3},$$

and the potential is $\frac{1}{r}$. By placing at $O$ a negative charge of electricity equal to unity, the potential will become zero over the original surface, and the density at $P$ will be

$$\sigma = \frac{1}{4\pi}\frac{a^2-\alpha^2}{\alpha r^3}\left(1 - \frac{\beta^3 r^3}{(\beta^2 r^2 + (b^2-\beta^2)(p^2-\beta^2))^{\frac{3}{2}}}\right).$$

This gives the distribution of electricity on one of the spherical segments due to a charge placed at $O$. The distribution on the other spherical segment may be found by exchanging $a$ and $b$, $\alpha$ and $\beta$, and putting $q$ or $AQ$ instead of $p$.

To find the total charge induced on the conductor by the electrified point at $O$, let us examine the inverted system.

In the inverted system we have charges $\alpha'$ at $A'$, and $\beta'$ at $B'$, and a negative charge $\frac{\alpha'\beta'}{\sqrt{\alpha'^2+\beta'^2}}$ at a point $C'$ in the line $A'B'$, such that

$$A'C' : C'B' :: \alpha'^2 : \beta'^2.$$

If $OA' = a'$, $OB' = b'$, $OC' = c'$, we find

$$c'^2 = \frac{a'^2\beta'^2 + b'^2\alpha'^2 - \alpha'^2\beta'^2}{\alpha'^2+\beta'^2}.$$

Inverting this system the charges become

$$\frac{\alpha'}{a'} = \frac{\alpha}{a}, \qquad \frac{\beta'}{b'} = \frac{\beta}{b},$$

and
$$-\frac{\alpha'\beta'}{\sqrt{\alpha'^2+\beta'^2}}\frac{1}{c'} = -\frac{\alpha\beta}{\sqrt{a^2\beta^2+b^2\alpha^2-\alpha^2\beta^2}}.$$

Hence the whole charge on the conductor due to a unit of negative electricity at $O$ is

$$\frac{\alpha}{a} + \frac{\beta}{b} - \frac{\alpha\beta}{\sqrt{a^2\beta^2+b^2\alpha^2-\alpha^2\beta^2}}.$$

*Distribution of Electricity on Three Spherical Surfaces which Intersect at Right Angles.*

169.] Let the radii of the spheres be $\alpha$, $\beta$, $\gamma$, then

$$BC = \sqrt{\beta^2+\gamma^2}, \quad CA = \sqrt{\gamma^2+\alpha^2}, \quad AB = \sqrt{\alpha^2+\beta^2}.$$

Let $PQR$, Fig. 13, be the feet of the perpendiculars from $ABC$

on the opposite sides of the triangle, and let $O$ be the intersection of perpendiculars.

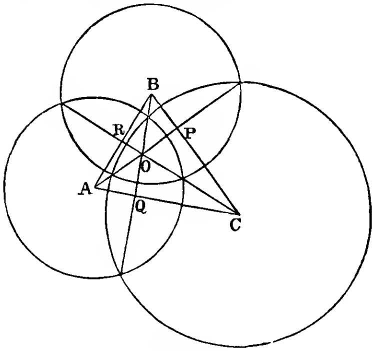



Fig. 13.

Then $P$ is the image of $B$ in the sphere $\gamma$, and also the image of $C$ in the sphere $\beta$. Also $O$ is the image of $P$ in the sphere $\alpha$.

Let charges $\alpha$, $\beta$, and $\gamma$ be placed at $A$, $B$, and $C$.

Then the charge to be placed at $P$ is

$$-\frac{\beta\gamma}{\sqrt{\beta^2+\gamma^2}} = -\frac{1}{\sqrt{\frac{1}{\beta^2}+\frac{1}{\gamma^2}}}.$$

Also $AP = \dfrac{\sqrt{\beta^2\gamma^2+\gamma^2\alpha^2+\alpha^2\beta^2}}{\sqrt{\beta^2+\gamma^2}}$, so that the charge at $O$, considered as the image of $P$, is

$$\frac{\alpha\beta\gamma}{\sqrt{\beta^2\gamma^2+\gamma^2\alpha^2+\alpha^2\beta^2}} = \frac{1}{\sqrt{\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}}}.$$

In the same way we may find the system of images which are electrically equivalent to four spherical surfaces at potential unity intersecting at right angles.

If the radius of the fourth sphere is $\delta$, and if we make the charge at the centre of this sphere $=\delta$, then the charge at the intersection of the line of centres of any two spheres, say $\alpha$ and $\beta$, with their plane of intersection, is

$$-\frac{1}{\sqrt{\frac{1}{\alpha^2}+\frac{1}{\beta^2}}}.$$

The charge at the intersection of the plane of any three centres $ABC$ with the perpendicular from the centre $D$ is

$$+\frac{1}{\sqrt{\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}}},$$

and the charge at the intersection of the four perpendiculars is

$$-\frac{1}{\sqrt{\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}+\frac{1}{\delta^2}}}.$$

*System of Four Spheres Intersecting at Right Angles, at zero potential, under the Action of an Electrified Unit Point.*

170.] Let the four spheres be $A$, $B$, $C$, $D$, and let the electrified point be $O$. Draw four spheres $A_1$, $B_1$, $C_1$, $D_1$, of which any one, $A_1$, passes through $O$ and cuts three of the spheres, in this case $B$, $C$, and $D$, at right angles. Draw six spheres $(ab)$, $(ac)$, $(ad)$, $(bc)$, $(bd)$, $(cd)$, of which each passes through $O$ and through the circle of intersection of two of the original spheres.

The three spheres $B_1$, $C_1$, $D_1$ will intersect in another point besides $O$. Let this point be called $A'$, and let $B'$, $C'$, and $D'$ be the intersections of $C_1$, $D_1$, $A_1$, of $D_1$, $A_1$, $B_1$, and of $A_1$, $B_1$, $C_1$ respectively. Any two of these spheres, $A_1$, $B_1$, will intersect one of the six $(cd)$ in a point $(a'\,b')$. There will be six such points.

Any one of the spheres, $A_1$, will intersect three of the six $(ab)$, $(ac)$, $(ad)$ in a point $a'$. There will be four such points. Finally, the six spheres $(ab)$, $(ac)$, $(ad)$, $(cd)$, $(db)$, $(bc)$, will intersect in one point $S$ in addition to $O$.

If we now invert the system with respect to a sphere of radius unity and centre $O$, the four spheres $A$, $B$, $C$, $D$ will be inverted into spheres, and the other ten spheres will become planes. Of the points of intersection the first four $A'$, $B'$, $C'$, $D'$ will become the centres of the spheres, and the others will correspond to the other eleven points described above. These fifteen points form the image of $O$ in the system of four spheres.

At the point $A'$, which is the image of $O$ in the sphere $A$, we must place a charge equal to the image of $O$, that is, $-\dfrac{\alpha}{a}$, where $\alpha$ is the radius of the sphere $A$, and $a$ is the distance of its centre from $O$. In the same way we must place the proper charges at $B'$, $C'$, $D'$.

The charge for any of the other eleven points may be found from the expressions in the last article by substituting $\alpha'$, $\beta'$, $\gamma'$, $\delta'$ for $\alpha$, $\beta$, $\gamma$, $\delta$, and multiplying the result for each point by the distance of the point from $O$, where

$$\alpha' = -\frac{\alpha}{a^2-\alpha^2}, \quad \beta' = -\frac{\beta}{b^2-\beta^2}, \quad \gamma' = -\frac{\gamma}{c^2-\gamma^2}, \quad \delta' = -\frac{\delta}{d^2-\delta^2}.$$

[The cases discussed in Arts. 169, 170 may be dealt with as follows: Taking three coordinate planes at right angles, let us place at the system of eight points $\left(\pm\frac{1}{2\alpha}, \quad \pm\frac{1}{2\beta}, \quad \pm\frac{1}{2\gamma}\right)$ charges $\pm e$, the minus charges being at the points which have 1 or 3 negative coordinates. Then it is obvious the coordinate planes are at potential zero. Now let us invert with regard to any point and we have the case of three spheres cutting orthogonally under the influence of an electrified point. If we invert with regard to one of the electrified points, we find the solution for the case of a conductor in the form of three spheres of radii $\alpha$, $\beta$, $\gamma$ cutting orthogonally and freely charged.

If to the above system of electrified points we superadd their images in a sphere with its centre at the origin we see that, in addition to the three coordinate planes, the surface of the sphere forms also a part of the surface of zero potential.]

*Two Spheres not Intersecting.*

171.] When a space is bounded by two spherical surfaces which do not intersect, the successive images of an influencing point within this space form two infinite series, none of which lie between the spherical surfaces, and therefore fulfil the condition of the applicability of the method of electrical images.

Any two non-intersecting spheres may be inverted into two concentric spheres by assuming as the point of inversion either of the two common inverse points of the pair of spheres.

We shall begin, therefore, with the case of two uninsulated concentric spherical surfaces, subject to the induction of an electrified point $P$ placed between them.

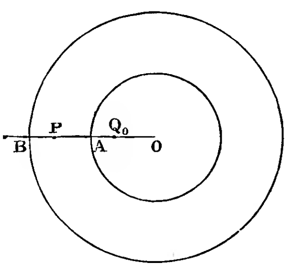



Fig. 14.

Let the radius of the first be $b$, and that of the second $be^{\varpi}$, and let the distance of the influencing point from the centre be $r = be^{u}$.

Then all the successive images will be on the same radius as the influencing point.

Let $Q_0$, Fig. 14, be the image of $P$ in the first sphere, $P_1$ that of $Q_0$ in the second sphere, $Q_1$ that of $P_1$ in the first sphere, and so on; then
$$OP_s . OQ_s = b^2,$$
and
$$OP_s . OQ_{s-1} = b^2 e^{2\varpi},$$

also
$$OQ_0 = be^{-u},$$
$$OP_1 = be^{u+2\varpi},$$
$$OQ_1 = be^{-(u+2)\varpi}, \text{ \&c.}$$
Hence
$$OP_s = be^{(u+2s\varpi)},$$
$$OQ_s = be^{-(u+2s\varpi)}.$$

If the charge of $P$ is denoted by $P$, that of $P_s$ by $P_s$, then
$$P_s = Pe^{\varpi}, \qquad Q_s = -Pe^{-(u+s\varpi)}.$$

Next, let $Q_1'$ be the image of $P$ in the second sphere, $P_1'$ that of $Q_1'$ in the first, &c., then
$$OQ_1' = be^{2\varpi-u}, \qquad OP_1' = be^{u-2\varpi},$$
$$OQ_2' = be^{4\varpi-u}, \qquad OP_2' = be^{u-4\varpi},$$
$$OQ_s' = be^{2s\varpi-u}, \qquad OP_s' = be^{u-2s\varpi},$$
$$Q_s' = -Pe^{s\varpi-u}, \qquad P_s' = Pe^{-s\varpi}.$$

Of these images all the $P$'s are positive, and all the $Q$'s negative, all the $P'$'s and $Q$'s belong to the first sphere, and all the $P$'s and $Q'$'s to the second.

The images within the first sphere form two converging series, the sum of which is
$$-P\frac{e^{\varpi-u}-1}{e^{\varpi}-1}.$$

This therefore is the quantity of electricity on the first or interior sphere. The images outside the second sphere form two diverging series, but the surface-integral due to each with respect to the spherical surface is zero. The charge of electricity on the exterior spherical surface is therefore
$$P\left(\frac{e^{\varpi-u}-1}{e^{\varpi}-1}-1\right) = -P\frac{e^{\varpi}-e^{\varpi-u}}{e^{\varpi}-1}.$$

If we substitute for these expressions their values in terms of $OA$, $OB$, and $OP$, we find
$$\text{charge on } A = -P\frac{OA}{OP}\frac{PB}{AB},$$
$$\text{charge on } B = -P\frac{OB}{OP}\frac{AP}{AB}.$$

If we suppose the radii of the spheres to become infinite, the case becomes that of a point placed between two parallel planes $A$ and $B$. In this case these expressions become
$$\text{charge on } A = -P\frac{PB}{AB},$$
$$\text{charge on } B = -P\frac{AP}{AB}.$$

172.] In order to pass from this case to that of any two spheres not intersecting each other, we begin by finding the two common inverse points O, O′ through which all circles pass that are orthogonal to both spheres. Then, if we invert the system with respect to either of these points, the spheres become concentric, as in the first case.

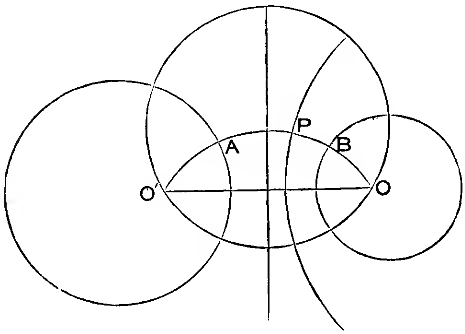


Fig. 15.

If we take the point O in Fig. 15 as centre of inversion this point will be situated in Fig. 14 somewhere between the two spherical surfaces.

Now in Art. 171 we solved the case where an electrified point is placed between two concentric conductors at zero potential. By inversion of that case with regard to the point O we shall therefore deduce the distributions induced on two spherical conductors at potential zero, exterior to one another, by an electrified point in their neighbourhood. In Art. 173 it will be shewn how the results thus obtained may be employed in finding the distributions on two spherical charged conductors subject to their mutual influence only.

The radius $OAPB$ in Fig. 14 on which the successive images lie becomes in Fig. 15 an arc of a circle through O and O′, and the ratio of O′P to OP is equal to $Ce^u$ where $C$ is a numerical quantity.

If we put $\theta = \log \frac{\mathrm{O'P}}{\mathrm{OP}}, \quad \alpha = \log \frac{\mathrm{O'A}}{\mathrm{OA}}, \quad \beta = \log \frac{\mathrm{O'B}}{\mathrm{OB}},$

then $\beta - \alpha = \varpi, \quad u + \alpha = \theta$*.

All the successive images of P will lie on the arc O′APBO.

The position of the image of P in A is $\mathrm{Q}_0$ where

$$\theta(\mathrm{Q}_0) = \log \frac{\mathrm{O'Q_0}}{\mathrm{OQ_0}} = 2\alpha - \theta.$$

* {Since O′ inverts into $O$, the common centre of the spheres, we have by Art. 162 $\frac{\mathrm{O'P}}{OP} = \frac{\mathrm{OP}}{\mathrm{OO'}}, \; \frac{\mathrm{O'A}}{OA} = \frac{\mathrm{OA}}{\mathrm{OO'}}$, so that $\frac{\mathrm{O'P.OA}}{\mathrm{OP.O'A}} = \frac{OP}{OA} = e^u$.}

That of $Q_0$ in B is $P_1$ where

$$\theta(P_1) = \log\frac{O'P_1}{OP_1} = \theta + 2\varpi.$$

Similarly

$$\theta(P_s) = \theta + 2s\varpi, \qquad \theta(Q_s) = 2a - \theta - 2s\varpi.$$

In the same way if the successive images of P in B, A, B, &c. are $Q_0'$, $P_1'$, $Q_1'$, &c.,

$$\theta(Q_0') = 2\beta - \theta, \qquad \theta(P_1') = \theta - 2\varpi;$$

$$\theta(P_s') = \theta - 2s\varpi, \qquad \theta(Q_s') = 2\beta - \theta + 2s\varpi.$$

To find the charge of any image $P_s$ we observe that in the inverted figure (14) its charge is

$$P\sqrt{\frac{OP_s}{OP}}.$$

In the original figure (15) we must multiply this by $OP_s$. Hence the charge of $P_s$ in the dipolar figure as $P = \mathrm{P}/OP$, is

$$\mathrm{P}\sqrt{\frac{OP_s \, . \, O'P_s}{OP \, . \, O'P}}.$$

If we make $\xi = \sqrt{OP \, . \, O'P}$, and call $\xi$ the parameter of the point P, then we may write

$$P_s = \frac{\xi_s}{\xi}\mathrm{P},$$

or the charge of any image is proportional to its parameter.

If we make use of the curvilinear coordinates $\theta$ and $\phi$, such that

$$e^{\theta + \sqrt{-1}\phi} = \frac{x + \sqrt{-1}\,y - k}{x + \sqrt{-1}\,y + k},$$

where $2k$ is the distance $OO'$, then

$$x = -\frac{k\sinh\theta}{\cosh\theta - \cos\phi}, \qquad y = \frac{k\sin\phi}{\cosh\theta - \cos\phi};$$

$$x^2 + (y - k\cot\phi)^2 = k^2\operatorname{cosec}^2\phi,$$

$$(x + k\coth\theta)^2 + y^2 = k^2\operatorname{cosech}^2\theta,$$

$$\cot\phi = \frac{x^2 + y^2 - k^2}{2ky}\,*, \qquad \coth\theta = -\frac{x^2 + y^2 + k^2}{2kx};$$

$$\xi = \frac{\sqrt{2}\,k}{\sqrt{\cosh\theta - \cos\phi}}\,\dagger.$$

* {Hence $\phi$ is constant for all points on the arc along which the images are situated.}

† In these expressions we must remember that

$$2\cosh\theta = e^{\theta} + e^{-\theta}, \qquad 2\sinh\theta = e^{\theta} - e^{-\theta},$$

Since the charge of each image is proportional to its parameter, $\xi$, and is to be taken positively or negatively according as it is of the form P or Q, we find

$$P_s = \frac{P\sqrt{\cosh\theta - \cos\phi}}{\sqrt{\cosh(\theta + 2s\varpi) - \cos\phi}},$$

$$Q_s = -\frac{P\sqrt{\cosh\theta - \cos\phi}}{\sqrt{\cosh(2\alpha - \theta - 2s\varpi) - \cos\phi}},$$

$$P_s' = \frac{P\sqrt{\cosh\theta - \cos\phi}}{\sqrt{\cosh(\theta - 2s\varpi) - \cos\phi}},$$

$$Q_s' = -\frac{P\sqrt{\cosh\theta - \cos\phi}}{\sqrt{\cosh(2\beta - \theta + 2s\varpi) - \cos\phi}}.$$

We have now obtained the positions and charges of the two infinite series of images. We have next to determine the total charge on the sphere A by finding the sum of all the images within it which are of the form Q or P′. We may write this

$$P\sqrt{\cosh\theta - \cos\phi}\sum_{s=1}^{s=\infty}\frac{1}{\sqrt{\cosh(\theta - 2s\varpi) - \cos\phi}},$$

$$P - \sqrt{\cosh\theta - \cos\phi}\sum_{s=0}^{s=\infty}\frac{1}{\sqrt{\cosh(2\alpha - \theta - 2s\varpi) - \cos\phi}}.$$

In the same way the total induced charge on B is

$$P\sqrt{\cosh\theta - \cos\phi}\sum_{s=1}^{s=\infty}\frac{1}{\sqrt{\cosh(\theta + 2s\varpi) - \cos\phi}},$$

$$-P\sqrt{\cosh\theta - \cos\phi}\sum_{s=0}^{s=\infty}\frac{1}{\sqrt{\cosh(2\beta - \theta + 2s\varpi) - \cos\phi}}.$$

173.] We shall apply these results to the determination of the coefficients of capacity and induction of two spheres whose radii are $a$ and $b$, and the distance between whose centres is $c$.

Let the sphere $A$ be at potential unity, and the sphere $B$ at potential zero.

Then the successive images of a charge $a$ placed at the centre

and the other functions of $\theta$ are derived from these by the same definitions as the corresponding trigonometrical functions.

The method of applying dipolar coordinates to this case was given by Thomson in *Liouville's Journal* for 1847. See Thomson's reprint of *Electrical Papers*, §§ 211, 212. In the text I have made use of the investigation of Prof. Betti, *Nuovo Cimento*, vol. xx, for the analytical method, but I have retained the idea of electrical images as used by Thomson in his original investigation, *Phil. Mag.*, 1853.

of the sphere $A$ will be those of the actual distribution of electricity. All the images will lie on the axis between the poles and the centres of the spheres, and it will be observed that of the four systems of images determined in Art. 172, only the third and fourth exist in this case.

If we put

$$k = \frac{\sqrt{a^4+b^4+c^4-2b^2c^2-2c^2a^2-2a^2b^2}}{2c},$$

$$\text{then} \quad \sinh\alpha = -\frac{k}{a}, \qquad \sinh\beta = \frac{k}{b}.$$

The values of $\theta$ and $\phi$ for the centre of the sphere $A$ are

$$\theta = 2\alpha, \qquad \phi = 0.$$

Hence in the equations we must substitute $a$ or $-k\frac{1}{\sinh\alpha}$ for $P$, $2\alpha$ for $\theta$ and 0 for $\phi$, remembering that $P$ itself forms part of the charge of $A$. We thus find for the coefficient of capacity of $A$

$$q_{aa} = \quad k\sum\nolimits_{s=0}^{s=\infty}\frac{1}{\sinh(s\varpi-\alpha)},$$

for the coefficient of induction of $A$ on $B$ or of $B$ on $A$

$$q_{ab} = -k\sum\nolimits_{s=1}^{s=\infty}\frac{1}{\sinh s\varpi}.$$

We might, in like manner, by supposing $B$ at potential unity and $A$ at potential zero, determine the value of $q_{bb}$. We should find, with our present notation,

$$q_{bb} = \quad k\sum\nolimits_{s=0}^{s=\infty}\frac{1}{\sinh(\beta+s\varpi)}.$$

To calculate these quantities in terms of $a$ and $b$, the radii of the spheres, and of $c$ the distance between their centres, we observe that if

$$K = \sqrt{a^4+b^4+c^4-2b^2c^2-2c^2a^2-2a^2b^2},$$

we may write

$$\sinh\alpha = -\frac{K}{2ac}, \quad \sinh\beta = \frac{K}{2bc}, \quad \sinh\varpi = \frac{K}{2ab},$$

$$\cosh\alpha = \frac{c^2+a^2-b^2}{2ca}, \quad \cosh\beta = \frac{c^2+b^2-a^2}{2cb}, \quad \cosh\varpi = \frac{c^2-a^2-b^2}{2ab};$$

and we may make use of

$$\sinh(\alpha+\beta) = \sinh\alpha\cosh\beta + \cosh\alpha\sinh\beta,$$
$$\cosh(\alpha+\beta) = \cosh\alpha\cosh\beta + \sinh\alpha\sinh\beta.$$

By this process or by the direct calculation of the successive images as shewn in Sir W. Thomson's paper, we find

$$q_{aa} = a + \frac{a^2 b}{c^2 - b^2} + \frac{a^3 b^2}{(c^2 - b^2 + ac)(c^2 - b^2 - a_c)} + \&c.,$$

$$q_{ab} = -\frac{ab}{c} - \frac{a^2 b^2}{c(c^2 - a^2 - b^2)} - \frac{a^3 b^3}{c(c^2 - a^2 - b^2 + ab)(c^2 - a^2 - b^2 - ab)} - \&c.,$$

$$q_{bb} = b + \frac{ab^2}{c^2 - a^2} + \frac{a^2 b^3}{(c^2 - a^2 + bc)(c^2 - a^2 - bc)} + \&c.$$

174.] We have then the following equations to determine the charges $E_a$ and $E_b$ of the two spheres when electrified to potentials $V_a$ and $V_b$ respectively,

$$E_a = V_a q_{aa} + V_b q_{ab},$$
$$E_b = V_a q_{ab} + V_b q_{bb}.$$

If we put $$q_{aa} q_{bb} - q_{ab}{}^2 = D = \frac{1}{D'},$$

and $$p_{aa} = q_{bb} D', \qquad p_{ab} = -q_{ab} D', \qquad p_{bb} = q_{aa} D',$$

whence $$p_{aa} p_{bb} - p_{ab}{}^2 = D';$$

then the equations to determine the potentials in terms of the charges are

$$V_a = p_{aa} E_a + p_{ab} E_b,$$
$$V_b = p_{ab} E_a + p_{bb} E_b,$$

and $p_{aa}$, $p_{ab}$, and $p_{bb}$ are the coefficients of potential.

The total energy of the system is, by Art. 85,

$$\begin{aligned} Q &= \tfrac{1}{2}(E_a V_a + E_b V_b) \\ &= \tfrac{1}{2}(V_a^2 q_{aa} + 2 V_a V_b q_{ab} + V_b{}^2 q_{bb}) \\ &= \tfrac{1}{2}(E_a{}^2 p_{aa} + 2 E_a E_b p_{ab} + E_b{}^2 p_{bb}). \end{aligned}$$

The repulsion between the spheres is therefore, by Arts. 92, 93,

$$\begin{aligned} F &= \tfrac{1}{2}\left\{V_a^2 \frac{dq_{aa}}{dc} + 2 V_a V_b \frac{dq_{ab}}{dc} + V_b^2 \frac{dq_{bb}}{dc}\right\} \\ &= -\tfrac{1}{2}\left\{E_a{}^2 \frac{dp_{aa}}{dc} + 2 E_a E_b \frac{dp_{ab}}{dc} + E_b{}^2 \frac{dp_{bb}}{dc}\right\}, \end{aligned}$$

where $c$ is the distance between the centres of the spheres.

Of these two expressions for the repulsion, the first, which expresses it in terms of the potentials of the spheres and the variations of the coefficients of capacity and induction, is the most convenient for calculation.

We have therefore to differentiate the $q$'s with respect to $c$. These quantities are expressed as functions of $k$, $\alpha$, $\beta$, and $\varpi$, and must be differentiated on the supposition that $a$ and $b$ are constant. From the equations

$$k = -a\sinh\alpha = b\sinh\beta = -c\frac{\sinh\alpha\sinh\beta}{\sinh\varpi},$$

we find

$$\frac{dk}{dc} = \frac{\cosh\alpha\cosh\beta}{\sinh\varpi},$$

$$\frac{d\alpha}{dc} = \frac{\sinh\alpha\cosh\beta}{k\sinh\varpi},$$

$$\frac{d\beta}{dc} = \frac{\cosh\alpha\sinh\beta}{k\sinh\varpi},$$

$$\frac{d\varpi}{dc} = \frac{1}{k};$$

whence we find

$$\frac{dq_{aa}}{dc} = \frac{\cosh\alpha\cosh\beta}{\sinh\varpi}\frac{q_{aa}}{k} - \sum_{s=0}^{s=\infty}\frac{(sc+b\cosh\beta)\cosh(s\varpi-\alpha)}{c(\sinh(s\varpi-\alpha))^2},$$

$$\frac{dq_{ab}}{dc} = \frac{\cosh\alpha\cosh\beta}{\sinh\varpi}\frac{q_{ab}}{k} + \sum_{s=1}^{s=\infty}\frac{s\cosh s\varpi}{(\sinh s\varpi)^2},$$

$$\frac{dq_{bb}}{dc} = \frac{\cosh\alpha\cosh\beta}{\sinh\varpi}\frac{q_{bb}}{k} - \sum_{s=0}^{s=\infty}\frac{(sc+a\cosh\alpha)\cosh(\beta+s\varpi)}{c(\sinh(\beta+s\varpi))^2}.$$

Sir William Thomson has calculated the force between two spheres of equal radius separated by any distance less than the diameter of one of them. For greater distances it is not necessary to use more than two or three of the successive images.

The series for the differential coefficients of the $q$'s with respect to $c$ are easily obtained by direct differentiation,

$$\frac{dq_{aa}}{dc} = -\frac{2a^2bc}{(c^2-b^2)^2} - \frac{2a^3b^2c(2c^2-2b^2-a^2)}{(c^2-b^2+ac)^2(c^2-b^2-ac)^2} - \&c.,$$

$$\frac{dq_{ab}}{dc} = \frac{ab}{c^2} + \frac{a^2b^2(3c^2-a^2-b^2)}{c^2(c^2-a^2-b^2)^2}$$
$$+\frac{a^3b^3\{(5c^2-a^2-b^2)(c^2-a^2-b^2)-a^2b^2\}}{c^2(c^2-a^2-b^2+ab)^2(c^2-a^2-b^2-ab)^2} - \&c.,$$

$$\frac{dq_{bb}}{dc} = -\frac{2ab^2c}{(c^2-a^2)^2} - \frac{2a^2b^3c(2c^2-2a^2-b^2)}{(c^2-a^2+bc)^2(c^2-a^2-bc)^2} - \&c.$$

### *Distribution of Electricity on Two Spheres in Contact.*

175.] If we suppose the two spheres at potential unity and not influenced by any point, then, if we invert the system with respect to the point of contact, we shall have two parallel planes, distant $\frac{1}{2a}$ and $\frac{1}{2b}$ from the point of inversion, and electrified by the action of a positive unit of electricity at that point.

There will be a series of positive images, each equal to unity, at distances $s\left(\frac{1}{a}+\frac{1}{b}\right)$ from the origin, where $s$ may have any integral value from $-\infty$ to $+\infty$.

There will also be a series of negative images each equal to $-1$, the distances of which from the origin, reckoned in the direction of $a$, are $\frac{1}{a}+s\left(\frac{1}{a}+\frac{1}{b}\right)$.

When this system is inverted back again into the form of the two spheres in contact, we have corresponding to the positive images a series of negative images, the distances of which from the point of contact are of the form $\dfrac{1}{s\left(\frac{1}{a}+\frac{1}{b}\right)}$, where $s$ is positive for the sphere $A$ and negative for the sphere $B$. The charge of each image, when the potential of the spheres is unity, is numerically equal to its distance from the point of contact, and is always negative.

There will also be a series of positive images corresponding to the negative ones for the two planes, whose distances from the point of contact measured in the direction of the centre of $a$, are of the form $\dfrac{1}{\frac{1}{a}+s\left(\frac{1}{a}+\frac{1}{b}\right)}$.

When $s$ is zero, or a positive integer, the image is inside the sphere $A$.

When $s$ is a negative integer the image is inside the sphere $B$.

The charge of each image is numerically equal to its distance from the origin and is always positive.

The total charge of the sphere $A$ is therefore

$$E_a = \sum_{s=0}^{s=\infty} \frac{1}{\frac{1}{a}+s\left(\frac{1}{a}+\frac{1}{b}\right)} - \frac{ab}{a+b}\sum_{s=1}^{s=\infty}\frac{1}{s}.$$

Each of these series is infinite, but if we combine them in the form

$$E_a = \sum_{s=1}^{s=\infty} \frac{a^2 b}{s(a+b)\{s(a+b)-a\}}$$

the series becomes convergent.

In the same way we find for the charge of the sphere $B$,

$$E_b = \sum_{s=1}^{s=\infty} \frac{ab}{s(a+b)-b} - \frac{ab}{a+b} \sum_{s=-1}^{s=-\infty} \frac{1}{s}$$

$$= \sum_{s=1}^{s=\infty} \frac{ab^2}{s(a+b)\{s(a+b)-b\}}.$$

The expression for $E_a$ is obviously equal to

$$\frac{ab}{a+b} \int_0^1 \frac{\theta^{\frac{b}{a+b}-1} - 1}{1-\theta} d\theta,$$

in which form the result in this case was given by Poisson.

It may also be shewn (Legendre, *Traité des Fonctions Elliptiques*, ii. 438) that the above series for $E_a$ is equal to

$$a - \left\{\gamma + \Psi\left(\frac{b}{a+b}\right)\right\} \frac{ab}{a+b},$$

where $\gamma = \cdot 57712\ldots$, and $\Psi(x) = \frac{d}{dx} \log \Gamma(1+x)$.

The values of $\Psi$ have been tabulated by Gauss (*Werke*, Band iii, pp. 161–162).

If we denote for an instant $b \div (a+b)$ by $x$, we find for the difference of the charges $E_a$ and $E_b$,

$$-\frac{d}{dx} \log \Gamma(x)\Gamma(1-x) \times \frac{ab}{a+b},$$

$$= \frac{ab}{a+b} \times \frac{d}{dx} \log \sin \pi x,$$

$$= \frac{\pi ab}{a+b} \cot \frac{\pi b}{a+b}.$$

When the spheres are equal the charge of each for potential unity is

$$E_a = a \sum_{s=1}^{s=\infty} \frac{1}{2s(2s-1)}$$

$$= a\left(1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \&c.\right)$$

$$= a \log_e 2 = \cdot 69314718\, a.$$

When the sphere $A$ is very small compared with the sphere $B$, the charge on $A$ is

$$E_a = \frac{a^2}{b} \sum_{s=1}^{s=\infty} \frac{1}{s^2} \text{ approximately},$$

$$\text{or} \quad E_a = \frac{\pi^2}{6} \frac{a^2}{b}.$$

The charge on $B$ is nearly the same as if $A$ were removed, or

$$E_b = b.$$

The mean density on each sphere is found by dividing the charge by the surface. In this way we get

$$\sigma_a = \frac{E_a}{4\pi a^2} = \frac{\pi}{24 b},$$

$$\sigma_b = \frac{E_b}{4\pi b^2} = \frac{1}{4\pi b},$$

$$\sigma_a = \frac{\pi^2}{6}\sigma_b.$$

Hence, if a very small sphere is made to touch a very large one, the mean density on the small sphere is equal to that on the large sphere multiplied by $\frac{\pi^2}{6}$, or 1·644936.

### *Application of Electrical Inversion to the case of a Spherical Bowl.*

176.] One of the most remarkable illustrations of the power of Sir W. Thomson's method of Electrical Images is furnished by his investigation of the distribution of electricity on a portion of a spherical surface bounded by a small circle. The results of this investigation, without proof, were communicated to M. Liouville and published in his *Journal* in 1847. The complete investigation is given in the reprint of Thomson's *Electrical Papers*, Article XV. I am not aware that a solution of the problem of the distribution of electricity on a finite portion of any curved surface has been given by any other mathematician.

As I wish to explain the method rather than to verify the calculation, I shall not enter at length into either the geometry or the integration, but refer my readers to Thomson's work.

### *Distribution of Electricity on an Ellipsoid.*

177.] It is shewn by a well-known method* that the attraction of a shell bounded by two similar and similarly situated and concentric ellipsoids is such that there is no resultant attraction on any point within the shell. If we suppose the thickness of the shell to diminish indefinitely while its density increases, we ultimately arrive at the conception of a surface-density varying as the perpendicular from the centre on the tangent plane, and

* Thomson and Tait's *Natural Philosophy*, § 520, or Art. 150 of this book.

since the resultant attraction of this superficial distribution on any point within the ellipsoid is zero, electricity, if so distributed on the surface, will be in equilibrium.

Hence, the surface-density at any point of an ellipsoid undisturbed by external influence varies as the distance of the tangent plane from the centre.

*Distribution of Electricity on a Disk.*

By making two of the axes of the ellipsoid equal, and making the third vanish, we arrive at the case of a circular disk, and at an expression for the surface-density at any point $P$ of such a disk when electrified to the potential $V$ and left undisturbed by external influence. If $\sigma$ be the surface-density on one side of the disk, and if $KPL$ be a chord drawn through the point $P$, then

$$\sigma = \frac{V}{2\pi^2\sqrt{KP.PL}}.$$

*Application of the Principle of Electric Inversion.*

178.] Take any point $Q$ as the centre of inversion, and let $R$ be the radius of the sphere of inversion. Then the plane of the disk becomes a spherical surface passing through $Q$, and the disk itself becomes a portion of the spherical surface bounded by a circle. We shall call this portion of the surface the *bowl*.

If $S'$ is the disk electrified to potential $V'$ and free from external influence, then its electrical image $S$ will be a spherical segment at potential zero, and electrified by the influence of a quantity $V'R$ of electricity placed at $Q$.

We have therefore by the process of inversion obtained the solution of the problem of the distribution of electricity on a bowl or a plane disk at zero potential when under the influence of an electrified point in the surface of the sphere or plane produced.

*Influence of an Electrified Point placed on the unoccupied part of the Spherical Surface.*

The form of the solution, as deduced by the principles already given and by the geometry of inversion, is as follows:

If $C$ is the central point or pole of the spherical bowl $S$, and if $a$ is the distance from $C$ to any point in the edge of the segment, then, if a quantity $q$ of electricity is placed at a point $Q$ in the surface of the sphere produced, and if the bowl $S$ is maintained at potential zero, the density $\sigma$ at any point $P$ of the bowl will be

$$\sigma = \frac{1}{2\pi^2}\frac{q}{QP^2}\sqrt{\frac{CQ^2-a^2}{a^2-CP^2}},$$

$CQ$, $CP$, and $QP$ being the straight lines joining the points, $C$, $Q$, and $P$.

It is remarkable that this expression is independent of the radius of the spherical surface of which the bowl is a part. It is therefore applicable without alteration to the case of a plane disk.

### *Influence of any Number of Electrified Points.*

Now let us consider the sphere as divided into two parts, one of which, the spherical segment on which we have determined the electric distribution, we shall call the *bowl*, and the other the remainder, or unoccupied part of the sphere on which the influencing point $Q$ is placed.

If any number of influencing points are placed on the remainder of the sphere, the electricity induced by these on any point of the bowl may be obtained by the summation of the densities induced by each separately.

179.] Let the whole of the remaining surface of the sphere be uniformly electrified, the surface-density being $\rho$, then the density at any point of the bowl may be obtained by ordinary integration over the surface thus electrified.

We shall thus obtain the solution of the case in which the bowl is at potential zero, and electrified by the influence of the remaining portion of the spherical surface rigidly electrified with density $\rho$.

Now let the whole system be insulated and placed within a sphere of diameter $f$, and let this sphere be uniformly and rigidly electrified so that its surface-density is $\rho'$.

There will be no resultant force within this sphere, and therefore the distribution of electricity on the bowl will be unaltered, but the potential of all points within the sphere will be increased by a quantity $V$ where $$V = 2\pi\rho' f.$$
Hence the potential at every point of the bowl will now be $V$.

Now let us suppose that this sphere is concentric with the sphere of which the bowl forms a part, and that its radius exceeds that of the latter sphere by an infinitely small quantity.

We have now the case of the bowl maintained at potential $V$ and influenced by the remainder of the sphere rigidly electrified with superficial density $\rho+\rho'$.

180.] We have now only to suppose $\rho+\rho' = 0$, and we get the case of the bowl maintained at potential $V$ and free from external influence.

If $\sigma$ is the density on either surface of the bowl at a given point when the bowl is at potential zero, and is influenced by the rest of the sphere electrified to density $\rho$, then, when the bowl is maintained at potential $V$, we must increase the density on the outside of the bowl by $\rho'$, the density on the supposed enveloping sphere.

The result of this investigation is that if $f$ is the diameter of the sphere, $a$ the chord of the radius of the bowl, and $r$ the chord of the distance of $P$ from the pole of the bowl, then the surface-density $\sigma$ on the *inside* of the bowl is

$$\sigma = \frac{V}{2\pi^2 f}\left\{\sqrt{\frac{f^2-a^2}{a^2-r^2}} - \tan^{-1}\sqrt{\frac{f^2-a^2}{a^2-r^2}}\right\},$$

and the surface-density on the outside of the bowl at the same point is

$$\sigma + \frac{V}{2\pi f}.$$

In the calculation of this result no operation is employed more abstruse than ordinary integration over part of a spherical surface. To complete the theory of the electrification of a spherical bowl we only require the geometry of the inversion of spherical surfaces.

181.] Let it be required to find the surface-density induced at any point of the uninsulated bowl by a quantity $q$ of electricity placed at a point $Q$, not now in the spherical surface produced.

Invert the bowl with respect to $Q$, the radius of the sphere of inversion being $R$. The bowl $S$ will be inverted into its image $S'$, and the point $P$ will have $P'$ for its image. We have now to determine the density $\sigma'$ at $P'$ when the bowl $S'$ is maintained at potential $V'$, such that $q = V'R$, and is not influenced by any external force.

The density $\sigma$ at the point $P$ of the original bowl is

$$\sigma = -\frac{\sigma' R^3}{QP^3},$$

this bowl being at potential zero, and influenced by a quantity $q$ of electricity placed at $Q$.

The result of this process is as follows:

Let the figure represent a section through the centre, $O$, of the sphere, the pole, $C$, of the bowl, and the influencing point $Q$. $D$ is a point which corresponds in the inverted figure to the unoccupied pole of the rim of the bowl, and may be found by the following construction.

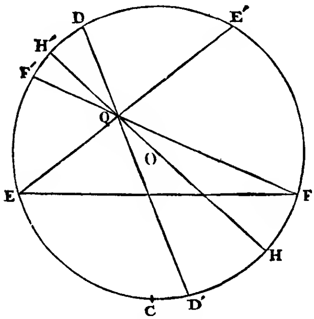


Fig. 16.

Draw through $Q$ the chords $EQE'$ and $FQF'$, then if we suppose the radius of the sphere of inversion to be a mean proportional between the segments into which a chord is divided at $Q$, $E'F'$ will be the image of $EF$. Bisect the arc $F'CE'$ in $D'$, so that $F'D' = D'E'$, and draw $D'QD$ to meet the sphere in $D$. $D$ is the point required. Also through $O$, the centre of the sphere, and $Q$ draw $HOQH'$ meeting the sphere in $H$ and $H'$. Then if $P$ be any point in the bowl, the surface-density at $P$ on the side which is separated from $Q$ by the completed spherical surface, induced by a quantity $q$ of electricity at $Q$, will be

$$\sigma = \frac{q}{2\pi^2}\frac{QH.QH'}{HH'.PQ^3}\left\{\frac{PQ}{DQ}\left(\frac{CD^2-a^2}{a^2-CP^2}\right)^{\frac{1}{2}} - \tan^{-1}\left[\frac{PQ}{DQ}\left(\frac{CD^2-a^2}{a^2-CP^2}\right)^{\frac{1}{2}}\right]\right\},$$

where $a$ denotes the chord drawn from $C$, the pole of the bowl, to the rim of the bowl*.

On the side next to $Q$ the surface-density is

$$\sigma + \frac{q}{2\pi}\frac{QH.QH'}{HH'.PQ^3}.$$

* {For further investigations of the electrical distribution on a bowl, see Ferrer's *Quarterly Journal of Math.* 1882; Gallop., *Quarterly Journal*, 1886, p. 229. In this paper it is shewn that the capacity of the bowl $= \frac{a\,(\alpha + \sin\alpha)}{\pi}$ where $a$ is the radius of the sphere of which the bowl forms a part and $\alpha$ the semi-vertical angle of the cone passing through the edge of the bowl whose apex is the centre of the sphere. See also Kruseman 'On the Potential of the Electric Field in the neighbourhood of a Spherical Bowl,' *Phil. Mag.* xxiv. 38, 1887. Basset, *Proc. Lond. Math. Soc.* xvi. p. 286.}

# APPENDIX TO CHAPTER XI.

{THE electrical distribution over two mutually influencing spheres has occupied the attention of many mathematicians. The first solution, which was expressed in terms of definite integrals, was given by Poisson in two most powerful and fascinating papers, *Mem. de l'Institut*, 1811, (1) p. 1, (2) p. 163. In addition to those mentioned in the text the following authors among others have considered the problem. Plana, *Mem. di Torino* 7, p. 71, 16, p. 57; Cayley, *Phil. Mag.* (4), 18, pp. 119, 193; Kirchhoff, *Crelle*, 59, p. 89, Wied. Ann. 27, p. 673; Mascart, C. R. 98, p. 222, 1884.

The series giving the charges on the spheres have been put in a very elegant form by Kirchhoff. They can easily be deduced as follows.

Suppose the radii of the spheres whose centres are $A$, $B$ are $a$, $b$, their potentials $U$, $V$ respectively, then if the spheres did not influence each other the electrical effect would be the same as that of two charges $aU$, $bV$ placed at the centres of the spheres. When the distance $c$ between the centres is finite this distribution of electricity would not make the potentials over the spheres constant; thus the charge at $A$ would alter the potential of the sphere $B$. If we wish to keep this potential unaltered we must take the image of $A$ in $B$ and place a charge there, this charge however will alter the potential of $A$, so we must take the image of this image and so on. Thus we shall get an infinite series of images which it will be convenient to divide into four sets $\alpha$, $\beta$, $\gamma$, $\delta$. The first two sets are due to the charge at the centre of $A$, $\alpha$ comprises the images inside $A$, $\beta$, the images inside the sphere $B$, the other two sets, $\gamma$ and $\delta$, are due to the charge at the centre of $B$; $\gamma$ consists of those inside $B$, $\delta$ of those inside $A$. Let $p_n$, $f_n$ denote the charge and the distance from $A$ of the $n^{\text{th}}$ image of the first set, $p_n'$, $f_n'$ the charge and the distance from $B$ of the $n^{\text{th}}$ image of the second set, then we have the following relations between the consecutive images,

$$f_n' = \frac{b^2}{c-f_n}, \qquad p_n' = -\frac{p_n f_n'}{b},$$

$$f_{n+1} = \frac{a^2}{c-f_n'}, \qquad p_{n+1} = -\frac{p_n' f_{n+1}}{a}.$$

Eliminating $f_n'$ and $p_n'$ from these equations we get

$$p_{n+1} = \frac{p_n(cf_{n+1}-a^2)}{ab}, \tag{1}$$

but
$$f_{n+1} = \frac{a^2}{c-\dfrac{b^2}{c-f_n}}, \quad \text{so that } cf_{n+1}-a^2 = \frac{a^2b^2}{c^2-cf_n-b^2},$$

$$p_{n+1} = p_n \frac{ab}{c^2 - cf_n - b^2},$$

or
$$\frac{p_n}{p_{n+1}} = \frac{c^2 - cf_n - b^2}{ab};$$

but from (1)
$$\frac{p_n}{p_{n-1}} = \frac{cf_n - a^2}{ab},$$

and thus
$$\frac{p_n}{p_{n+1}} + \frac{p_n}{p_{n-1}} = \frac{c^2 - b^2 - a^2}{ab},$$

or if we put $p_n = \frac{1}{P_n}$, $p_{n-1} = \frac{1}{P_{n-1}}$, $p_{n+1} = \frac{1}{P_{n+1}}$, we get

$$P_{n+1} + P_{n-1} = \frac{c^2 - b^2 - a^2}{ab} P_n.$$

From the symmetry of the equations we see that if we put $p_n' = \frac{1}{P_n'}$, we shall get the same sequence equation for $P_n'$ as for $P_n$.

From the sequence equation we see that

$$P_n = A\alpha^n + \frac{B}{\alpha^n};$$

where $\alpha$ and $1/\alpha$ are the roots of the equation

$$x^2 - x\frac{(c^2 - a^2 - b^2)}{ab} + 1 = 0.$$

We shall suppose that $\alpha$ is the root which is less than unity. Then

$$p_n = \frac{\alpha^n}{A\alpha^{2n} + B},$$

and the charge on the sphere due to this series of images is

$$\sum\nolimits_{n=0}^{n=\infty} \frac{\alpha^n}{A\alpha^{2n} + B}.$$

To determine $A$ and $B$ we have the equations

$$P_0 = \frac{1}{Ua} \quad = A + B,$$

$$P_1 = \frac{c^2 - b^2}{Ua^2 b} = A\alpha + \frac{B}{\alpha}; \text{ (Art. 164)}$$

hence
$$\frac{A}{B} = -\frac{(a + b\alpha)^2}{c^2} = -\xi^2, \text{ say,}$$

$$p_n = aU\{1 - \xi^2\}\frac{\alpha^n}{1 - \xi^2\alpha^{2n}},$$

$$\Sigma p_n = aU(1 - \xi^2)\left\{\frac{1}{1 - \xi^2} + \frac{\alpha}{1 - \xi^2\alpha^2} + \frac{\alpha^2}{1 - \xi^2\alpha^4} + \cdots\right\};$$

$$p_n' = \frac{a^n}{A'a^{2n}+B'},$$

$$p_0' = -\frac{abU}{c} = \frac{1}{A'+B'},$$

$$p_1' = -\frac{a^2b^2U}{c(c^2-(a^2+b^2))} = \frac{a}{A'a^2+B'}.$$

Hence $$A'/B' = -a^2,$$

and $$\Sigma p_n' = -\frac{abU}{c}\{1-a^2\}\left\{\frac{1}{1-a^2}+\frac{a}{1-a^4}+\frac{a^3}{1-a^6}+\dots\right\}.$$

Hence if $E_1$ and $E_2$ are the charges on the sphere, and if

$$E_1 = q_{11}U + q_{12}V,$$

$$E_2 = q_{12}U + q_{22}V;$$

$$q_{11} = a(1-\xi^2)\left\{\frac{1}{1-\xi^2}+\frac{a}{1-\xi^2a^2}+\frac{a^2}{1-\xi^2a^4}+\dots\right\},$$

$$q_{12} = -\frac{ab}{c}(1-a^2)\left\{\frac{1}{1-a^2}+\frac{a}{1-a^4}+\frac{a^2}{1-a^6}+\dots\right\},$$

$$q_{22} = b(1-\eta^2)\left\{\frac{1}{1-\eta^2}+\frac{a}{1-\eta^2a^2}+\frac{a^2}{1-\eta^2a^4}+\dots\right\},$$

where $$\eta^2 = \frac{(b+aa^2)}{c^2}.$$

These are the series given by Poisson and Kirchhoff.

Since $$\frac{\epsilon^p+1}{\epsilon^p-1} = \frac{2}{p} + 4\int_0^\infty \frac{\sin pt}{\epsilon^{2\pi t}-1}\,dt\,*,$$

$$\frac{1}{1-\epsilon^p} = \frac{1}{2} - \frac{1}{p} - 2\int_0^\infty \frac{\sin pt}{\epsilon^{2\pi t}-1}\,dt,$$

$$\frac{a^n}{1-\xi^2a^{2n}} = \frac{1}{2}a^n - \frac{1}{2n\log a+2\log\xi} - 2\int_0^\infty \frac{a^n\sin(2n\log a+2\log\xi)t}{\epsilon^{2\pi t}-1}\,dt,$$

$$\Sigma\frac{a^n}{1-\xi^2a^{2n}} = \frac{1}{2}\frac{1}{1-a} - \Sigma\frac{a^n}{2n\log a+2\log\xi} - 2\int_0^\infty \Sigma\frac{a^n\sin(2n\log a+2\log\xi)t}{\epsilon^{2\pi t}-1}\,dt.$$

Now $$\Sigma\frac{a^n}{2n\log a+2\log\xi} = \int_0^\infty \frac{\epsilon^{2t\log\xi}}{1-a\epsilon^{2t\log a}}\,dt,$$

and

$$\Sigma a^n\sin(2n\log a+2\log\xi)t = \frac{\sin(2t\log\xi)-a\sin(2t\log\xi/a)}{1-2a\cos(2t\log a)+a^2};$$

hence

$$q_{11} = a(1-\xi^2)\left\{\frac{1}{2}\frac{1}{1-a} - \int_0^\infty \frac{\epsilon^{2t\log\xi}}{1-a\epsilon^{2t\log a}}\,dt - 2\int_0^\infty \frac{\sin(2t\log\xi)-a\sin(2t\log\xi/a)}{(\epsilon^{2\pi t}-1)(1-2a\cos(2t\log a)+a^2)}\right\}dt,$$

which is Poisson's integral for these expressions.}

* {De Morgan, *Diff. and Int. Cal.* p. 672.}

## CHAPTER XII.

### THEORY OF CONJUGATE FUNCTIONS IN TWO DIMENSIONS.

182.] The number of independent cases in which the problem of electrical equilibrium has been solved is very small. The method of spherical harmonics has been employed for spherical conductors, and the methods of electrical images and of inversion are still more powerful in the cases to which they can be applied. The case of surfaces of the second degree is the only one, as far as I know, in which both the equipotential surfaces and the lines of force are known when the lines of force are not plane curves.

But there is an important class of problems in the theory of electrical equilibrium, and in that of the conduction of currents, in which we have to consider space of two dimensions only.

For instance, if throughout the part of the electric field under consideration, and for a considerable distance beyond it, the surfaces of all the conductors are generated by the motion of straight lines parallel to the axis of $z$, and if the part of the field where this ceases to be the case is so far from the part considered that the electrical action of the distant part of the field may be neglected, then the electricity will be uniformly distributed along each generating line, and if we consider a part of the field bounded by two planes perpendicular to the axis of $z$ and at distance unity, the potential and the distributions of electricity will be functions of $x$ and $y$ only.

If $\rho\, dx\, dy$ denotes the quantity of electricity in an element whose base is $dx\, dy$ and height unity, and $\sigma\, ds$ the quantity on an element of area whose base is the linear element $ds$ and height unity, then the equation of Poisson may be written

$$\frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} + 4\pi\rho = 0.$$

When there is no free electricity, this is reduced to the equation of Laplace,
$$\frac{d^2V}{dx^2}+\frac{d^2V}{dy^2}=0.$$

The general problem of electric equilibrium may be stated as follows:—

A continuous space of two dimensions, bounded by closed curves $C_1$, $C_2$, &c. being given, to find the form of a function, $V$, such that at these boundaries its value may be $V_1$, $V_2$, &c. respectively, being constant for each boundary, and that within this space $V$ may be everywhere finite, continuous, and single valued, and may satisfy Laplace's equation.

I am not aware that any perfectly general solution of even this problem has been given, but the method of transformation given in Art. 190 is applicable to this case, and is much more powerful than any known method applicable to three dimensions.

The method depends on the properties of conjugate functions of two variables.

### *Definition of Conjugate Functions.*

183.] Two quantities $a$ and $\beta$ are said to be conjugate functions of $x$ and $y$, if $a+\sqrt{-1}\,\beta$ is a function of $x+\sqrt{-1}\,y$.

It follows from this definition that
$$\frac{da}{dx}=\frac{d\beta}{dy},\quad\text{and}\quad \frac{da}{dy}+\frac{d\beta}{dx}=0; \tag{1}$$
$$\frac{d^2a}{dx^2}+\frac{d^2a}{dy^2}=0,\quad \frac{d^2\beta}{dx^2}+\frac{d^2\beta}{dy^2}=0. \tag{2}$$

Hence both functions satisfy Laplace's equation. Also
$$\frac{da}{dx}\frac{d\beta}{dy}-\frac{da}{dy}\frac{d\beta}{dx}=\left|\frac{da}{dx}\right|^2+\left|\frac{da}{dy}\right|^2=\left|\frac{d\beta}{dx}\right|^2+\left|\frac{d\beta}{dy}\right|^2=R^2. \tag{3}$$

If $x$ and $y$ are rectangular coordinates, and if $ds_1$ is the intercept of the curve ($\beta=$ constant) between the curves ($a$) and ($a+da$), and $ds_2$ the intercept of $a$ between the curves ($\beta$) and ($\beta+d\beta$), then
$$-\frac{ds_1}{da}=\frac{ds_2}{d\beta}=\frac{1}{R}, \tag{4}$$
and the curves intersect at right angles.

If we suppose the potential $V=V_0+ka$, where $k$ is some constant, then $V$ will satisfy Laplace's equation, and the curves ($a$)

will be equipotential curves. The curves ($\beta$) will be lines of force, and the surface-integral of $R$ over unit-length of a cylindrical surface whose projection on the plane of $xy$ is the curve $AB$ will be $k\,(\beta_B - \beta_A)$, where $\beta_A$ and $\beta_B$ are the values of $\beta$ at the extremities of the curve.

If there be drawn on the plane one series of curves corresponding to values of $\alpha$ in arithmetical progression, and another series corresponding to a series of values of $\beta$ having the same common difference, then the two series of curves will everywhere intersect at right angles, and, if the common difference is small enough, the elements into which the plane is divided will be ultimately little squares, whose sides, in different parts of the field, are in different directions and of different magnitudes, being inversely proportional to $R$.

If two or more of the equipotential lines ($\alpha$) are closed curves enclosing a continuous space between them, we may take these for the surfaces of conductors at potentials $V_0 + k\alpha_1$, $V_0 + k\alpha_2$, &c. respectively. The quantity of electricity upon any one of these between the lines of force ($\beta_1$) and ($\beta_2$) will be $\frac{k}{4\pi}(\beta_2 - \beta_1)$.

The number of equipotential lines between two conductors will therefore indicate their difference of potential, and the number of lines of force which emerge from a conductor will indicate the quantity of electricity upon it.

We must next state some of the most important theorems relating to conjugate functions, and in proving them we may use either the equations (1), containing the differential coefficients, or the original definition, which makes use of imaginary symbols.

184.] THEOREM I. *If $x'$ and $y'$ are conjugate functions with respect to $x$ and $y$, and if $x''$ and $y''$ are also conjugate functions with respect to $x$ and $y$, then the functions $x' + x''$ and $y' + y''$ will be conjugate functions with respect to $x$ and $y$.*

For $$\frac{dx'}{dx} = \frac{dy'}{dy}, \text{ and } \frac{dx''}{dx} = \frac{dy''}{dy};$$

therefore $$\frac{d\,(x' + x'')}{dx} = \frac{d\,(y' + y'')}{dy}.$$

Also $$\frac{dx'}{dy} = -\frac{dy'}{dx}, \text{ and } \frac{dx''}{dy} = -\frac{dy''}{dx};$$

therefore $$\frac{d(x'+x'')}{dy} = -\frac{d(y'+y'')}{dx};$$

or $x'+x''$ and $y'+y''$ are conjugate with respect to $x$ and $y$.

*Graphic Representation of a Function which is the Sum of Two Given Functions.*

Let a function $(\alpha)$ of $x$ and $y$ be graphically represented by a series of curves in the plane of $xy$, each of these curves corresponding to a value of $\alpha$ which belongs to a series of such values increasing by a common difference, $\delta$.

Let any other function, $(\beta)$, of $x$ and $y$ be represented in the same way by a series of curves corresponding to a series of values of $\beta$ having the same common difference as those of $\alpha$.

Then to represent the function $(\alpha+\beta)$ in the same way, we must draw a series of curves through the intersections of the two former series, from the intersection of the curves $(\alpha)$ and $(\beta)$ to that of the curves $(\alpha+\delta)$ and $(\beta-\delta)$, then through the intersection of $(\alpha+2\,\delta)$ and $(\beta-2\,\delta)$, and so on. At each of these points the function will have the same value, namely $(\alpha+\beta)$. The next curve must be drawn through the points of intersection of $(\alpha)$ and $(\beta+\delta)$, of $(\alpha+\delta)$ and $(\beta)$, of $(\alpha+2\,\delta)$ and $(\beta-\delta)$, and so on. The function belonging to this curve will be $(\alpha+\beta+\delta)$.

In this way, when the series of curves $(\alpha)$ and the series $(\beta)$ are drawn, the series $(\alpha+\beta)$ may be constructed. These three series of curves may be drawn on separate pieces of transparent paper, and when the first and second have been properly superposed, the third may be drawn.

The combination of conjugate functions by addition in this way enables us to draw figures of many interesting cases with very little trouble when we know how to draw the simpler cases of which they are compounded. We have, however, a far more powerful method of transformation of solutions, depending on the following theorem.

185.] THEOREM II. *If $x''$ and $y''$ are conjugate functions with respect to the variables $x'$ and $y'$, and if $x'$ and $y'$ are conjugate functions with respect to $x$ and $y$, then $x''$ and $y''$ will be conjugate functions with respect to $x$ and $y$.*

For
$$\frac{dx''}{dx} = \frac{dx''}{dx'}\frac{dx'}{dx} + \frac{dx''}{dy'}\frac{dy'}{dx},$$
$$= \frac{dy''}{dy'}\frac{dy'}{dy} + \frac{dy''}{dx'}\frac{dx'}{dy},$$
$$= \frac{dy''}{dy};$$
and
$$\frac{dx''}{dy} = \frac{dx''}{dx'}\frac{dx'}{dy} + \frac{dx''}{dy'}\frac{dy'}{dy},$$
$$= -\frac{dy''}{dy'}\frac{dy'}{dx} - \frac{dy''}{dx'}\frac{dx'}{dx},$$
$$= -\frac{dy''}{dy};$$
and these are the conditions that $x''$ and $y''$ should be conjugate functions of $x$ and $y$.

This may also be shewn from the original definition of conjugate functions. For $x'' + \sqrt{-1}\, y''$ is a function of $x' + \sqrt{-1}\, y'$, and $x' + \sqrt{-1}\, y'$ is a function of $x + \sqrt{-1}\, y$. Hence, $x'' + \sqrt{-1}\, y''$ is a function of $x + \sqrt{-1}\, y$.

In the same way we may shew that if $x'$ and $y'$ are conjugate functions of $x$ and $y$, then $x$ and $y$ are conjugate functions of $x'$ and $y'$.

This theorem may be interpreted graphically as follows:—

Let $x'$, $y'$ be taken as rectangular coordinates, and let the curves corresponding to values of $x''$ and of $y''$ taken in regular arithmetical series be drawn on paper. A double system of curves will thus be drawn cutting the paper into little squares. Let the paper be also ruled with horizontal and vertical lines at equal intervals, and let these lines be marked with the corresponding values of $x'$ and $y'$.

Next, let another piece of paper be taken in which $x$ and $y$ are made rectangular coordinates and a double system of curves $x'$, $y'$ is drawn, each curve being marked with the corresponding value of $x'$ or $y'$. This system of curvilinear coordinates will correspond, point for point, to the rectilinear system of coordinates $x'$, $y'$ on the first piece of paper.

Hence, if we take any number of points on the curve $x''$ on the first paper, and note the values of $x'$ and $y'$ at these points, and mark the corresponding points on the second paper, we shall find a number of points on the transformed curve $x''$. If we do the same for all the curves $x''$, $y''$ on the first paper, we shall obtain on the second paper a double series of curves $x''$, $y''$ of a different form, but having the same property of cutting the paper into little squares.

186.] THEOREM III. *If $V$ is any function of $x'$ and $y'$, and if $x'$ and $y'$ conjugate functions of $x$ and $y$, then*

$$\iint\left(\frac{d^2V}{dx^2}+\frac{d^2V}{dy^2}\right)dx\,dy=\iint\left(\frac{d^2V}{dx'^2}+\frac{d^2V}{dy'^2}\right)dx'\,dy',$$

*the integration being between the same limits.*

For
$$\frac{dV}{dx}=\frac{dV}{dx'}\frac{dx'}{dx}+\frac{dV}{dy'}\frac{dy'}{dx},$$

$$\frac{d^2V}{dx^2}=\frac{d^2V}{dx'^2}\left(\frac{dx'}{dx}\right)^2+2\frac{d^2V}{dx'dy'}\frac{dx'}{dx}\frac{dy'}{dx}+\frac{d^2V}{dy'^2}\left(\frac{dy'}{dx}\right)^2$$
$$+\frac{dV}{dx'}\frac{d^2x'}{dx^2}+\frac{dV}{dy'}\frac{d^2y'}{dx^2};$$

and
$$\frac{d^2V}{dy^2}=\frac{d^2V}{dx'^2}\left(\frac{dx'}{dy}\right)^2+2\frac{d^2V}{dx'dy'}\frac{dx'}{dy}\frac{dy'}{dy}+\frac{d^2V}{dy'^2}\left(\frac{dy'}{dy}\right)^2$$
$$+\frac{dV}{dx'}\frac{d^2x'}{dy^2}+\frac{dV}{dy'}\frac{d^2y'}{dy^2}.$$

Adding the last two equations, and remembering the conditions of conjugate functions (1), we find

$$\frac{d^2V}{dx^2}+\frac{d^2V}{dy^2}=\frac{d^2V}{dx'^2}\left\{\left(\frac{dx'}{dx}\right)^2+\left(\frac{dx'}{dy}\right)^2\right\}+\frac{d^2V}{dy'^2}\left\{\left(\frac{dy'}{dx}\right)^2+\left(\frac{dy'}{dy}\right)^2\right\},$$
$$=\left(\frac{d^2V}{dx'^2}+\frac{d^2V}{dy'^2}\right)\left(\frac{dx'}{dx}\frac{dy'}{dy}-\frac{dx'}{dy}\frac{dy'}{dx}\right).$$

Hence

$$\iint\left(\frac{d^2V}{dx^2}+\frac{d^2V}{dy^2}\right)dx\,dy=\iint\left(\frac{d^2V}{dx'^2}+\frac{d^2V}{dy'^2}\right)\left(\frac{dx'}{dx}\frac{dy'}{dy}-\frac{dx'}{dy}\frac{dy'}{dx}\right)dx\,dy,$$
$$=\iint\left(\frac{d^2V}{dx'^2}+\frac{d^2V}{dy'^2}\right)dx'\,dy'.$$

If $V$ is a potential, then, by Poisson's equation

$$\frac{d^2V}{dx^2}+\frac{d^2V}{dy^2}+4\pi\rho=0,$$

and we may write the result

$$\iint\rho\,dx\,dy=\iint\rho'\,dx'\,dy',$$

or the quantity of electricity in corresponding portions of two systems is the same if the coordinates of one system are conjugate functions of those of the other.

### *Additional Theorems on Conjugate Functions.*

187.] THEOREM IV. *If $x_1$ and $y_1$, and also $x_2$ and $y_2$, are conjugate functions of $x$ and $y$, then, if*

$$X = x_1 x_2 - y_1 y_2, \text{ and } Y = x_1 y_2 + x_2 y_1,$$

*$X$ and $Y$ will be conjugate functions of $x$ and $y$.*

For $$X + \sqrt{-1}\,Y = (x_1 + \sqrt{-1}\,y_1)(x_2 + \sqrt{-1}\,y_2).$$

THEOREM V. *If $\phi$ be a solution of the equation*

$$\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} = 0,$$

*and if* $$2R = \log\left(\left|\frac{d\phi}{dx}\right|^2 + \left|\frac{d\phi}{dy}\right|^2\right), \text{ and } \Theta = -\tan^{-1}\frac{\frac{d\phi}{dx}}{\frac{d\phi}{dy}},$$

*$R$ and $\Theta$ will be conjugate functions of $x$ and $y$.*

For $R$ and $\Theta$ are conjugate functions of $\frac{d\phi}{dy}$ and $\frac{d\phi}{dx}$, and these are conjugate functions of $x$ and $y$.

### EXAMPLE I.—*Inversion.*

188.] As an example of the general method of transformation let us take the case of inversion in two dimensions.

If $O$ is a fixed point in a plane, and $OA$ a fixed direction, and if $r = OP = a e^\rho$, and $\theta = AOP$, and if $x$, $y$ are the rectangular coordinates of $P$ with respect to $O$,

$$\left.\begin{aligned} \rho &= \log\frac{1}{a}\sqrt{x^2+y^2}, & \theta &= \tan^{-1}\frac{y}{x}, \\ x &= a e^\rho \cos\theta, & y &= a e^\rho \sin\theta, \end{aligned}\right\} \tag{5}$$

thus $\rho$ and $\theta$ are conjugate functions of $x$ and $y$.

If $\rho' = n\rho$ and $\theta' = n\theta$, $\rho'$ and $\theta'$ will be conjugate functions of $\rho$ and $\theta$. In the case in which $n = -1$ we have

$$r' = \frac{a^2}{r}, \text{ and } \theta' = -\theta, \tag{6}$$

which is the case of ordinary inversion combined with turning the figure 180° from $OA$.

*Inversion in Two Dimensions.*

In this case if $r$ and $r'$ represent the distances of corresponding points from $O$, $e$ and $e'$ the total electrification of a body, $S$ and $S'$ superficial elements, $V$ and $V'$ solid elements, $\sigma$ and $\sigma'$ surface-densities, $\rho$ and $\rho'$ volume densities, $\phi$ and $\phi'$ corresponding potentials,

$$\left.\begin{array}{l} \dfrac{r'}{r} = \dfrac{S'}{S} = \dfrac{a^2}{r^2} = \dfrac{r'^2}{a^2}, \quad \dfrac{V'}{V} = \dfrac{a^4}{r^4} = \dfrac{r'^4}{a^4}, \\ \dfrac{e'}{e} = 1, \quad \dfrac{\sigma'}{\sigma} = \dfrac{r^2}{a^2} = \dfrac{a^2}{r'^2}, \quad \dfrac{\rho'}{\rho} = \dfrac{r^4}{a^4} = \dfrac{a^4}{r'^4}, \\ \text{and since by hypothesis } \phi' \text{ is got from } \phi \text{ by expressing} \\ \text{the old variables in terms of the new, } \dfrac{\phi'}{\phi} = 1. \end{array}\right\} \quad (7)$$

EXAMPLE II.—*Electric Images in Two Dimensions.*

189.] Let $A$ be the centre of a circle of radius $AQ = b$ at zero potential, and let $E$ be a charge at $A$, then the potential at any point $P$ is

$$\phi = 2E \log \frac{b}{AP}; \quad (8)$$

and if the circle is a section of a hollow conducting cylinder, the surface-density at any point $Q$ is $-\dfrac{E}{2\pi b}$.

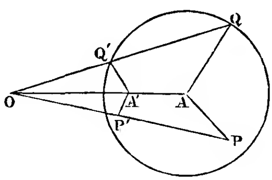



Fig. 17.

Invert the system with respect to a point $O$, making

$$AO = mb, \text{ and } a^2 = (m^2 - 1)\, b^2;$$

then the circle inverts into itself and we have a charge at $A'$ equal to that at $A$, where

$$AA' = \frac{b}{m}.$$

The density at $Q'$ is

$$-\frac{E}{2\pi b} \frac{b^2 - \overline{AA'}|^2}{A'Q'^2},$$

and the potential at any point $P'$ within the circle is

$$\begin{aligned} \phi' = \phi &= 2E(\log b - \log AP), \\ &= 2E(\log OP' - \log A'P' - \log m). \end{aligned} \quad (9)$$

This is equivalent to the potential arising from a combination of a charge $E$ at $A'$, and a charge $-E$ at $O$, which is the image of $A'$ with respect to the circle. The imaginary charge at $O$ is thus equal and opposite to that at $A'$.

If the point $P'$ is defined by its polar coordinates referred to the centre of the circle, and if we put

$$\rho = \log r - \log b, \quad \text{and} \quad \rho_0 = \log AA' - \log b,$$

then $$AP' = be^{\rho}, \quad AA' = be^{\rho_0}, \quad AO = be^{-\rho_0}; \tag{10}$$

and the potential at the point $(\rho, \theta)$ is

$$\phi = E\log(e^{-2\rho_0} - 2e^{-\rho_0}e^{\rho}\cos\theta + e^{2\rho}) \\ - E\log(e^{2\rho_0} - 2e^{\rho_0}e^{\rho}\cos\theta + e^{2\rho}) + 2E\rho_0. \tag{11}$$

This is the potential at the point $(\rho, \theta)$ due to a charge $E$, placed at the point $(\rho_0, 0)$, with the condition that when $\rho = 0$, $\phi = 0$.

In this case $\rho$ and $\theta$ are the conjugate functions in equations (5): $\rho$ is the logarithm of the ratio of the radius vector of a point to the radius of the circle, and $\theta$ is an angle.

The centre is the only singular point in this system of coordinates, and the line-integral $\int \frac{d\theta}{ds}\,ds$ round a closed curve is zero or $2\pi$, according as the closed curve excludes or includes the centre.

Example III.—*Neumann's Transformation of this Case**.

190.] Now let $\alpha$ and $\beta$ be any conjugate functions of $x$ and $y$, such that the curves $(\alpha)$ are equipotential curves, and the curves $(\beta)$ are lines of force due to a system consisting of a charge of half a unit per unit length at the origin, and an electrified system disposed in any manner at a certain distance from the origin.

Let us suppose that the curve for which the potential is $\alpha_0$ is a closed curve, such that no part of the electrified system except the half-unit at the origin lies within this curve.

Then all the curves $(\alpha)$ between this curve and the origin will be closed curves surrounding the origin, and all the curves $(\beta)$ will meet in the origin, and will cut the curves $(\alpha)$ orthogonally.

The coordinates of any point within the curve $(\alpha_0)$ will be determined by the values of $\alpha$ and $\beta$ at that point, and if the point travels round one of the curves $(\alpha)$ in the positive direction, the value of $\beta$ will increase by $2\pi$ for each complete circuit.

If we now suppose the curve $(\alpha_0)$ to be the section of the inner

* See Crelle's *Journal*, lix. p. 335, 1861, also Schwarz Crelle, lxxiv. p. 218, 1872.

surface of a hollow cylinder of any form maintained at potential zero under the influence of a charge of linear density $E$ on a line of which the origin is the projection, then we may leave the external electrified system out of consideration, and we have for the potential at any point $(a)$ within the curve

$$\phi = 2E(a - a_0), \tag{12}$$

and for the quantity of electricity on any part of the curve $a_0$ between the points corresponding to $\beta_1$ and $\beta_2$,

$$Q = \frac{1}{2\pi} E(\beta_1 - \beta_2). \tag{13}$$

If in this way, or in any other, we have determined the distribution of potential for the case of a curve of given section when the charge is placed at a given point taken as origin, we may pass to the case in which the charge is placed at any other point by an application of the general method of transformation.

Let the values of $a$ and $\beta$ for the point at which the charge is placed be $a_1$ and $\beta_1$, then substituting in equation (11) $a - a_0$ for $\rho$, $a_1 - a_0$ for $\rho_0$, since both vanish at the surface $a = a_0$, and $\beta - \beta_1$ for $\theta$, we find for the potential at any point whose coordinates are $a$ and $\beta$,

$$\begin{aligned}\phi = {} & E\log\left(1 - 2e^{a+a_1-2a_0}\cos(\beta-\beta_1) + e^{2(a+a_1-2a_0)}\right) \\ & - E\log\left(1 - 2e^{a-a_1}\cos(\beta-\beta_1) + e^{2(a-a_1)}\right) - 2E(a_1 - a_0).\end{aligned} \tag{14}$$

This expression for the potential becomes zero when $a = a_0$, and is finite and continuous within the curve $a_0$ except at the point $(a_1, \beta_1)$, at which point the second term becomes infinite, and in the immediate neighbourhood of that point this term is ultimately equal to $-2E\log r'$, where $r'$ is the distance from that point.

We have therefore obtained the means of deducing the solution of Green's problem for a charge at any point within a closed curve when the solution for a charge at any other point is known.

The charge induced upon an element of the curve $a_0$ between the points $\beta$ and $\beta + d\beta$ by a charge $E$ placed at the point $(a_1, \beta_1)$ is, with the notation of Art. 183,

$$-\frac{1}{4\pi}\frac{d\phi}{ds_1} ds_2,$$

where $ds_1$ is measured inwards and $a$ is to be put equal to $a_0$ after differentiation.

This becomes, by (4) of Art. 183,

$$\frac{1}{4\pi}\frac{d\phi}{d\alpha}d\beta, \quad (\alpha = \alpha_0);$$

i.e. $$-\frac{E}{2\pi}\frac{1-e^{2(\alpha_1-\alpha_0)}}{1-2e^{(\alpha_1-\alpha_0)}\cos(\beta-\beta_1)+e^{2(\alpha_1-\alpha_0)}}d\beta. \tag{15}$$

From this expression we may find the potential at any point $(\alpha_1, \beta_1)$ within the closed curve, when the value of the potential at every point of the closed curve is given as a function of $\beta$, and there is no electrification within the closed curve.

For, by Art. 86, the part of the potential at $(\alpha_1, \beta_1)$, due to the maintenance of the portion $d\beta$ of the closed curve at the potential $V$ is $nV$, where $n$ is the charge induced on $d\beta$ by unit of electrification at $(\alpha_1, \beta_1)$. Hence, if $V$ is the potential at a point on the closed curve defined as a function of $\beta$, and $\phi$ the potential at the point $(\alpha_1, \beta_1)$ within the closed curve, there being no electrification within the curve,

$$\phi = \frac{1}{2\pi}\int_0^{2\pi}\frac{(1-e^{2(\alpha_1-\alpha_0)})\,V d\beta}{1-2e^{(\alpha_1-\alpha_0)}\cos(\beta-\beta_1)+e^{2(\alpha_1-\alpha_0)}}. \tag{16}$$

EXAMPLE IV.—*Distribution of Electricity near an Edge of a Conductor formed by Two Plane Faces.*

191.] In the case of an infinite plane face $y = 0$ of a conductor, extending to infinity in the negative direction of $y$, charged with electricity to the surface-density $\sigma_0$, we find for the potential at a distance $y$ from the plane

$$V = C - 4\pi\sigma_0 y,$$

where $C$ is the value of the potential of the conductor itself.

Assume a straight line in the plane as a polar axis, and transform into polar coordinates, and we find for the potential

$$V = C - 4\pi\sigma_0 a e^{\rho}\sin\theta,$$

and for the quantity of electricity on a parallelogram of breadth unity, and length $ae^{\rho}$ measured along the axis

$$E = \sigma_0 a e^{\rho}.$$

Now let us make $\rho = n\rho'$ and $\theta = n\theta'$, then, since $\rho'$ and $\theta'$ are conjugate to $\rho$ and $\theta$, the equations

$$V = C - 4\pi\sigma_0 a e^{n\rho'}\sin n\theta'$$

and $$E = \sigma_0 a e^{n\rho'}$$

express a possible distribution of potential and of electricity.

If we write $r$ for $ae^{\rho'}$, $r$ will be the distance from the axis; we may also put $\theta$ instead of $\theta'$ for the angle. We shall then have

$$V = C - 4\pi\sigma_0 \frac{r^n}{a^{n-1}} \sin n\theta,$$

$$E = \sigma_0 \frac{r^n}{a^{n-1}}.$$

$V$ will be equal to $C$ whenever $n\theta = \pi$ or a multiple of $\pi$.

Let the edge be a salient angle of the conductor, the inclination of the faces being $a$, then the angle of the dielectric is $2\pi - a$, so that when $\theta = 2\pi - a$ the point is in the other face of the conductor. We must therefore make

$$n(2\pi - a) = \pi, \quad \text{or} \quad n = \frac{\pi}{2\pi - a}.$$

Then
$$V = C - 4\pi\sigma_0 a \left(\frac{r}{a}\right)^{\frac{\pi}{2\pi - a}} \sin \frac{\pi\theta}{2\pi - a},$$

$$E = \sigma_0 a \left(\frac{r}{a}\right)^{\frac{\pi}{2\pi - a}}.$$

The surface-density $\sigma$ at any distance $r$ from the edge is

$$\sigma = \frac{dE}{dr} = \frac{\pi}{2\pi - a} \sigma_0 \left(\frac{r}{a}\right)^{\frac{a - \pi}{2\pi - a}}.$$

When the angle is a salient one $a$ is less than $\pi$, and the surface-density varies according to some inverse power of the distance from the edge, so that at the edge itself the density becomes infinite, although the whole charge reckoned from the edge to any finite distance from it is always finite.

Thus, when $a = 0$ the edge is infinitely sharp, like the edge of a mathematical plane. In this case the density varies inversely as the square root of the distance from the edge.

When $a = \frac{\pi}{3}$ the edge is like that of an equilateral prism, and the density varies inversely as the $\frac{2}{5}$th power of the distance.

When $a = \frac{\pi}{2}$ the edge is a right angle, and the density is inversely as the cube root of the distance.

When $a = \frac{2\pi}{3}$ the edge is like that of a regular hexagonal prism, and the density is inversely as the fourth root of the distance.

When $a = \pi$ the edge is obliterated, and the density is constant.

When $a = \frac{4}{3}\pi$ the edge is like that of the outside of the hexagonal prism, and the density is *directly* as the square root of the distance from the edge.

When $a = \frac{3}{2}\pi$ the edge is a re-entrant right angle, and the density is directly as the distance from the edge.

When $a = \frac{5}{3}\pi$ the edge is a re-entrant angle of 60°, and the density is directly as the square of the distance from the edge.

In reality, in all cases in which the density becomes infinite at any point, there is a discharge of electricity into the dielectric at that point, as is explained in Art. 55.

### Example V.—*Ellipses and Hyperbolas.* Fig. X.

192.] We see that if

$$x_1 = e^{\phi}\cos\psi, \qquad y_1 = e^{\phi}\sin\psi, \tag{1}$$

$x_1$ and $y_1$ will be conjugate functions of $\phi$ and $\psi$.

Also, if $$x_2 = e^{-\phi}\cos\psi, \qquad y_2 = -e^{-\phi}\sin\psi, \tag{2}$$

$x_2$ and $y_2$ will be conjugate functions of $\phi$ and $\psi$. Hence, if

$$2x = x_1 + x_2 = (e^{\phi} + e^{-\phi})\cos\psi, \; 2y = y_1 + y_2 = (e^{\phi} - e^{-\phi})\sin\psi, \tag{3}$$

$x$ and $y$ will also be conjugate functions of $\phi$ and $\psi$.

In this case the points for which $\phi$ is constant lie on the ellipse whose axes are $e^{\phi} + e^{-\phi}$ and $e^{\phi} - e^{-\phi}$.

The points for which $\psi$ is constant lie on the hyperbola whose axes are $2\cos\psi$ and $2\sin\psi$.

On the axis of $x$, between $x = -1$ and $x = +1$,

$$\phi = 0, \qquad \psi = \cos^{-1} x. \tag{4}$$

On the axis of $x$, beyond these limits on either side, we have

$$\begin{aligned} x &> \;\;1, \qquad \psi = 2n\pi, \qquad && \phi = \log(x + \sqrt{x^2-1}), \\ x &< -1, \qquad \psi = (2n+1)\pi, \qquad && \phi = \log(\sqrt{x^2-1} - x). \end{aligned} \tag{5}$$

Hence, if $\phi$ is the potential function, and $\psi$ the function of flow, we have the case of electricity flowing from the positive to the negative side of the axis of $x$ through the space between the points $-1$ and $+1$, the parts of the axis beyond these limits being impervious to electricity.

Since, in this case, the axis of $y$ is a line of flow, we may suppose it also impervious to electricity.

We may also consider the ellipses to be sections of the equipotential surfaces due to an indefinitely long flat conductor of breadth 2, charged with half a unit of electricity per unit of length. {This includes the charge on both sides of the flat conductor.}

If we make $\psi$ the potential function, and $\phi$ the function of flow, the case becomes that of an infinite plane from which a strip of breadth 2 has been cut away and the plane on one side charged to potential $\pi$ while the other remains at zero potential.

These cases may be considered as particular cases of the quadric surfaces treated of in Chapter X. The forms of the curves are given in Fig. X.

EXAMPLE VI.—Fig. XI.

193.] Let us next consider $x'$ and $y'$ as functions of $x$ and $y$, where

$$x' = b\log\sqrt{x^2+y^2}, \qquad y' = b\tan^{-1}\frac{y}{x}, \tag{6}$$

$x'$ and $y'$ will be also conjugate functions of the $\phi$ and $\psi$ of Art. 192.

The curves resulting from the transformation of Fig. X with respect to these new coordinates are given in Fig. XI.

If $x'$ and $y'$ are rectangular coordinates, then the properties of the axis of $x$ in the first figure will belong to a series of lines parallel to $x'$ in the second figure for which $y' = bn'\pi$, where $n'$ is any integer.

The positive values of $x'$ on these lines will correspond to values of $x$ greater than unity, for which, as we have already seen,

$$\psi = n\pi, \qquad \phi = \log(x+\sqrt{x^2-1}) = \log\left(e^{\frac{x'}{b}} + \sqrt{e^{\frac{2x'}{b}}-1}\right). \tag{7}$$

The negative values of $x'$ on the same lines will correspond to values of $x$ less than unity, for which, as we have seen,

$$\phi = 0, \qquad \psi = \cos^{-1}x = \cos^{-1}e^{\frac{x'}{b}}. \tag{8}$$

The properties of the axis of $y$ in the first figure will belong to a series of lines in the second figure parallel to $x'$, for which

$$y' = b\pi(n'+\tfrac{1}{2}). \tag{9}$$

The value of $\psi$ along these lines is $\psi = \pi(n+\frac{1}{2})$ for all points both positive and negative, and

$$\phi = \log(y+\sqrt{y^2+1}) = \log\left(e^{\frac{x'}{b}} + \sqrt{e^{\frac{2x'}{b}}+1}\right). \qquad (10)$$

[The curves for which $\phi$ and $\psi$ are constant may be traced directly from the equations

$$x' = \tfrac{1}{2} b \log \tfrac{1}{4}\left(e^{2\phi} + e^{-2\phi} + 2\cos 2\psi\right),$$

$$y' = b\tan^{-1}\left(\frac{e^{\phi}-e^{-\phi}}{e^{\phi}+e^{-\phi}}\tan\psi\right).$$

As the figure repeats itself for intervals of $\pi b$ in the values of $y'$ it will be sufficient to trace the lines for one such interval.

Now there will be two cases, according as $\phi$ or $\psi$ changes sign with $y'$. Let us suppose that $\phi$ so changes sign. Then any curve for which $\psi$ is constant will be symmetrical about the axis of $x'$, cutting that axis orthogonally at some point on its negative side. If we begin with this point for which $\phi = 0$, and gradually increase $\phi$, the curve will bend round from being at first orthogonal to being, for large values of $\phi$, at length parallel to the axis of $x'$. The positive side of the axis of $x'$ is one of the system, viz. $\psi$ is there zero, and when $y' = \pm\frac{1}{2}\pi b$, $\psi = \frac{1}{2}\pi$. The lines for which $\psi$ has constant values ranging from 0 to $\frac{1}{2}\pi$ form therefore a system of curves embracing the positive side of the axis of $x'$.

The curves for which $\phi$ has constant values cut the system $\psi$ orthogonally, the values of $\phi$ ranging from $+\infty$ to $-\infty$. For any one of the curves $\phi$ drawn above the axis of $x'$ the value of $\phi$ is positive, along the negative side of the axis of $x'$ the value is zero, and for any curve below the axis of $x'$ the value is negative.

We have seen that the system $\psi$ is symmetrical about the axis of $x'$; let $PQR$ be any curve cutting that system orthogonally and terminating in $P$ and $R$ in the lines $y' = \pm\frac{1}{2}\pi b$, the point $Q$ being in the axis of $x'$. Then the curve $PQR$ is symmetrical about the axis of $x'$, but if $c$ be the value of $\phi$ along $PQ$, the value of $\phi$ along $QR$ will be $-c$. This discontinuity in the value of $\phi$ will be accounted for by an electrical distribution in the case which will be discussed in Art. 195.

If we next suppose that $\psi$ and not $\phi$ changes sign with $y'$, the values of $\phi$ will range from 0 to $\infty$. When $\phi = 0$ we have the negative side of the axis of $x'$, and when $\phi = \infty$ we have a line at an infinite distance perpendicular to the axis of $x'$. Along any line $PQR$ between these two cutting the $\psi$ system orthogonally the value of $\phi$ is constant throughout its entire length and positive.

Any value $\psi$ now experiences an abrupt change at the point where the curve along which it is constant crosses the negative side of the axis of $x'$, the sign of $\psi$ changing there. The significance of this discontinuity will appear in Art. 197.

The lines we have shewn how to trace are drawn in Fig. XI if we limit ourselves to two-thirds of that diagram, cutting off the uppermost third.]

194.] If we consider $\phi$ as the potential function, and $\psi$ as the function of flow, we may consider the case to be that of an indefinitely long strip of metal of breadth $\pi b$ with a non-conducting division extending from the origin indefinitely in the positive direction, and thus dividing the positive part of the strip into two separate channels. We may suppose this division to be a narrow slit in the sheet of metal.

If a current of electricity is made to flow along one of these divisions and back again along the other, the entrance and exit of the current being at an infinite distance on the positive side of the origin, the distribution of potential and of current will be given by the functions $\phi$ and $\psi$ respectively.

If, on the other hand, we make $\psi$ the potential, and $\phi$ the function of flow, then the case will be that of a current in the general direction of $y'$, flowing through a sheet in which a number of non-conducting divisions are placed parallel to $x'$, extending from the axis of $y'$ to an infinite distance in the negative direction.

195.] We may also apply the results to two important cases in statical electricity.

(1) Let a conductor in the form of a plane sheet, bounded by a straight edge but otherwise unlimited, be placed in the plane of $xz$ on the positive side of the origin, and let two infinite conducting planes be placed parallel to it and at distances $\frac{1}{2}\pi b$ on either side. Then, if $\psi$ is the potential function, its value is 0 for the middle conductor and $\frac{1}{2}\pi$ for the two planes.

Let us consider the quantity of electricity on a part of the middle conductor, extending to a distance 1 in the direction of $z$, and from the origin to $x' = a$.

The electricity on the part of this strip extending from $x_1'$ to $x_2'$ is $\frac{1}{4\pi}(\phi_2 - \phi_1)$.

Hence from the origin to $x' = a$ the amount on one side of the middle plate is

$$E = \frac{1}{4\pi}\log\left(e^{\frac{a}{b}} + \sqrt{e^{\frac{2a}{b}} - 1}\right). \tag{11}$$

If $a$ is large compared with $b$, this becomes

$$\begin{aligned} E &= \frac{1}{4\pi}\log 2e^{\frac{a}{b}}, \\ &= \frac{a + b\log_e 2}{4\pi b}. \end{aligned} \tag{12}$$

Hence the quantity of electricity on the plane bounded by the straight edge is greater than it would have been if the electricity had been uniformly distributed over it with the same density that it has at a distance from the boundary, and it is equal to the quantity of electricity having the same uniform surface-density, but extending to a breadth equal to $b\log_e 2$ beyond the actual boundary of the plate.

This imaginary uniform distribution is indicated by the dotted straight lines in Fig. XI. The vertical lines represent lines of force, and the horizontal lines equipotential surfaces, on the hypothesis that the density is uniform over both planes, produced to infinity in all directions.

196.] Electrical condensers are sometimes formed of a plate placed midway between two parallel plates extending considerably beyond the intermediate one on all sides. If the radius of curvature of the boundary of the intermediate plate is great compared with the distance between the plates, we may treat the boundary as approximately a straight line, and calculate the capacity of the condenser by supposing the intermediate plate to have its area extended by a strip of uniform breadth round its boundary, and assuming the surface-density on the extended plate the same as it is in the parts not near the boundary.

Thus, if $S$ be the actual area of the plate, $L$ its circumference, and $B$ the distance between the large plates, we have

$$b = \frac{1}{\pi}B, \tag{13}$$

and the breadth of the additional strip is

$$a = \frac{\log_e 2}{\pi} \cdot B, \tag{14}$$

so that the extended area is

$$S' = S + \frac{\log_e 2}{\pi} BL. \tag{15}$$

The capacity of one side of the middle plate is

$$\frac{1}{2\pi}\frac{S'}{B} = \frac{1}{2\pi}\left\{\frac{S}{B} + L\frac{1}{\pi}\log_e 2\right\}. \tag{16}$$

*Corrections for the Thickness of the Plate.*

Since the middle plate is generally of a thickness which cannot be neglected in comparison with the distance between the plates, we may obtain a better representation of the facts of the case by supposing the section of the intermediate plate to correspond with the curve $\psi = \psi'$.

The plate will be of nearly uniform thickness, $\beta = 2b\psi'$, at a distance from the boundary, but will be rounded near the edge.

The position of the actual edge of the plate is found by putting $y' = 0$, whence

$$x' = b\log_e \cos\psi'. \tag{17}$$

The value of $\phi$ at this edge is 0, and at a point for which $x' = a$ ($a/b$ being large) it is approximately

$$\frac{a + b\log_e 2}{b}.$$

Hence, altogether, the quantity of electricity on the plate is the same as if a strip of breadth

$$\frac{B}{\pi}\left(\log_e 2 + \log_e \cos\frac{\pi\beta}{2B}\right),$$

$$\text{i.e. } \frac{B}{\pi}\log_e\left(2\cos\frac{\pi\beta}{2B}\right), \tag{18}$$

had been added to the plate, the density being assumed to be everywhere the same as it is at a distance from the boundary.

*Density near the Edge.*

The surface-density at any point of the plate is

$$\frac{1}{4\pi}\frac{d\phi}{dx'} = \frac{1}{4\pi b}\frac{e^{\frac{x'}{b}}}{\sqrt{e^{\frac{2x'}{b}} - 1}}$$

$$= \frac{1}{4\pi b}\left(1 + \tfrac{1}{2}e^{-\frac{2x'}{b}} + \tfrac{3}{8}e^{-\frac{4x'}{b}} + \&c.\right). \tag{19}$$

The quantity within brackets rapidly approaches unity as $x'$ increases, so that at a distance from the boundary equal to $n$ times the breadth of the strip $a$, the actual density is greater than the normal density by about $\frac{1}{2^{2n+1}}$ of the normal density.

In like manner we may calculate the density on the infinite planes

$$= \frac{1}{4\pi b} \frac{e^{\frac{x'}{b}}}{\sqrt{e^{\frac{2x'}{b}} + 1}}. \tag{20}$$

When $x' = 0$, the density is $2^{-\frac{1}{2}}$ of the normal density.

At $n$ times the breadth of the strip on the positive side, the density is less than the normal density by about $\frac{1}{2^{2n+1}}$ of the normal density.

At $n$ times the breadth of the strip on the negative side, the density is about $\frac{1}{2^n}$ of the normal density.

These results indicate the degree of accuracy to be expected in applying this method to plates of limited extent, or in which irregularities may exist not very far from the boundary. The same distribution would exist in the case of an infinite series of similar plates at equal distances, the potentials of these plates being alternately $+V$ and $-V$. In this case we must take the distance between the plates equal to $B$.

197.] (2) The second case we shall consider is that of an infinite series of planes parallel to $x'z$ at distances $B = \pi b$, and all cut off by the plane of $y'z$, so that they extend only on the negative side of this plane. If we make $\phi$ the potential function, we may regard these planes as conductors at potential zero.

Let us consider the curves for which $\phi$ is constant.

When $y' = n\pi b$, that is, in the prolongation of each of the planes, we have

$$x' = b \log \tfrac{1}{2}(e^{\phi} + e^{-\phi}), \tag{21}$$

when $y' = (n + \frac{1}{2})\pi b$, that is in the intermediate positions

$$x' = b \log \tfrac{1}{2}(e^{\phi} - e^{-\phi}). \tag{22}$$

Hence, when $\phi$ is large, the curve for which $\phi$ is constant is an undulating line whose mean distance from the axis of $y'$ is approximately

$$a = b(\phi - \log_e 2), \tag{23}$$

and the amplitude of the undulations on either side of this line is

$$\tfrac{1}{2} b \log \frac{e^{\phi}+e^{-\phi}}{e^{\phi}-e^{-\phi}}. \tag{24}$$

When $\phi$ is large this becomes $be^{-2\phi}$, so that the curve approaches to the form of a straight line parallel to the axis of $y'$ at a distance $a$ from that axis on the positive side.

If we suppose a plane for which $x' = a$, kept at a constant potential while the system of parallel planes is kept at a different potential, then, since $b\phi = a + b\log_e 2$, the surface-density of the electricity induced on the plane is equal to that which would have been induced on it by a plane parallel to itself at a potential equal to that of the series of planes, but at a distance greater than that of the edges of the planes by $b\log_e 2$.

If $B$ is the distance between two of the planes of the series, $B = \pi b$, so that the additional distance is

$$a = B\frac{\log_e 2}{\pi}. \tag{25}$$

198.] Let us next consider the space included between two of the equipotential surfaces, one of which consists of a series of parallel waves, while the other corresponds to a large value of $\phi$, and may be considered as approximately plane.

If $D$ is the depth of these undulations from the crest to the trough of each wave, then we find for the corresponding value of $\phi$,

$$\phi = \tfrac{1}{2}\log\frac{e^{\frac{D}{b}}+1}{e^{\frac{D}{b}}-1}. \tag{26}$$

The value of $x'$ at the crest of the wave is

$$b\log\tfrac{1}{2}(e^{\phi}+e^{-\phi}). \tag{27}$$

*Hence, if $A$ is the distance from the crests of the waves to

* Let $\Phi$ be the potential of the plane, $\phi$ of the undulating surface. The quantity of electricity on the plane per unit area is $1 \div 4\pi b$. Hence the capacity

$$= 1 \div 4\pi b(\Phi - \phi),$$

$$= 1 \div 4\pi(A + a'), \text{ suppose.}$$

Then $\qquad A + a' = b(\Phi - \phi).$

But $\qquad A + b\log\tfrac{1}{2}(e^{\phi}+e^{-\phi}) = b(\Phi - \log 2);$

$$\therefore \quad a' = -b\phi + b(\log 2 + \log\tfrac{1}{2}(e^{\phi}+e^{-\phi}))$$

$$= b\log(1+e^{-2\phi})$$

$$= b\log\frac{2}{1+e^{-\frac{D}{b}}}, \text{ by (26).}$$

the opposite plane, the capacity of the system composed of the plane surface and the undulating surface is the same as that of two planes at a distance $A+a'$, where

$$a' = \frac{B}{\pi}\log_e \frac{2}{1+e^{-\pi\frac{D}{B}}}. \tag{28}$$

199.] If a single groove of this form be made in a conductor having the rest of its surface plane, and if the other conductor is a plane surface at a distance $A$, the capacity of the one conductor with respect to the other will be diminished. The amount of this diminution will be less than the $\frac{1}{n}$th part of the diminution due to $n$ such grooves side by side, for in the latter case the average electrical force between the conductors will be less than in the former case, so that the induction on the surface of each groove will be diminished on account of the neighbouring grooves.

If $L$ is the length, $B$ the breadth, and $D$ the depth of the groove, the capacity of a portion of the opposite plane whose area is $S$ will be

$$\frac{S-LB}{4\pi A} + \frac{LB}{4\pi(A+a')} = \frac{S}{4\pi A} - \frac{LB}{4\pi A}\cdot\frac{a'}{A+a'}. \tag{29}$$

If $A$ is large compared with $B$ or $a'$, the correction becomes by (28)

$$\frac{L}{4\pi^2}\frac{B^2}{A^2}\log_e \frac{2}{1+e^{-\pi\frac{D}{B}}}, \tag{30}$$

and for a slit of infinite depth, putting $D=\infty$, the correction is

$$\frac{L}{4\pi^2}\frac{B^2}{A^2}\log_e 2. \tag{31}$$

To find the surface-density on the series of parallel plates we must find $\sigma = \frac{1}{4\pi}\frac{d\psi}{dx'}$ when $\phi = 0$. We find

$$\sigma = \frac{1}{4\pi b}\frac{1}{\sqrt{e^{-2\frac{x'}{b}}-1}}. \tag{32}$$

The average density on the plane plate at distance $A$ from the edges of the series of plates is $\bar{\sigma} = \frac{1}{4\pi b}$. Hence at a distance from the edge of one of the plates equal to $n a$ the surface-density is $\frac{1}{\sqrt{2^{2n}-1}}$ of this average density.

200.] Let us next attempt to deduce from these results the distribution of electricity in the figure {a series of co-axial cylinders in front of a plane} formed by rotating the plane of the figure in Art. 197 about the axis $y' = -R$. In this case, Poisson's equation will assume the form

$$\frac{d^2V}{dx'^2} + \frac{d^2V}{dy'^2} + \frac{1}{R+y'}\frac{dV}{dy'} + 4\pi\rho = 0. \tag{33}$$

Let us assume $V = \phi$, the function given in Art. 193, and determine the value of $\rho$ from this equation. We know that the first two terms disappear, and therefore

$$\rho = -\frac{1}{4\pi}\frac{1}{R+y'}\frac{d\phi}{dy'}. \tag{34}$$

If we suppose that, in addition to the surface-density already investigated, there is a distribution of electricity in space according to the law just stated, the distribution of potential will be represented by the curves in Fig. XI.

Now from this figure it is manifest that $\frac{d\phi}{dy'}$ is generally very small except near the boundaries of the plates, so that the new distribution may be approximately represented by a certain superficial distribution of electricity near the edges of the plates.

If therefore we integrate $\iint \rho\, dx' dy'$ between the limits $y' = 0$ and $y' = \frac{\pi}{2} b$, and from $x' = -\infty$ to $x = +\infty$, we shall find the whole additional charge on one side of the plates due to the curvature.

Since $\frac{d\phi}{dy'} = -\frac{d\psi}{dx'}$, we have

$$\begin{aligned}\int_{-\infty}^{\infty} \rho\, dx' &= \int_{-\infty}^{\infty} \frac{1}{4\pi}\frac{1}{R+y}\frac{d\psi}{dx'}\, dx' \\ &= \frac{1}{4\pi}\frac{1}{R+y'}(\psi_\infty - \psi_{-\infty}) \\ &= \frac{1}{8}\frac{1}{R+y'}\left(2\frac{y'}{B} - 1\right).\end{aligned} \tag{35}$$

Integrating with respect to $y'$, we find

$$\int_0^{\frac{B}{2}}\int_{-\infty}^{\infty}\rho\,dx'dy' = \frac{1}{8} - \frac{1}{8}\frac{2R+B}{B}\log\frac{2R+B}{2R} \tag{36}$$

$$= -\frac{1}{32}\frac{B}{R} + \frac{1}{192}\frac{B^2}{R^2} + \&c. \tag{37}$$

This is half the total quantity of electricity which we must suppose distributed in space near the edge of one of the cylinders per unit of circumference. Since it is only close to the edge of the plate that the density is sensible, we may suppose the electricity all condensed on the surface of the plate without altering sensibly its action on the opposed plane surface, and in calculating the attraction between that surface and the cylindric surface we may suppose this electricity to belong to the cylindric surface.

If there had been no curvature the superficial charge on the positive surface of the plate per unit of length would have been

$$-\int_{-\infty}^{0}\frac{1}{4\pi}\frac{d\phi}{dy'}dx' = \frac{1}{4\pi}(\psi_0 - \psi_{-\infty}) = -\frac{1}{8}.$$

Hence, if we add to it the whole of the above distribution, this charge must be multiplied by the factor $\left(1+\frac{1}{2}\frac{B}{R}\right)$ to get the total charge on the positive side*.

† In the case of a disk of radius $R$ placed midway between two

* {Since there is a charge on the negative side of the plate equal to that on the positive side, it would seem that the total charge on the cylinders per unit circumference is $-\frac{1}{4}\left(1+\frac{1}{4}\frac{B}{R}\right)$, so that the correction for curvature is $\left(1+\frac{1}{4}\frac{B}{R}\right)$ and not $\left(1+\frac{1}{2}\frac{B}{R}\right)$ as in the text.}

† [In Art. 200, in estimating the total space distribution we might perhaps more correctly take for it the integral $\iint\rho\, 2\pi\,(R+y')\,dx'\,dy'$, which gives, per unit circumference of the edge of radius $R$, $-\frac{1}{32}\frac{B}{R}$, thus leading to the same correction as in the text.

The case of the disk may be treated in like manner as follows:

Let the figure of Art. 195 revolve round a line perpendicular to the plates and at a distance $+R$ from the edge of the middle one. That edge will therefore envelope a circle, which will be the edge of the disk. As in Art. 200, we begin with Poisson's equation, which in this case will be

$$\frac{d^2V}{dy'^2} + \frac{d^2V}{dx'^2} - \frac{1}{R-x'}\frac{dV}{dx'} + 4\pi\rho = 0.$$

We now assume that $V=\psi$, the potential function of Art. 195. We must therefore suppose electricity to exist in the region between the plates whose volume density $\rho$ is

$$\frac{1}{4\pi}\frac{1}{R-x'}\frac{d\psi}{dx'}.$$

infinite parallel plates at a distance $B$, we find for the capacity of a disk

$$\frac{R^2}{B}+2\frac{\log_e 2}{\pi}R+\tfrac{1}{2}B. \tag{38}$$

The total amount is

$$2\int_0^{\frac{B}{2}}\int_{-\infty}^{R}\rho.2\pi(R-x')\,dx'\,dy'.$$

Now if $R$ is large in comparison with the distance between the plates this result will be seen, on an examination of the potential lines in Fig. XI, to be sensibly the same as

$$\int_0^{\frac{B}{2}}\int_{-\infty}^{\infty}\frac{d\psi}{dx'}\,dx'\,dy'; \text{ that is, } -\tfrac{1}{8}\pi B.$$

The total surface distribution if we include both sides of the disk is

$$2\int_0^R\left(-\frac{1}{4\pi}\frac{d\psi}{dy'}\right)_{y'=0}2\pi(R-x')\,dx'$$

$$=-\int_0^R(R-x')\left(\frac{d\phi}{dx'}\right)_{y'=0}dx'$$

$$=-\int_0^R\phi_{y'=0}\,dx'$$

$$=-\int_0^R\log\left(e^{\frac{x'}{b}}+\sqrt{e^{\frac{2x'}{b}}-1}\right)dx'$$

$$=-\int_0^R\left\{\frac{x'}{b}+\log\left(1+\sqrt{1-e^{-\frac{2x'}{b}}}\right)\right\}dx'$$

$$=-\frac{\pi R^2}{2b}-\int_0^{\frac{R}{b}}b\log\left(1+\sqrt{1-e^{-2\xi}}\right)d\xi.$$

To evaluate the latter integral put

$$\sqrt{1-e^{-2\xi}}=1-t,$$

we get approximately if $R/b$ is large

$$\int_0^{\frac{R}{b}}\log\left(1+\sqrt{1-e^{-2\xi}}\right)d\xi=\tfrac{1}{2}\int_1^{\frac{1}{2}e^{-\frac{2R}{b}}}\log(2-t)\left(\frac{1}{2-t}-\frac{1}{t}\right)dt$$

$$=-\frac{1}{4}\{\log 2\}^2-\frac{1}{2}\log 2\left(-\log 2-\frac{2R}{b}\right)-\sum_{n=1}^{n=\infty}\frac{1}{2^{n+1}}\frac{1}{n^2}$$

$$=\frac{R}{b}\log 2+\frac{1}{4}\{\log 2\}^2-\sum_{n=1}^{n=\infty}\frac{1}{2^{n+1}}\frac{1}{n^2};$$

so that the quantity of electricity on the plate

$$=-\frac{R^2}{2b}-R\log 2-\frac{1}{8}\pi B-\frac{b}{4}\{\log 2\}^2+\sum_{n=1}^{n=\infty}\frac{b}{2^{n+1}}\frac{1}{n^2}.$$

Since the difference of potential of the plates $=\frac{\pi}{2}$ and $B=\pi b$, the capacity is

$$\frac{R^2}{B}+\frac{2}{\pi}R\log 2+\frac{B}{4}+\frac{B}{2\pi^2}(\log 2)^2-\frac{B}{\pi^2}\sum_{n=1}^{n=\infty}\frac{1}{2^n}\frac{1}{n^2}=\frac{\pi^2}{12}-\frac{1}{2}(\log 2)^2,$$

a result which is less than that in the text by $\cdot 28\,B$ nearly.]

*Theory of Thomson's Guard-ring.*

201.] In some of Sir W. Thomson's electrometers, a large plane surface is kept at one potential, and at a distance $A$ from this surface is placed a plane disk of radius $R$ surrounded by a large plane plate called a Guard-ring with a circular aperture of radius $R'$ concentric with the disk. This disk and plate are kept at potential zero.

The interval between the disk and the guard-plate may be regarded as a circular groove of infinite depth, and of breadth $R'-R$, which we denote by $B$.

The charge on the disk due to unit potential of the large disk, supposing the density uniform, would be $\frac{R^2}{4A}$.

The charge on one side of a straight groove of breadth $B$ and length $L = 2\pi R$, and of infinite depth, may be estimated by the number of lines of force emanating from the large disk and falling upon the side of the groove. Referring to Art. 197 and footnote we see that the charge will therefore be

$$\tfrac{1}{2} LB \times \frac{1}{4\pi b},$$

$$\text{i.e.}\quad \tfrac{1}{4}\frac{RB}{A+a'},$$

since in this case $\Phi = 1$, $\phi = 0$, and therefore $b = A + a'$.

But since the groove is not straight, but has a radius of curvature $R$, this must be multiplied by the factor $\left(1 + \tfrac{1}{2}\frac{B}{R}\right)$*.

The whole charge on the disk is therefore

$$\frac{R^2}{4A} + \tfrac{1}{4}\frac{RB}{A+a'}\left(1 + \frac{B}{2R}\right) \tag{39}$$

$$= \frac{R^2 + R'^2}{8A} - \frac{R'^2 - R^2}{8A} \cdot \frac{a'}{A+a'}. \tag{40}$$

The value of $a'$ cannot be greater than

$$\frac{B\log 2}{\pi}, = 0.22\,B \text{ nearly}.$$

If $B$ is small compared with either $A$ or $R$ this expression will give a sufficiently good approximation to the charge on the disk due to unity of difference of potential. The ratio of $A$ to $R$

* {If we take the correction for curvature to be $\left(1 + \frac{1}{4}\frac{B}{R}\right)$, see footnote p. 306, the charge on the disk will be less than that given in the text by $B^2/16\,(A+a')$.}

may have any value, but the radii of the large disk and of the guard-ring must exceed $R$ by several multiples of $A$.

EXAMPLE VII.—Fig. XII.

202.] Helmholtz, in his memoir on discontinuous fluid motion*, has pointed out the application of several formulae in which the coordinates are expressed as functions of the potential and its conjugate function.

One of these may be applied to the case of an electrified plate of finite size placed parallel to an infinite plane surface connected with the earth.

Since $x_1 = A\phi$ and $y_1 = A\psi$,
and also $x_2 = Ae^{\phi}\cos\psi$ and $y_2 = Ae^{\phi}\sin\psi$,
are conjugate functions of $\phi$ and $\psi$, the functions formed by adding $x_1$ to $x_2$ and $y_1$ to $y_2$ will be also conjugate. Hence, if

$$x = A\phi + Ae^{\phi}\cos\psi,$$
$$y = A\psi + Ae^{\phi}\sin\psi,$$

then $x$ and $y$ will be conjugate with respect to $\phi$ and $\psi$, and $\phi$ and $\psi$ will be conjugate with respect to $x$ and $y$.

Now let $x$ and $y$ be rectangular coordinates, and let $k\psi$ be the potential, then $k\phi$ will be conjugate to $k\psi$, $k$ being any constant.

Let us put $\psi = \pi$, then $y = A\pi$, $x = A(\phi - e^{\phi})$.

If $\phi$ varies from $-\infty$ to 0, and then from 0 to $+\infty$, $x$ varies from $-\infty$ to $-A$ and from $-A$ to $-\infty$. Hence the equipotential surface, for which $\psi = \pi$, is a plane parallel to $xz$ at a distance $b = \pi A$ from the origin, and extending from $x = -\infty$ to $x = -A$.

Let us consider a portion of this plane, extending from

$$x = -(A+a) \text{ to } x = -A \text{ and from } z = 0 \text{ to } z = c,$$

let us suppose its distance from the plane of $xz$ to be $y = b = A\pi$, and its potential to be $V = k\psi = k\pi$.

The charge of electricity on the portion of the plane considered is found by ascertaining the values of $\phi$ at its extremities.

We have therefore to determine $\phi$ from the equation

$$x = -(A+a) = A(\phi - e^{\phi}),$$

$\phi$ will have a negative value $\phi_1$ and a positive value $\phi_2$; at the edge of the plane, where $x = -A$, $\phi = 0$.

Hence the charge on the one side of the plane is $-ck\phi_1 \div 4\pi$, and that on the other side is $ck\phi_2 \div 4\pi$.

* *Monatsberichte der Königl. Akad. der Wissenschaften*, zu Berlin, April 23, 1868, p. 215.

Both these charges are positive and their sum is

$$\frac{ck(\phi_2-\phi_1)}{4\pi}.$$

If we suppose that $a$ is large compared with $A$,

$$\phi_1 = -\frac{a}{A} - 1 + e^{-\frac{a}{A} - 1 + e^{-\frac{a}{A} - 1 + \&c.}}$$

$$\phi_2 = \log\left\{\frac{a}{A} + 1 + \log\left(\frac{a}{A} + 1 + \&c.\right)\right\}.$$

If we neglect the exponential terms in $\phi_1$ we shall find that the charge on the negative surface exceeds that which it would have if the superficial density had been uniform and equal to that at a distance from the boundary, by a quantity equal to the charge on a strip of breadth $A = \frac{b}{\pi}$ with the uniform superficial density.

The total capacity of the part of the plane considered is

$$C = \frac{c}{4\pi^2}(\phi_2 - \phi_1).$$

The total charge is $CV$, and the attraction towards the infinite plane, whose equation is $y = 0$ and potential $\psi = 0$, is

$$-\tfrac{1}{2} V^2 \frac{dC}{db} = V^2 \frac{ac}{8\pi^3 A^2}\left(1 + \frac{\frac{A}{a}}{1 + \frac{A}{a}\log\frac{a}{A}} + e^{-\frac{a}{A}} + \&c.\right)$$

$$= \frac{V^2 c}{8\pi b^2}\left\{a + \frac{b}{\pi} - \frac{b^2}{\pi^2 a}\log\frac{a\pi}{b} + \&c.\right\}.$$

The equipotential lines and lines of force are given in Fig. XII.

EXAMPLE VIII. *Theory of a Grating of Parallel Wires.* Fig. XIII.

203.] In many electrical instruments a wire grating is used to prevent certain parts of the apparatus from being electrified by induction. We know that if a conductor be entirely surrounded by a metallic vessel at the same potential with itself, no electricity can be induced on the surface of the conductor by any electrified body outside the vessel. The conductor, however, when completely surrounded by metal, cannot be seen, and therefore, in certain cases, an aperture is left which is covered with a grating of fine wire. Let us investigate the effect of this grating in diminishing the effect of electrical induction. We shall suppose the grating to consist of a series of parallel wires in one plane and at equal intervals, the diameter of the wires being small compared with the distance between them, while the nearest portions of the electrified bodies on the one side and of the protected conductor on the other are at distances from the plane of the screen, which are considerable compared with the distance between consecutive wires.

204.] The potential at a distance $r'$ from the axis of a straight wire of infinite length charged with a quantity of electricity $\lambda$ per unit of length is $V = -2\lambda \log r' + C.$ (1)

We may express this in terms of polar coordinates referred to an axis whose distance from the wire is unity, in which case we must make

$$r'^2 = 1 - 2r\cos\theta + r^2, \tag{2}$$

and if we suppose that the axis of reference is also charged with the linear density $\lambda'$, we find

$$V = -\lambda \log(1 - 2r\cos\theta + r^2) - 2\lambda' \log r + C. \tag{3}$$

If we now make

$$r = e^{2\pi\frac{y}{a}}, \qquad \theta = \frac{2\pi x}{a}, \tag{4}$$

then, by the theory of conjugate functions,

$$V = -\lambda \log\left(1 - 2e^{\frac{2\pi y}{a}}\cos\frac{2\pi x}{a} + e^{\frac{4\pi y}{a}}\right) - 2\lambda' \log e^{\frac{2\pi y}{a}} + C, \tag{5}$$

where $x$ and $y$ are rectangular coordinates, will be the value of the potential due to an infinite series of fine wires parallel to $z$ in the plane of $xz$, and passing through points in the axis of $x$ for which $x$ is a multiple of $a$, and to planes perpendicular to the axis of $y$.

Each of these wires is charged with a linear density $\lambda$.

The term involving $\lambda'$ indicates an electrification, producing a constant force $\frac{4\pi\lambda'}{a}$ in the direction of $y$.

The forms of the equipotential surfaces and lines of force when $\lambda' = 0$ are given in Fig. XIII. The equipotential surfaces near the wires are nearly cylinders, so that we may consider the solution approximately true, even when the wires are cylinders of a diameter which is finite but small compared with the distance between them.

The equipotential surfaces at a distance from the wires become more and more nearly planes parallel to that of the grating.

If in the equation we make $y=b_1$, a quantity large compared with $a$, we find approximately,

$$V_1 = -\frac{4\pi b_1}{a}(\lambda+\lambda')+C \text{ nearly.} \tag{6}$$

If we next make $y=-b_2$, where $b_2$ is a positive quantity large compared with $a$, we find approximately,

$$V_2 = \frac{4\pi b_2}{a}\lambda' + C \text{ nearly.} \tag{7}$$

If $c$ is the radius of the wires of the grating, $c$ being small compared with $a$, we may find the potential of the grating itself by supposing that the surface of the wire coincides with the equipotential surface which cuts the plane of $xz$ at a distance $c$ from the axis of $z$. To find the potential of the grating we therefore put $x=c$, and $y=0$, whence

$$V = -2\lambda \log_e 2\sin\frac{\pi c}{a} + C. \tag{8}$$

205.] We have now obtained expressions representing the electrical state of a system consisting of a grating of wires whose diameter is small compared with the distance between them, and two plane conducting surfaces, one on each side of the grating, and at distances which are great compared with the distance between the wires.

The surface-density $\sigma_1$ on the first plane is got from the equation (6)

$$4\pi\sigma_1 = \frac{dV_1}{db_1} = -\frac{4\pi}{a}(\lambda+\lambda'), \tag{9}$$

that on the second plane $\sigma_2$ from the equation (7)

$$4\pi\sigma_2 = \frac{dV_2}{db_2} = \frac{4\pi}{a}\lambda'. \tag{10}$$

If we now write

$$\alpha = -\frac{a}{2\pi}\log_e\left(2\sin\frac{\pi c}{a}\right), \tag{11}$$

and eliminate $c$, $\lambda$ and $\lambda'$ from the equations (6), (7), (8), (9), (10), we find

$$4\pi\sigma_1\left(b_1+b_2+\frac{b_1 b_2}{\alpha}\right) = V_1\left(1+\frac{b_2}{\alpha}\right) - V_2 - V\frac{b_2}{\alpha}, \tag{12}$$

$$4\pi\sigma_2\left(b_1+b_2+\frac{b_1 b_2}{\alpha}\right) = -V_1 + V_2\left(1+\frac{b_1}{\alpha}\right) - V\frac{b_1}{\alpha}. \tag{13}$$

When the wires are infinitely thin, $\alpha$ becomes infinite, and the terms in which it is the denominator disappear, so that the case is reduced to that of two parallel planes without a grating interposed.

If the grating is in metallic communication with one of the planes, say the first, $V = V_1$, and the right-hand side of the equation for $\sigma_1$ becomes $V_1 - V_2$. Hence the density $\sigma_1$ induced on the first plane when the grating is interposed is to that which would be induced on it if the grating were removed, the second plane being maintained at the same potential, as 1 to $1 + \dfrac{b_1 b_2}{\alpha(b_1 + b_2)}$.

We should have found the same value for the effect of the grating in diminishing the electrical influence of the first surface on the second, if we had supposed the grating connected with the second surface. This is evident since $b_1$ and $b_2$ enter into the expression in the same way. It is also a direct result of the theorem of Art. 88.

The induction of the one electrified plane on the other through the grating is the same as if the grating were removed, and the distance between the planes increased from $b_1 + b_2$ to

$$b_1 + b_2 + \frac{b_1 b_2}{\alpha}.$$

If the two planes are kept at potential zero, and the grating electrified to a given potential, the quantity of electricity on the grating will be to that which would be induced on a plane of equal area placed in the same position as

$$b_1 b_2 : b_1 b_2 + \alpha(b_1 + b_2).$$

This investigation is approximate only when $b_1$ and $b_2$ are large compared with $\alpha$, and when $a$ is large compared with $c$. The quantity $\alpha$ is a line which may be of any magnitude. It becomes infinite when $c$ is indefinitely diminished.

If we suppose $c = \frac{1}{2} a$ there will be no apertures between the wires of the grating, and therefore there will be no induction through it. We ought therefore to have for this case $\alpha = 0$. The formula (11), however, gives in this case

$$\alpha = -\frac{a}{2\pi}\log_e 2, \qquad = -0{\cdot}11\,a,$$

which is evidently erroneous, as the induction can never be altered in sign by means of the grating. It is easy, however, to proceed to a higher degree of approximation in the case of a grating of cylindrical wires. I shall merely indicate the steps of this process.

### *Method of Approximation.*

206.] Since the wires are cylindrical, and since the distribution of electricity on each is symmetrical with respect to the diameter parallel to $y$, the proper expansion of the potential is of the form

$$V = C_0 \log r + \Sigma C_i\, r^i \cos i\,\theta, \tag{14}$$

where $r$ is the distance from the axis of one of the wires, and $\theta$ the angle between $r$ and $y$; and, since the wire is a conductor, when $r$ is made equal to the radius $V$ must be constant, and therefore the coefficient of each of the multiple cosines of $\theta$ must vanish.

For the sake of conciseness let us assume new coordinates $\xi$, $\eta$, &c. such that

$$a\xi = 2\pi x, \quad a\eta = 2\pi y, \quad a\rho = 2\pi r, \quad a\beta = 2\pi b, \text{ \&c.}, \tag{15}$$

and let

$$F_\beta = \log\left(e^{\eta+\beta} + e^{-(\eta+\beta)} - 2\cos\xi\right). \tag{16}$$

Then if we make

$$V = A_0 F_\beta + A_1 \frac{dF_\beta}{d\eta} + A_2 \frac{d^2 F_\beta}{d\eta^2} + \text{\&c.} \tag{17}$$

by giving proper values to the coefficients $A$ we may express any potential which is a function of $\eta$ and $\cos\xi$, and does not become infinite except when $\eta + \beta = 0$ and $\cos\xi = 1$.

When $\beta = 0$ the expansion of $F$ in terms of $\rho$ and $\theta$ is*

$$F_0 = 2\log\rho + \tfrac{1}{12}\rho^2 \cos 2\theta - \tfrac{1}{1440}\rho^4 \cos 4\theta + \text{\&c.} \tag{18}$$

For finite values of $\beta$ the expansion of $F$ is

$$F_\beta = \beta + 2\log\left(1 - e^{-\beta}\right) + \frac{1 + e^{-\beta}}{1 - e^{-\beta}}\rho\cos\theta - \frac{e^{-\beta}}{(1 - e^{-\beta})^2}\rho^2 \cos 2\theta + \text{\&c.} \tag{19}$$

In the case of the grating with two conducting planes whose equations are $\eta = \beta_1$ and $\eta = -\beta_2$, that of the plane of the grating being $\eta = 0$, there will be two infinite series of images

* {The expansion of $F$ can be got by noticing that $\log(e^{-\eta} + \epsilon^{\eta} - 2\cos\xi)$ only differs by a constant from $\log r^2 + \log r_1^2 + \log r_2^2 + \ldots$ where $r, r_1, r_2 \ldots$ are the distances of $P$ from the wires.

We can apply the same method to expand $F_\beta$ since this corresponds to moving the wires through a distance $-b$ parallel to $y$, the expansion however is not of the same form as that given in the text.}

of the grating. The first series will consist of the grating itself together with an infinite series of images on both sides, equal and similarly electrified. The axes of these imaginary cylinders lie in planes whose equations are of the form

$$\eta = \pm 2n(\beta_1 + \beta_2), \tag{20}$$

$n$ being an integer.

The second series will consist of an infinite series of images for which the coefficients $A_0$, $A_2$, $A_4$, &c. are equal and opposite to the same quantities in the grating itself, while $A_1$, $A_3$, &c. are equal and of the same sign. The axes of these images are in planes whose equations are of the form

$$\eta = 2\beta_2 \pm 2m(\beta_1 + \beta_2), \tag{21}$$

$m$ being an integer.

The potential due to any infinite series of such images will depend on whether the number of images is odd or even. Hence the potential due to an infinite series is indeterminate, but if we add to it the function $B\eta + C$, the conditions of the problem will be sufficient to determine the electrical distribution.

We may first determine $V_1$ and $V_2$, the potentials of the two conducting planes, in terms of the coefficients $A_0$, $A_1$, &c., and of $B$ and $C$. We must then determine $\sigma_1$ and $\sigma_2$, the surface-densities at any points of these planes. The mean values of $\sigma_1$ and $\sigma_2$ are given by the equations

$$4\pi\sigma_1 = \frac{2\pi}{a}(A_0 - B), \qquad 4\pi\sigma_2 = \frac{2\pi}{a}(A_0 + B). \tag{22}$$

We must then expand the potentials due to the grating itself and to all the images in terms of $\rho$ and cosines of multiples of $\theta$, adding to the result

$$B\rho\cos\theta + C.$$

The terms independent of $\theta$ then give $V$ the potential of the grating, and the coefficient of the cosine of each multiple of $\theta$ equated to zero gives an equation between the indeterminate coefficients.

In this way as many equations may be found as are sufficient to eliminate all these coefficients and to leave two equations to determine $\sigma_1$ and $\sigma_2$ in terms of $V_1$, $V_2$, and $V$.

These equations will be of the form

$$\begin{aligned} V_1 - V &= 4\pi\sigma_1(b_1 + a - \gamma) + 4\pi\sigma_2(a + \gamma), \\ V_2 - V &= 4\pi\sigma_1(a + \gamma) + 4\pi\sigma_2(b_2 + a - \gamma). \end{aligned} \tag{23}$$

The quantity of electricity induced on one of the planes protected by the grating, the other plane being at a given difference of potential, will be the same as if the planes had been at a distance

$$\frac{(\alpha-\gamma)(b_1+b_2)+b_1 b_2-4\alpha\gamma}{\alpha+\gamma} \text{ instead of } b_1+b_2.$$

The values of $\alpha$ and $\gamma$ are approximately as follows,

$$\alpha = \frac{a}{2\pi}\Big\{\log\frac{a}{2\pi c} - \frac{5}{3}\cdot\frac{\pi^4 c^4}{15 a^4+\pi^4 c^4}$$
$$+2e^{-4\pi\frac{b_1+b_2}{a}}\Big(1+e^{-4\pi\frac{b_1}{a}}+e^{-4\pi\frac{b_2}{a}}+\&\text{c.}\Big)+\&\text{c.}\Big\}, \qquad (24)$$

$$\gamma = \frac{3\pi a c^2}{3a^2+\pi^2 c^2}\left(\frac{e^{-4\pi\frac{b_1}{a}}}{1-e^{-4\pi\frac{b_1}{a}}} - \frac{e^{-4\pi\frac{b_2}{a}}}{1-e^{-4\pi\frac{b_2}{a}}}\right)+\&\text{c.}^* \qquad (25)$$

* {In the Supplementary Volume another method of employing conjugate functions, by which the capacity of finite plane surfaces etc. can be calculated, will be described}.

# CHAPTER XIII.

## ELECTROSTATIC INSTRUMENTS.

### *On Electrostatic Instruments.*

THE instruments which we have to consider at present may be divided into the following classes:

(1) Electrical machines for the production and augmentation of electrification.

(2) Multipliers, for increasing electrification in a known ratio.

(3) Electrometers, for the measurement of electric potentials and charges.

(4) Accumulators, for holding large electrical charges.

### *Electrical Machines.*

207.] In the common electrical machine a plate or cylinder of glass is made to revolve so as to rub against a surface of leather, on which is spread an amalgam of zinc and mercury. The surface of the glass becomes electrified positively and that of the rubber negatively. As the electrified surface of the glass moves away from the negative electrification of the rubber it acquires a high positive potential. It then comes opposite to a set of sharp metal points in connexion with the conductor of the machine. The positive electrification of the glass induces a negative electrification of the points, which is the more intense the sharper the points and the nearer they are to the glass.

When the machine works properly there is a discharge through the air between the glass and the points, the glass loses part of its positive charge, which is transferred to the points and so to the insulated prime conductor of the machine, or to any other body with which it is in electric communication.

The portion of the glass which is advancing towards the rubber has thus a smaller positive charge than that which is leaving it at the same time, so that the rubber, and the conductors in communication with it, become negatively electrified.

The highly positive surface of the glass where it leaves the rubber is more attracted by the negative charge of the rubber than the partially discharged surface which is advancing towards the rubber. The electrical forces therefore act as a resistance to the force employed in turning the machine. The work done in turning the machine is therefore greater than that spent in overcoming ordinary friction and other resistances, and the excess is employed in producing a state of electrification whose energy is equivalent to this excess.

The work done in overcoming friction is at once converted into heat in the bodies rubbed together. The electrical energy may be also converted either into mechanical energy or into heat.

If the machine does not store up mechanical energy, all the energy will be converted into heat, and the only difference between the heat due to friction and that due to electrical action is that the former is generated at the rubbing surfaces while the latter may be generated in conductors at a distance*.

We have seen that the electrical charge on the surface of the glass is attracted by the rubber. If this attraction were sufficiently intense there would be a discharge between the glass and the rubber, instead of between the glass and the collecting points. To prevent this, flaps of silk are attached to the rubber. These become negatively electrified and adhere to the glass, and so diminish the potential near the rubber.

The potential therefore increases more gradually as the glass moves away from the rubber, and therefore at any one point there is less attraction of the charge on the glass towards the rubber, and consequently less danger of direct discharge to the rubber.

In some electrical machines the moving part is of ebonite instead of glass, and the rubbers of wool or fur. The rubber is then electrified positively and the prime conductor negatively.

* It is probable that in many cases where dynamical energy is converted into heat by friction, part of the energy may be first transformed into electrical energy and then converted into heat as the electrical energy is spent in maintaining currents of short circuit close to the rubbing surfaces. See Sir W. Thomson, 'On the Electro-dynamic Qualities of Metals.' *Phil. Trans.*, 1856, p. 649.

## *The Electrophorus of Volta.*

208.] The electrophorus consists of a plate of resin or of ebonite backed with metal, and a plate of metal of the same size. An insulating handle can be screwed to the back of either of these plates. The ebonite plate has a metal pin which connects the metal plate with the metal back of the ebonite plate when the ebonite and metal plates are in contact.

The ebonite plate is electrified negatively by rubbing it with wool or cat's skin. The metal plate is then brought near the ebonite by means of the insulating handle. No direct discharge passes between the ebonite and the metal plate, but the potential of the metal plate is rendered negative by induction, so that when it comes within a certain distance of the metal pin a spark passes, and if the metal plate be now carried to a distance it is found to have a positive charge which may be communicated to a conductor. The metal at the back of the ebonite plate is found to have a negative charge equal and opposite to the charge of the metal plate.

In using the instrument to charge a condenser or accumulator one of the plates is laid on a conductor in communication with the earth, and the other is first laid on it, then removed and applied to the electrode of the condenser, then laid on the fixed plate and the process repeated. If the ebonite plate is fixed the condenser will be charged positively. If the metal plate is fixed the condenser will be charged negatively.

The work done by the hand in separating the plates is always greater than the work done by the electrical attraction during the approach of the plates, so that the operation of charging the condenser involves the expenditure of work. Part of this work is accounted for by the energy of the charged condenser, part is spent in producing the noise and heat of the sparks, and the rest in overcoming other resistances to the motion.

## *On Machines producing Electrification by Mechanical Work.*

209.] In the ordinary frictional electrical machine the work done in overcoming friction is far greater than that done in increasing the electrification. Hence any arrangement by which the electrification may be produced entirely by mechanical work against the electrical forces is of scientific importance if not of practical value. The first machine of this kind seems to have been Nicholson's Revolving Doubler, described in the *Philosophical Transactions* for 1788 as 'an Instrument which, by the turning of a Winch, produces the Two States of Electricity without Friction or Communication with the Earth.'

210.] It was by means of the revolving doubler that Volta succeeded in developing from the electrification of the pile an electrification capable of affecting his electrometer. Instruments on the same principle have been invented independently by Mr. C. F. Varley * and Sir W. Thomson.

These instruments consist essentially of insulated conductors of various forms, some fixed and others moveable. The moveable conductors are called Carriers, and the fixed ones may be called Inductors, Receivers, and Regenerators. The inductors and receivers are so formed that when the carriers arrive at certain points in their revolution they are almost completely surrounded by a conducting body. As the inductors and receivers cannot completely surround the carrier and at the same time allow it to move freely in and out without a complicated arrangement of moveable pieces, the instrument is not theoretically perfect without a pair of regenerators, which store up the small amount of electricity which the carriers retain when they emerge from the receivers.

For the present, however, we may suppose the inductors and receivers to surround the carrier completely when it is within them, in which case the theory is much simplified.

We shall suppose the machine to consist of two inductors $A$ and $C$, and of two receivers $B$ and $D$, with two carriers $F$ and $G$.

Suppose the inductor $A$ to be positively electrified so that its potential is $A$, and that the carrier $F$ is within it and is at potential $F$. Then, if $Q$ is the coefficient of induction (taken positive) between $A$ and $F$, the quantity of electricity on the carrier will be $Q\,(F-A)$.

If the carrier, while within the inductor, is put in connexion with the earth, then $F=0$, and the charge on the carrier will be $-QA$, a negative quantity. Let the carrier be carried round till it is within the receiver $B$, and let it then come in contact with a spring so as to be in electrical connexion with $B$. It

* Specification of Patent, Jan. 27, 1860, No. 206.

will then, as was shewn in Art. 32, become completely discharged, and will communicate its whole negative charge to the receiver $B$.

The carrier will next enter the inductor $C$, which we shall suppose charged negatively. While within $C$ it is put in connexion with the earth and thus acquires a positive charge, which it carries off and communicates to the receiver $D$, and so on.

In this way, if the potentials of the inductors remain always constant, the receivers $B$ and $D$ receive successive charges, which are the same for every revolution of the carrier, and thus every revolution produces an equal increment of electricity in the receivers.

But by putting the inductor $A$ in communication with the receiver $D$, and the inductor $C$ with the receiver $B$, the potentials of the inductors will be continually increased, and the quantity of electricity communicated to the receivers in each revolution will continually increase.

For instance, let the potential of $A$ and $D$ be $U$, and that of $B$ and $C$, $V$, then, since the potential of the carrier is zero when it is within $A$, being in contact with earth, its charge is $=-QU$. The carrier enters $B$ with this charge and communicates it to $B$. If the capacity of $B$ and $C$ is $B$, their potential will be changed from $V$ to $V-\frac{Q}{B}U$.

If the other carrier has at the same time carried a charge $-QV$ from $C$ to $D$, it will change the potential of $A$ and $D$ from $U$ to $U-\frac{Q'}{A}V$, if $Q'$ is the coefficient of induction between the carrier and $C$, and $A$ the capacity of $A$ and $D$. If, therefore, $U_n$ and $V_n$ be the potentials of the two inductors after $n$ half revolutions, and $U_{n+1}$ and $V_{n+1}$ after $n+1$ half revolutions,

$$U_{n+1} = U_n - \frac{Q'}{A}V_n,$$

$$V_{n+1} = V_n - \frac{Q}{B}U_n.$$

If we write $p^2 = \frac{Q}{B}$ and $q^2 = \frac{Q'}{A}$, we find

$$pU_{n+1}+qV_{n+1} = (pU_n+qV_n)(1-pq) = (pU_0+qV_0)(1-pq)^{n+1},$$
$$pU_{n+1}-qV_{n+1} = (pU_n-qV_n)(1+pq) = (pU_0-qV_0)(1+pq)^{n+1}.$$

Hence

$$2U_n = \quad U_0\left((1-pq)^n+(1+pq)^n\right)+\frac{q}{p}V_0\left((1-pq)^n-(1+pq)^n\right),$$

$$2V_n = \frac{p}{q}-U_0\left((1-pq)^n-(1+pq)^n\right)+V_0\left((1-pq)^n+(1+pq)^n\right).$$

It appears from these equations that the quantity $pU+qV$ continually diminishes, so that whatever be the initial state of electrification the receivers are ultimately oppositely electrified, so that the potentials of $A$ and $B$ are in the ratio of $q$ to $-p$.

On the other hand, the quantity $pU-qV$ continually increases, so that, however little $pU$ may exceed or fall short of $qV$ at first, the difference will be increased in a geometrical ratio in each revolution till the electromotive forces become so great that the insulation of the apparatus is overcome.

Instruments of this kind may be used for various purposes.—

For producing a copious supply of electricity at a high potential, as is done by means of Mr. Varley's large machine.

For adjusting the charge of a condenser, as in the case of Thomson's electrometer, the charge of which can be increased or diminished by a few turns of a very small machine of this kind, which is called a Replenisher.

For multiplying small differences of potential. The inductors may be charged at first to an exceedingly small potential, as, for instance, that due to a thermo-electric pair, then, by turning the machine, the difference of potentials may be continually multiplied till it becomes capable of measurement by an ordinary electrometer. By determining by experiment the ratio of increase of this difference due to each turn of the machine, the original electromotive force with which the inductors were charged may be deduced from the number of turns and the final electrification.

In most of these instruments the carriers are made to revolve about an axis and to come into the proper positions with respect to the inductors by turning an axle. The connexions are made by means of springs so placed that the carriers come in contact with them at the proper instants.

211.] Sir W. Thomson *, however, has constructed a machine for multiplying electrical charges in which the carriers are drops of water falling into an insulated receiver out of an uninsulated

* *Proc. R. S.*, June 20, 1867.

vessel placed inside but not touching an inductor. The receiver is thus continually supplied with electricity of opposite sign to that of the inductor. If the inductor is electrified positively, the receiver will receive a continually increasing charge of negative electricity.

The water is made to escape from the receiver by means of a funnel, the nozzle of which is almost surrounded by the metal of the receiver. The drops falling from this nozzle are therefore nearly free from electrification. Another inductor and receiver of the same construction are arranged so that the inductor of the one system is in connexion with the receiver of the other. The rate of increase of charge of the receivers is thus no longer constant, but increases in a geometrical progression with the time, the charges of the two receivers being of opposite signs. This increase goes on till the falling drops are so diverted from their course by the electrical action that they fall outside of the receiver or even strike the inductor.

In this instrument the energy of the electrification is drawn from that of the falling drops.

212.] Several other electrical machines have been constructed in which the principle of electric induction is employed. Of these the most remarkable is that of Holtz, in which the carrier is a glass plate varnished with gum-lac and the inductors are pieces of pasteboard. Sparks are prevented from passing between the parts of the apparatus by means of two glass plates, one on each side of the revolving carrier plate. This machine is found to be very effective, and not to be much affected by the state of the atmosphere. The principle is the same as in the revolving doubler and the instruments developed out of the same idea, but as the carrier is an insulating plate and the inductors are imperfect conductors, the complete explanation of the action is more difficult than in the case where the carriers are good conductors of known form and are charged and discharged at definite points*.

213.] In the electrical machines already described sparks occur whenever the carrier comes in contact with a conductor at a different potential from its own.

* {The induction machines most frequently used at present are those of Voss and Wimshurst. A description of these with diagrams will be found in *Nature*, vol. xxviii. p. 12.}

Now we have shewn that whenever this occurs there is a loss of energy, and therefore the whole work employed in turning the machine is not converted into electrification in an available form, but part is spent in producing the heat and noise of electric sparks.

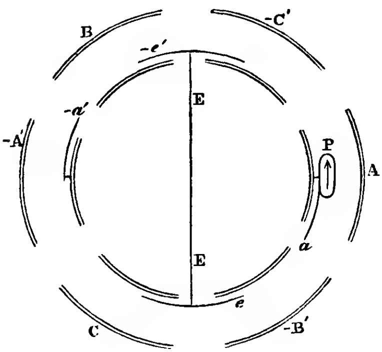


Fig. 18.

I have therefore thought it desirable to shew how an electrical machine may be constructed which is not subject to this loss of efficiency. I do not propose it as a useful form of machine, but as an example of the method by which the contrivance called in heat-engines a regenerator may be applied to an electrical machine to prevent loss of work.

In the figure let $A$, $B$, $C$, $A'$, $B'$, $C'$ represent hollow fixed conductors, so arranged that the carrier $P$ passes in succession within each of them. Of these $A$, $A'$ and $B$, $B'$ nearly surround the carrier when it is at the middle point of its passage, but $C$ and $C'$ do not cover it so much.

We shall suppose $A$, $B$, $C$ to be connected with a Leyden jar of great capacity at potential $V$, and $A'$, $B'$, $C'$ to be connected with another jar at potential $-V'$.

$P$ is one of the carriers moving in a circle from $A$ to $C'$, &c., and touching in its course certain springs, of which $a$ and $a'$ are connected with $A$ and $A'$ respectively, and $e$, $e'$ are connected with the earth.

Let us suppose that when the carrier $P$ is in the middle of $A$ the coefficient of induction between $P$ and $A$ is $-A$. The capacity of $P$ in this position is greater than $A$, since it is not completely surrounded by the receiver $A$. Let it be $A+\alpha$.

Then if the potential of $P$ is $U$, and that of $A$, $V$, the charge on $P$ will be $(A+\alpha)\,U-A\,V$.

Now let $P$ be in contact with the spring $a$ when in the middle of the receiver $A$, then the potential of $P$ is $V$, the same as that of $A$, and its charge is therefore $\alpha V$.

If $P$ now leaves the spring $a$ it carries with it the charge $\alpha V$. As $P$ leaves $A$ its potential diminishes, and it diminishes still more when it comes within the influence of $C''$, which is negatively electrified.

If when $P$ comes within $C'$ its coefficient of induction on $C'$ is $-C'$ and its capacity is $C'+c'$, then, if $U$ is the potential of $P$, the charge on $P$ is

$$(C'+c')\,U+C'V'=aV.$$

If
$$C'V'=aV,$$

then at this point $U$ the potential of $P$ will be reduced to zero.

Let $P$ at this point come in contact with the spring $e'$ which is connected with the earth. Since the potential of $P$ is equal to that of the spring there will be no spark at contact.

This conductor $C'$, by which the carrier is enabled to be connected to earth without a spark, answers to the contrivance called a regenerator in heat-engines. We shall therefore call it a Regenerator.

Now let $P$ move on, still in contact with the earth-spring $e'$, till it comes into the middle of the inductor $B$, the potential of which is $V$. If $-B$ is the coefficient of induction between $P$ and $B$ at this point, then, since $U=0$ the charge on $P$ will be $-BV$.

When $P$ moves away from the earth-spring it carries this charge with it. As it moves out of the positive inductor $B$ towards the negative receiver $A'$ its potential will be increasingly negative. At the middle of $A'$, if it retained its charge, its potential would be

$$-\frac{A'V'+BV}{A'+a'},$$

and if $BV$ is greater than $a'V'$ its numerical value will be greater than that of $V'$. Hence there is some point before $P$ reaches the middle of $A'$ where its potential is $-V'$. At this point let it come in contact with the negative receiver-spring $a'$. There will be no spark since the two bodies are at the same potential. Let $P$ move on to the middle of $A'$, still in contact with the spring, and therefore at the same potential with $A'$. During this motion it communicates a negative charge to $A'$. At the middle of $A'$ it leaves the spring and carries away a charge $-a'V'$ towards the positive regenerator $C$, where its potential is reduced to zero and it touches the earth-spring $e$. It then slides along the earth-spring into the negative inductor $B'$, during which motion it acquires a positive charge $B'V'$ which it finally communicates to the positive receiver $A$, and the cycle of operations is completed.

During this cycle the positive receiver has lost a charge $aV$ and gained a charge $B'V'$. Hence the total gain of positive electricity is
$$B'V' - aV.$$

Similarly the total gain of negative electricity is $BV - a'V'$.

By making the inductors so as to be as close to the surface of the carrier as is consistent with insulation, $B$ and $B'$ may be made large, and by making the receivers so as nearly to surround the carrier when it is within them, $a$ and $a'$ may be made very small, and then the charges of both the Leyden jars will be increased in every revolution.

The conditions to be fulfilled by the regenerators are
$$C'V' = aV, \quad \text{and} \quad CV = a'V'.$$

Since $a$ and $a'$ are small the regenerators must neither be large nor very close to the carriers.

*On Electrometers and Electroscopes.*

214.] An electrometer is an instrument by means of which electric charges or electric potentials may be measured. Instruments by means of which the existence of electric charges or of differences of potential may be indicated, but which are not capable of affording numerical measures, are called Electroscopes.

An electroscope if sufficiently sensitive may be used in electrical measurements, provided we can make the measurement depend on the absence of electrification. For instance, if we have two charged bodies $A$ and $B$ we may use the method described in Chapter I to determine which body has the greater charge. Let the body $A$ be carried by an insulating support into the interior of an insulated closed vessel $C$. Let $C$ be connected to earth and again insulated. There will then be no external electrification on $C$. Now let $A$ be removed, and $B$ introduced into the interior of $C$, and the electrification of $C$ tested by an electroscope. If the charge of $B$ is equal to that of $A$ there will be no electrification, but if it is greater or less there will be electrification of the same kind as that of $B$, or the opposite kind.

Methods of this kind, in which the thing to be observed is the non-existence of some phenomenon, are called *null* or *zero* methods. They require only an instrument capable of detecting the existence of the phenomenon.

In another class of instruments for the registration of phenomena the instruments may be depended upon to give always the same indication for the same value of the quantity to be registered, but the readings of the scale of the instrument are not proportional to the values of the quantity, and the relation between these readings and the corresponding value is unknown, except that the one is some continuous function of the other. Several electrometers depending on the mutual repulsion of parts of the instrument which are similarly electrified are of this class. The use of such instruments is to register phenomena, not to measure them. Instead of the true values of the quantity to be measured, a series of numbers is obtained, which may be used afterwards to determine these values when the scale of the instrument has been properly investigated and tabulated.

In a still higher class of instruments the scale readings are proportional to the quantity to be measured, so that all that is required for the complete measurement of the quantity is a knowledge of the coefficient by which the scale readings must be multiplied to obtain the true value of the quantity.

Instruments so constructed that they contain within themselves the means of independently determining the true values of quantities are called Absolute Instruments.

*Coulomb's Torsion Balance.*

215.] A great number of the experiments by which Coulomb established the fundamental laws of electricity were made by measuring the force between two small spheres charged with electricity, one of which was fixed while the other was held in equilibrium by two forces, the electrical action between the spheres, and the torsional elasticity of a glass fibre or metal wire. See Art. 38.

The balance of torsion consists of a horizontal arm of gum-lac, suspended by a fine wire or glass fibre, and carrying at one end a little sphere of elder pith, smoothly gilt. The suspension wire is fastened above to the vertical axis of an arm which can be moved round a horizontal graduated circle, so as to twist the upper end of the wire about its own axis any number of degrees.

The whole of this apparatus is enclosed in a case. Another little sphere is so mounted on an insulating stem that it can be charged and introduced into the case through a hole, and brought so that its centre coincides with a definite point in the horizontal circle described by the suspended sphere. The position of the suspended sphere is ascertained by means of a graduated circle engraved on the cylindrical glass case of the instrument.

Now suppose both spheres charged, and the suspended sphere in equilibrium in a known position such that the torsion-arm makes an angle $\theta$ with the radius through the centre of the fixed sphere. The distance of the centres is then $2a \sin \frac{1}{2}\theta$, where $a$ is the radius of the torsion-arm, and if $F$ is the force between the spheres the moment of this force about the axis of torsion is

$$Fa \cos \tfrac{1}{2}\theta.$$

Let both spheres be completely discharged, and let the torsion-arm now be in equilibrium at an angle $\phi$ with the radius through the fixed sphere.

Then the angle through which the electrical force twisted the torsion-arm must have been $\theta - \phi$, and if $M$ is the moment of the torsional elasticity of the fibre, we shall have the equation

$$Fa \cos \tfrac{1}{2}\theta = M(\theta - \phi).$$

Hence, if we can ascertain $M$, we can determine $F$, the actual force between the spheres at the distance $2a \sin \frac{1}{2}\theta$.

To find $M$, the moment of torsion, let $I$ be the moment of inertia of the torsion-arm, and $T$ the time of a double vibration of the arm under the action of the torsional elasticity, then

$$M = 4\pi^2 \frac{I}{T^2}.$$

In all electrometers it is of the greatest importance to know what force we are measuring. The force acting on the suspended sphere is due partly to the direct action of the fixed sphere, but partly also to the electrification, if any, of the sides of the case.

If the case is made of glass it is impossible to determine the electrification of its surface otherwise than by very difficult measurements at every point. If, however, either the case is made of metal, or if a metallic case which almost completely encloses the apparatus is placed as a screen between the spheres and the glass case, the electrification of the inside of the metal screen will depend entirely on that of the spheres, and the electrification of the glass case will have no influence on the spheres. In this way we may avoid any indefiniteness due to the action of the case.

To illustrate this by an example in which we can calculate all the effects, let us suppose that the case is a sphere of radius $b$, that the centre of motion of the torsion-arm coincides with the centre of the sphere and that its radius is $a$; that the charges on the two spheres are $E_1$ and $E$, and that the angle between their positions is $\theta$; that the fixed sphere is at a distance $a_1$ from the centre, and that $r$ is the distance between the two small spheres.

Neglecting for the present the effect of induction on the distribution of electricity on the small spheres, the force between them will be a repulsion

$$= \frac{EE_1}{r^2},$$

and the moment of this force round a vertical axis through the centre will be

$$\frac{EE_1\, aa_1 \sin\theta}{r^3}.$$

The image of $E_1$ due to the spherical surface of the case is a point in the same radius at a distance from the centre $\dfrac{b^2}{a_1}$ with a charge $-E_1\dfrac{b}{a_1}$, and the moment of the attraction between $E$ and this image about the axis of suspension is

$$EE_1\frac{b}{a_1}\,\frac{a\dfrac{b^2}{a_1}\sin\theta}{\left\{a^2-2\dfrac{ab^2}{a_1}\cos\theta+\dfrac{b^4}{a_1^{\,2}}\right\}^{\frac{3}{2}}}$$

$$= EE_1\frac{aa_1\sin\theta}{b^3\left\{1-2\dfrac{aa_1}{b^2}\cos\theta+\dfrac{a^2a^2_{\,1}}{b^4}\right\}^{\frac{3}{2}}}.$$

If $b$, the radius of the spherical case, is large compared with $a$ and $a_1$, the distances of the spheres from the centre, we may neglect the second and third terms of the factor in the denominator. Equating the moments tending to turn the torsion-arm, we get

$$EE_1\, aa_1 \sin\theta\left\{\frac{1}{r^3}-\frac{1}{b^3}\right\} = M(\theta-\phi).$$

*Electrometers for the Measurement of Potentials.*

216.] In all electrometers the moveable part is a body charged with electricity, and its potential is different from that of certain of the fixed parts round it. When, as in Coulomb's method, an insulated body having a certain charge is used, it is the charge which is the direct object of measurement. We may, however, connect the balls of Coulomb's electrometer, by means of fine wires, with different conductors. The charges of the balls will then depend on the values of the potentials of these conductors and on the potential of the case of the instrument. The charge on each ball will be approximately equal to its radius multiplied by the excess of its potential over that of the case of the instrument, provided the radii of the balls are small compared with their distances from each other and from the sides or opening of the case.

Coulomb's form of apparatus, however, is not well adapted for measurements of this kind, owing to the smallness of the force between spheres at the proper distances when the difference of potentials is small. A more convenient form is that of the Attracted Disk Electrometer. The first electrometers on this principle were constructed by Sir W. Snow Harris *. They have since been brought to great perfection, both in theory and construction, by Sir W. Thomson †.

When two disks at different potentials are brought face to face with a small interval between them there will be a nearly uniform electrification on the opposite faces and very little electrification on the backs of the disks, provided there are no other conductors or electrified bodies in the neighbourhood. The charge on the positive disk will be approximately proportional to its area, and to the difference of potentials of the disks, and inversely as the distance between them. Hence, by making the areas of the disks large and the distance between them small, a small difference of potential may give rise to a measurable force of attraction.

The mathematical theory of the distribution of electricity over two disks thus arranged is given at Art. 202, but since

* *Phil. Trans.* 1834.

† See an excellent report on Electrometers by Sir W. Thomson. *Report of the British Association*, Dundee, 1867.

it is impossible to make the case of the apparatus so large that we may suppose the disks insulated in an infinite space, the indications of the instrument in this form are not easily interpreted numerically.

217.] The addition of the guard-ring to the attracted disk is one of the chief improvements which Sir W. Thomson has made on the apparatus.

Instead of suspending the whole of one of the disks and determining the force acting upon it, a central portion of the disk is separated from the rest to form the attracted disk, and the outer ring forming the remainder of the disk is fixed. In this way the force is measured only on that part of the disk where it is most regular, and the want of uniformity of the

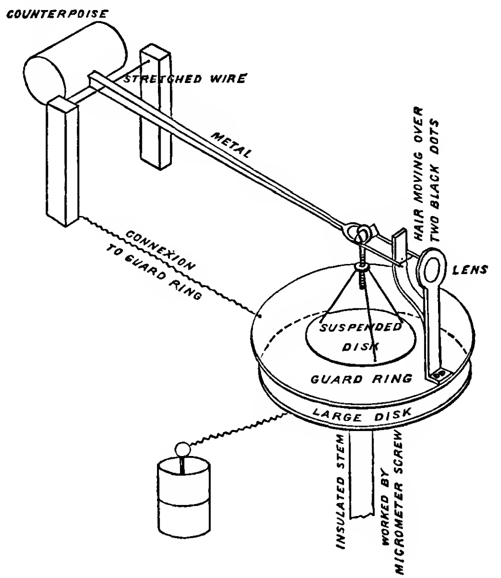



Fig. 19.

electrification near the edge is of no importance, as it occurs on the guard-ring and not on the suspended part of the disk.

Besides this, by connecting the guard-ring with a metal case surrounding the back of the attracted disk and all its suspending apparatus, the electrification of the back of the disk is rendered impossible, for it is part of the inner surface of a closed hollow conductor all at the same potential.

Thomson's Absolute Electrometer therefore consists essentially of two parallel plates at different potentials, one of which is made so that a certain area, no part of which is near the edge of the plate, is moveable under the action of electric force. To fix our ideas we may suppose the attracted disk and guard-ring uppermost. The fixed disk is horizontal, and is mounted on an insulating stem which has a measurable vertical motion given to it by means of a micrometer screw. The guard-ring is at least as large as the fixed disk; its lower surface is truly plane and parallel to the fixed disk. A delicate balance is erected on the guard-ring to which is suspended a light moveable disk which almost fills the circular aperture in the guard-ring without rubbing against its sides. The lower surface of the suspended disk must be truly plane, and we must have the means of knowing when its plane coincides with that of the lower surface of the guard-ring, so as to form a single plane interrupted only by the narrow interval between the disk and its guard-ring.

For this purpose the lower disk is screwed up till it is in contact with the guard-ring, and the suspended disk is allowed to rest upon the lower disk, so that its lower surface is in the same plane as that of the guard-ring. Its position with respect to the guard-ring is then ascertained by means of a system of fiducial marks. Sir W. Thomson generally uses for this purpose a black hair attached to the moveable part. This hair moves up or down just in front of two black dots on a white enamelled ground and is viewed along with these dots by means of a plano-convex lens with the plane side next the eye. If the hair as seen through the lens appears straight and bisects the interval between the black dots it is said to be in its *sighted position*, and indicates that the suspended disk with which it moves is in its proper position as regards height. The horizontality of the suspended disk may be tested by comparing the reflexion of part of any object from its upper surface with that of the remainder of the same object from the upper surface of the guard-ring.

The balance is then arranged so that when a known weight is placed on the centre of the suspended disk it is in equilibrium in its sighted position, the whole apparatus being freed from electrification by putting every part in metallic communication. A metal case is placed over the guard-ring so as to enclose the balance and suspended disk, sufficient apertures being left to see the fiducial marks.

The guard-ring, case, and suspended disk are all in metallic communication with each other, but are insulated from the other parts of the apparatus.

Now let it be required to measure the difference of potentials of two conductors. The conductors are put in communication with the upper and lower disks respectively by means of wires, the weight is taken off the suspended disk, and the lower disk is moved up by means of the micrometer screw till the electrical attraction brings the suspended disk down to its sighted position. We then know that the attraction between the disks is equal to the weight which brought the disk to its sighted position.

If $W$ be the numerical value of the weight, and $g$ the force of gravity, the force is $Wg$, and if $A$ is the area of the suspended disk, $D$ the distance between the disks, and $V$ the difference of the potential of the disks *,

$$Wg = \frac{V^2 A}{8\pi D^2}, \quad \text{or} \quad V = D\sqrt{\frac{8\pi g W}{A}}.$$

* Let us denote the radius of the suspended disk by $R$, and that of the aperture of the guard-ring by $R'$, then the breadth of the annular interval between the disk and the ring will be $B = R' - R$.

If the distance between the suspended disk and the large fixed disk is $D$, and the difference of potentials between these disks is $V$, then, by the investigation in Art. 201, the quantity of electricity on the suspended disk will be

$$Q = V\left\{\frac{R^2 + R'^2}{8D} - \frac{R'^2 - R^2}{8D}\frac{a}{D+a}\right\},$$

$$\text{where} \quad a = B\frac{\log_e 2}{\pi}, \quad \text{or} \quad a = 0{\cdot}220635\,(R' - R).$$

If the surface of the guard-ring is not exactly in the plane of the surface of the suspended disk, let us suppose that the distance between the fixed disk and the guard-ring is not $D$ but $D + z = D'$, then it appears from the investigation in Art. 225 that there will be an additional charge of electricity near the edge of the disk on account of its height $z$ above the general surface of the guard-ring. The whole charge in this case is therefore, approximately,

$$Q = V\left\{\frac{R^2 + R'^2}{8D} - \frac{R'^2 - R^2}{8D}\frac{a}{D+a} + \frac{R+R'}{D}(D' - D)\log_e\frac{4\pi(R+R')}{D'-D}\right\},$$

and in the expression for the attraction we must substitute for $A$, the area of the disk, the corrected quantity

$$A = \tfrac{1}{2}\pi\left\{R^2 + R'^2 - (R'^2 - R^2)\frac{a}{D+a} + 8(R+R')(D'-D)\log_e\frac{4\pi(R+R')}{D'-D}\right\},$$

where $R$ = radius of suspended disk,
$R'$ = radius of aperture in the guard-ring,
$D$ = distance between fixed and suspended disks,
$D'$ = distance between fixed disk and guard-ring,
$a$ = $0{\cdot}220635\,(R' - R)$.

When $a$ is small compared with $D$ we may neglect the second term, and when $D' - D$ is small we may neglect the last term. {For another investigation of this see Supplementary Volume.}

If the suspended disk is circular, of radius $R$, and if the radius of the aperture of the guard-ring is $R'$, then

$$A = \tfrac{1}{2}\pi\,(R^2+R'^2), \text{ and } V = 4D\sqrt{\frac{gW}{R^2+R'^2}}.$$

218.] Since there is always some uncertainty in determining the micrometer reading corresponding to $D = 0$, and since any error in the position of the suspended disk is most important when $D$ is small, Sir W. Thomson prefers to make all his measurements depend on differences of the electromotive force $V$. Thus, if $V$ and $V'$ are two potentials, and $D$ and $D'$ the corresponding distances,

$$V-V' = (D-D')\sqrt{\frac{8\pi gW}{A}}.$$

For instance, in order to measure the electromotive force of a galvanic battery, two electrometers are used.

By means of a condenser, kept charged if necessary by a replenisher, the lower disk of the principal electrometer is maintained at a constant potential. This is tested by connecting the lower disk of the principal electrometer with the lower disk of a secondary electrometer, the suspended disk of which is connected with the earth. The distance between the disks of the secondary electrometer and the force required to bring the suspended disk to its sighted position being constant, if we raise the potential of the condenser till the secondary electrometer is in its sighted position, we know that the potential of the lower disk of the principal electrometer exceeds that of the earth by a constant quantity which we may call $V$.

If we now connect the positive electrode of the battery to earth, and connect the suspended disk of the principal electrometer to the negative electrode, the difference of potentials between the disks will be $V+v$, if $v$ is the electromotive force of the battery. Let $D$ be the reading of the micrometer in this case, and let $D'$ be the reading when the suspended disk is connected with earth, then

$$v = (D-D')\sqrt{\frac{8\pi gW}{A}}.$$

In this way a small electromotive force $v$ may be measured by the electrometer with the disks at a conveniently measurable distance. When the distance is too small a small change of absolute distance makes a great change in the force, since the force varies inversely as the square of the distance, so that any error in the absolute distance introduces a large error in the result unless the distance is large compared with the limits of error of the micrometer screw.

The effects of small irregularities of form in the surfaces of the disks and of the interval between them diminish according to the inverse cube and higher inverse powers of the distance, and whatever be the form of a corrugated surface, the eminences of which just reach a plane surface, the electrical effect at any distance which is considerable compared to the breadth of the corrugations, is the same as that of a plane at a certain small distance behind the plane of the tops of the eminences. See Arts. 197, 198.

By means of the auxiliary electrification, tested by the auxiliary electrometer, a proper interval between the disks is secured.

The auxiliary electrometer may be of a simpler construction, in which there is no provision for the determination of the force of attraction in absolute measure, since all that is wanted is to secure a constant electrification. Such an electrometer may be called a *gauge* electrometer.

This method of using an auxiliary electrification besides the electrification to be measured is called the Heterostatic method of electrometry, in opposition to the Idiostatic method in which the whole effect is produced by the electrification to be measured.

In several forms of the attracted disk electrometer, the attracted disk is placed at one end of an arm which is supported by being attached to a platinum wire passing through its centre of gravity and kept stretched by means of a spring. The other end of the arm carries the hair which is brought to a sighted position by altering the distance between the disks, and so adjusting the force of the electric attraction to a constant value. In these electrometers this force is not in general determined in absolute measure, but is known to be constant, provided the torsional elasticity of the platinum wire does not change.

The whole apparatus is placed in a Leyden jar, of which the inner surface is charged and connected with the attracted disk and guard-ring. The other disk is worked by a micrometer screw, and is connected first with the earth and then with the conductor whose potential is to be measured. The difference of readings multiplied by a constant to be determined for each electrometer gives the potential required.

219.] The electrometers already described are not self-acting, but require for each observation an adjustment of a micrometer screw, or some other movement which must be made by the observer. They are therefore not fitted to act as self-registering instruments, which must of themselves move into the proper position. This condition is fulfilled by Thomson's Quadrant Electrometer.

The electrical principle on which this instrument is founded may be thus explained:—

$A$ and $B$ are two fixed conductors which may be at the same or at different potentials. $C$ is a moveable conductor at a high potential, which is so placed that part of it is opposite to the surface of $A$ and part opposite to that of $B$, and that the proportions of these parts are altered as $C$ moves.

For this purpose it is most convenient to make $C$ moveable about an axis, and make the opposed surfaces of $A$, of $B$, and of $C$ portions of surfaces of revolution about the same axis.

In this way the distance between the surface of $C$ and the opposed surfaces of $A$ or of $B$ remains always the same, and the motion of $C$ in the positive direction simply increases the area opposed to $B$ and diminishes the area opposed to $A$.

If the potentials of $A$ and $B$ are equal there will be no force urging $C$ from $A$ to $B$, but if the potential of $C$ differs from that of $B$ more than from that of $A$, then $C$ will tend to move so as to increase the area of its surface opposed to $B$.

By a suitable arrangement of the apparatus this force may be made nearly constant for different positions of $C$ within certain limits, so that if $C$ is suspended by a torsion fibre, its deflexions will be nearly proportional to the difference of potential between $A$ and $B$ multiplied by the difference of the potential of $C$ from the mean of those of $A$ and $B$.

$C$ is maintained at a high potential by means of a condenser provided with a replenisher and tested by a gauge electrometer, and $A$ and $B$ are connected with the two conductors the difference of whose potentials is to be measured. The higher the potential of $C$ the more sensitive is the instrument. This electrification of $C$, being independent of the electrification to be measured, places this electrometer in the heterostatic class.

We may apply to this electrometer the general theory of systems of conductors given in Arts. 93, 127.

Let $A$, $B$, $C$ denote the potentials of the three conductors respectively. Let $a$, $b$, $c$ be their respective capacities, $p$ the coefficient of induction between $B$ and $C$, $q$ that between $C$ and $A$, and $r$ that between $A$ and $B$. All these coefficients will in general vary with the position of $C$, and if $C$ is so arranged that the extremities of $A$ and $B$ are not near those of $C$ as long as the motion of $C$ is confined within certain limits, we may ascertain the form of these coefficients. If $\theta$ represents the deflexion of $C$ from $A$ towards $B$, then the part of the surface of $A$ opposed to $C$ will diminish as $\theta$ increases. Hence if $A$ is kept at potential 1 while $B$ and $C$ are kept at potential 0, the charge on $A$ will be $a = a_0 - \alpha\theta$, where $a_0$ and $\alpha$ are constants, and $a$ is the capacity of $A$.

If $A$ and $B$ are symmetrical, the capacity of $B$ is $b = b_0 + \alpha\theta$.

The capacity of $C$ is not altered by the motion, for the only effect of the motion is to bring a different part of $C$ opposite to the interval between $A$ and $B$. Hence $c = c_0$.

The quantity of electricity induced on $C$ when $B$ is raised to potential unity is $p = p_0 - \alpha\theta$.

The coefficient of induction between $A$ and $C$ is $q = q_0 + \alpha\theta$.

The coefficient of induction between $A$ and $B$ is not altered by the motion of $C$, but remains $r = r_0$.

Hence the electrical energy of the system is

$$W = \tfrac{1}{2}A^2a + \tfrac{1}{2}B^2b + \tfrac{1}{2}C^2c + BCp + CAq + ABr,$$

and if $\Theta$ is the moment of the force tending to increase $\theta$,

$$\Theta = \frac{dW}{d\theta}, \; A,\, B,\, C \text{ being supposed constant,}$$

$$= \tfrac{1}{2}A^2\frac{da}{d\theta} + \tfrac{1}{2}B^2\frac{db}{d\theta} + \tfrac{1}{2}C^2\frac{dc}{d\theta} + BC\frac{dp}{d\theta} + CA\frac{dq}{d\theta} + AB\frac{dr}{d\theta},$$

$$= -\tfrac{1}{2}A^2\alpha + \tfrac{1}{2}B^2\alpha - BC\alpha + CA\alpha\,;$$

$$\text{or} \qquad \Theta = \alpha\,(A-B)\,\{C - \tfrac{1}{2}(A+B)\}^*.$$

* {This can also be deduced as follows: If the needle is symmetrically placed within the quadrants there will be no couple when $A = B$. Since $dW/d\theta$ vanishes in this case for all possible values of $C$, we must have

$$\tfrac{1}{2}\frac{da}{d\theta} + \tfrac{1}{2}\frac{db}{d\theta} + \frac{dr}{d\theta} = 0,$$

$$\frac{dp}{d\theta} + \frac{dq}{d\theta} = 0,$$

$$\frac{dc}{d\theta} = 0.$$

In the present form of Thomson's Quadrant Electrometer the conductors $A$ and $B$ are in the form of a cylindrical box completely divided into four quadrants, separately insulated, but joined by wires so that two opposite quadrants $A$ and $A'$ are connected together as are also the two others $B$ and $B'$.

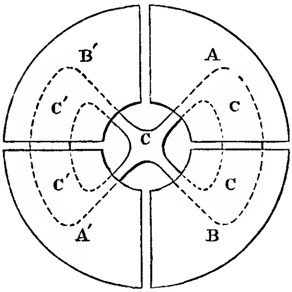



Fig. 20.

The conductor $C$ is suspended so as to be capable of turning about a vertical axis, and may consist of two opposite flat quadrantal arcs supported by radii at their extremities. In the position of equilibrium these quadrants should be partly within $A$ and partly within $B$, and the supporting radii should be near the middle of the quadrants of the hollow base, so that the divisions of the box and the extremities and supports of $C$ may be as far from each other as possible.

The conductor $C$ is kept permanently at a high potential by being connected with the inner coating of the Leyden jar which forms the case of the instrument. $B$ and $A$ are connected, the first with the earth, and the other with the body whose potential is to be measured.

If the potential of this body is zero, and if the instrument be in adjustment, there ought to be no force tending to make $C$ move, but if the potential of $A$ is of the same sign as that of $C$, then $C$ will tend to move from $A$ to $B$ with a nearly uniform force, and the suspension apparatus will be twisted till an equal force is called into play and produces equilibrium. Within

So that $$\frac{dW}{d\theta} = \tfrac{1}{2}(A-B)\left(A\frac{da}{d\theta} - B\frac{db}{d\theta} + 2C\frac{dq}{d\theta}\right).$$

If the quadrants entirely surround the needle the couple will not be affected by increasing all the potentials by the same amount, hence

$$\frac{da}{d\theta} - \frac{db}{d\theta} + 2\frac{dq}{d\theta} = 0.$$

So that $$\frac{dW}{d\theta} = \tfrac{1}{2}(A-B)\left\{(A-C)\frac{da}{d\theta} - (B-C)\frac{db}{d\theta}\right\}.$$

If the quadrants are symmetrical $\frac{da}{d\theta} = -\frac{db}{d\theta}$ and we get the expression in the text.

The student should also consult Dr. G. Hopkinson's Paper on the Quadrant Electrometer, *Phil. Mag.* 5th series, xix. p. 291, and Hallwachs Wied. Ann. xxix. p. 11.}

certain limits the deflexions of $C$ will be proportional to the product

$$(A-B)\{C-\tfrac{1}{2}(A+B)\}.$$

By increasing the potential of $C$ the sensibility of the instrument may be increased, and for small values of $\frac{1}{2}(A+B)$ the deflexions will be nearly proportional to $(A-B)\,C$.

## *On the Measurement of Electric Potential.*

220.] In order to determine large differences of potential in absolute measure we may employ the attracted disk electrometer, and compare the attraction with the effect of a weight. If at the same time we measure the difference of potential of the same conductors by means of the quadrant electrometer, we shall ascertain the absolute value of certain readings of the scale of the quadrant electrometer, and in this way we may deduce the value of the scale readings of the quadrant electrometer in terms of the potential of the suspended part, and the moment of torsion of the suspension apparatus*.

To ascertain the potential of a charged conductor of finite size we may connect the conductor with one electrode of the electrometer, while the other is connected to earth or to a body of constant potential. The electrometer reading will give the potential of the conductor after the division of its electricity between it and the part of the electrometer with which it is put in contact. If $K$ denote the capacity of the conductor, and $K'$ that of this part of the electrometer, and if $V$, $V'$ denote the potentials of these bodies before making contact, then their common potential after making contact will be

$$\overline{V} = \frac{KV+K'V'}{K+K'}.$$

Hence the original potential of the conductor was

$$V = \overline{V} + \frac{K'}{K}(\overline{V}-V').$$

If the conductor is not large compared with the electrometer, $K'$ will be comparable with $K$, and unless we can ascertain the values of $K$ and $K'$ the second term of the expression will have a doubtful value. But if we can make the potential of the

* {Large differences of potential are more conveniently measured by means of Sir William Thomson's new Voltmeter.}

electrode of the electrometer very nearly equal to that of the body before making contact, then the uncertainty of the values of $K$ and $K'$ will be of little consequence.

If we know the value of the potential of the body approximately, we may charge the electrode by means of a 'replenisher' or otherwise to this approximate potential, and the next experiment will give a closer approximation. In this way we may measure the potential of a conductor whose capacity is small compared with that of the electrometer.

*To Measure the Potential at any Point in the Air.*

221.] *First Method.* Place a sphere, whose radius is small compared with the distance of electrified conductors, with its centre at the given point. Connect it by means of a fine wire with the earth, then insulate it, and carry it to an electrometer and ascertain the total charge on the sphere.

Then, if $V$ be the potential at the given point, and $a$ the radius of the sphere, the charge on the sphere will be $-Va = Q$, and if $V'$ be the potential of the sphere as measured by an electrometer when placed in a room whose walls are connected with the earth, then
$$Q = V'a,$$
whence
$$V + V' = 0,$$
or the potential of the air at the point where the centre of the sphere was placed is equal but of opposite sign to the potential of the sphere after being connected to earth, then insulated, and brought into a room.

This method has been employed by M. Delmann of Creuznach in measuring the potential at a certain height above the earth's surface.

*Second Method.* We have supposed the sphere placed at the given point and first connected to earth, and then insulated, and carried into a space surrounded with conducting matter at potential zero.

Now let us suppose a fine insulated wire carried from the electrode of the electrometer to the place where the potential is to be measured. Let the sphere be first discharged completely. This may be done by putting it into the inside of a vessel of the same metal which nearly surrounds it and making it touch the vessel. Now let the sphere thus discharged be carried to the end of the wire and made to touch it. Since the sphere is not electrified it will be at the potential of the air at the place. If the electrode wire is at the same potential it will not be affected by the contact, but if the electrode is at a different potential it will by contact with the sphere be made nearer to that of the air than it was before. By a succession of such operations, the sphere being alternately discharged and made to touch the electrode, the potential of the electrode of the electrometer will continually approach that of the air at the given point.

222.] To measure the potential of a conductor without touching it, we may measure the potential of the air at any point in the neighbourhood of the conductor, and calculate that of the conductor from the result. If there be a hollow nearly surrounded by the conductor, then the potential at any point of the air in this hollow will be very nearly that of the conductor.

In this way it has been ascertained by Sir W. Thomson that if two hollow conductors, one of copper and the other of zinc, are in metallic contact, then the potential of the air in the hollow surrounded by zinc is positive with reference to that of the air in the hollow surrounded by copper.

*Third Method.* If by any means we can cause a succession of small bodies to detach themselves from the end of the electrode, the potential of the electrode will approximate to that of the surrounding air. This may be done by causing shot, filings, sand, or water to drop out of a funnel or pipe connected with the electrode. The point at which the potential is measured is that at which the stream ceases to be continuous and breaks into separate parts or drops.

Another convenient method is to fasten a slow match to the electrode. The potential is very soon made equal to that of the air at the burning end of the match. Even a fine metallic point is sufficient to create a discharge by means of the particles of the air {or dust?} when the difference of potentials is considerable, but if we wish to reduce this difference to zero, we must use one of the methods stated above.

If we only wish to ascertain the sign of the difference of the potentials at two places, and not its numerical value, we may cause drops or filings to be discharged at one of the places from a nozzle connected with the other place, and catch the drops or filings in an insulated vessel. Each drop as it falls is charged with a certain amount of electricity, and it is completely discharged into the vessel. The charge of the vessel therefore is continually accumulating, and after a sufficient number of drops have fallen, the charge of the vessel may be tested by the roughest methods. The sign of the charge is positive if the potential of the place connected to the nozzle is positive relatively to that of the other place.

## MEASUREMENT OF SURFACE-DENSITY OF ELECTRIFICATION.

### *Theory of the Proof Plane.*

223.] In testing the results of the mathematical theory of the distribution of electricity on the surface of conductors, it is necessary to be able to measure the surface-density at different points of the conductor. For this purpose Coulomb employed a small disk of gilt paper fastened to an insulating stem of gum-lac. He applied this disk to various points of the conductor by placing it so as to coincide as nearly as possible with the surface of the conductor. He then removed it by means of the insulating stem, and measured the charge of the disk by means of his electrometer.

Since the surface of the disk, when applied to the conductor, nearly coincided with that of the conductor, he concluded that the surface-density on the outer surface of the disk was nearly equal to that on the surface of the conductor at that place, and that the charge on the disk when removed was nearly equal to that on an area of the surface of the conductor equal to that of one side of the disk. A disk, when employed in this way, is called a Coulomb's Proof Plane.

As objections have been raised to Coulomb's use of the proof plane, I shall make some remarks on the theory of the experiment.

This experiment consists in bringing a small conducting body into contact with the surface of the conductor at the point where the density is to be measured, and then removing the body and determining its charge.

We have first to shew that the charge on the small body when in contact with the conductor is proportional to the surface-density which existed at the point of contact before the small body was placed there.

We shall suppose that all the dimensions of the small body, and especially its dimension in the direction of the normal at the point of contact, are small compared with either of the radii of curvature of the conductor at the point of contact. Hence the variation of the resultant force due to the conductor supposed rigidly electrified within the space occupied by the small body may be neglected, and we may treat the surface of the conductor near the small body as a plane surface.

Now the charge which the small body will take by contact with a plane surface will be proportional to the resultant force normal to the surface, that is, to the surface-density. We shall ascertain the amount of the charge for particular forms of the body.

We have next to shew that when the small body is removed no spark will pass between it and the conductor, so that it will carry its charge with it. This is evident, because when the bodies are in contact their potentials are the same, and therefore the density on the parts nearest to the point of contact is extremely small. When the small body is removed to a very short distance from the conductor, which we shall suppose to be electrified positively, then the electrification at the point nearest to the small body is no longer zero but positive, but, since the charge of the small body is positive, the positive electrification close to the small body will be less than at other neighbouring points of the surface. Now the passage of a spark depends in general on the magnitude of the resultant force, and this on the surface-density. Hence, since we suppose that the conductor is not so highly electrified as to be discharging electricity from the other parts of its surface, it will not discharge a spark to the small body from a part of its surface which we have shewn to have a smaller surface-density.

224.] We shall now consider various forms of the small body.

Suppose it to be a small hemisphere applied to the conductor so as to touch it at the centre of its flat side.

Let the conductor be a large sphere, and let us modify the form of the hemisphere so that its surface is a little more than a hemisphere, and meets the surface of the sphere at right angles. Then we have a case of which we have already obtained the exact solution. See Art. 168.

If $A$ and $B$ be the centres of the two spheres cutting each other at right angles, $DD'$ a diameter of the circle of intersection, and $C$ the centre of that circle, then if $V$ is the potential of a conductor whose outer surface coincides with that of the two spheres, the quantity of electricity on the exposed surface of the sphere $A$ is
$$\tfrac{1}{2}V(AD+BD+AC-CD-BC),$$
and that on the exposed surface of the sphere $B$ is
$$\tfrac{1}{2}V(AD+BD+BC-CD-AC),$$
the total charge being the sum of these, or
$$V(AD+BD-CD).$$

If $\alpha$ and $\beta$ are the radii of the spheres, then, when $\alpha$ is large compared with $\beta$, the charge on $B$ is to that on $A$ in the ratio of
$$\frac{3}{4}\frac{\beta^2}{\alpha^2}\left(1+\frac{1}{3}\frac{\beta}{\alpha}+\frac{1}{6}\frac{\beta^2}{\alpha^2}+\&c.\right) \text{ to } 1.$$

Now let $\sigma$ be the uniform surface-density on $A$ when $B$ is removed, then the charge on $A$ is
$$4\pi\alpha^2\sigma,$$
and therefore the charge on $B$ is
$$3\pi\beta^2\sigma\left(1+\frac{1}{3}\frac{\beta}{\alpha}+\&c.\right),$$
or, when $\beta$ is very small compared with $\alpha$, the charge on the hemisphere $B$ is equal to three times that due to a surface-density $\sigma$ extending over an area equal to that of the circular base of the hemisphere.

It appears from Art. 175 that if a small sphere is made to touch an electrified body, and is then removed to a distance from it, the mean surface-density on the sphere is to the surface-density of the body at the point of contact as $\pi^2$ is to 6, or as 1.645 to 1.

225.] The most convenient form for the proof plane is that of a circular disk. We shall therefore shew how the charge on a circular disk laid on an electrified surface is to be measured.

For this purpose we shall construct a value of the potential function so that one of the equipotential surfaces resembles a circular flattened protuberance whose general form is somewhat like that of a disk lying on a plane.

Let $\sigma$ be the surface-density of a plane, which we shall suppose to be that of $xy$.

The potential due to this electrification will be

$$V = -4\pi\sigma z.$$

Now let two disks of radius $a$ be rigidly electrified with surface-densities $-\sigma'$ and $+\sigma'$. Let the first of these be placed on the plane of $xy$ with its centre at the origin, and the second parallel to it at the very small distance $c$.

Then it may be shewn, as we shall see in the theory of magnetism, that the potential of the two disks at any point is $\omega\,\sigma' c$, where $\omega$ is the solid angle subtended by the edge of either disk at the point. Hence the potential of the whole system will be

$$V = -4\pi\sigma z + \sigma' c\omega.$$

The forms of the equipotential surfaces and lines of induction are given on the left-hand side of Fig. XIII A.

Let us trace the form of the surface for which $V=0$. This surface is indicated by the dotted line.

Putting the distance of any point from the axis of $z = r$, then, when $r$ is much less than $a$, and $z$ is small, we find

$$\omega = 2\pi - 2\pi\frac{z}{a} + \&c.$$

Hence, for values of $r$ considerably less than $a$, the equation of the zero equipotential surface is

$$0 = -4\pi\sigma z_0 + 2\pi\sigma' c - 2\pi\sigma'\frac{z_0 c}{a} + \&c.;$$

$$\text{or} \quad z_0 = \frac{\sigma' c}{2\sigma + \sigma'\dfrac{c}{a}}.$$

Hence this equipotential surface near the axis is nearly flat.

Outside the disk, where $r$ is greater than $a$, $\omega$ is zero when $z$ is zero, so that the plane of $xy$ is part of the equipotential surface.

To find where these two parts of the surface meet, let us find at what point of this plane $\dfrac{dV}{dz} = 0$.

When $r$ is very nearly equal to $a$, the solid angle $\omega$ becomes approximately a lune of the sphere of unit radius whose angle is $\tan^{-1}\{z \div (r-a)\}$, that is, $\omega$ is $2\tan^{-1}\{z \div (r-a)\}$, so that when $z = 0$ $\dfrac{dV}{dz} = -4\pi\sigma + \dfrac{2\sigma' c}{r-a}$, approximately.

Hence, when

$$\frac{dV}{dz} = 0, \quad r_0 = a + \frac{\sigma' c}{2\pi\sigma} = a + \frac{z_0}{\pi}, \text{ nearly.}$$

The equipotential surface $V=0$ is therefore composed of a disk-like figure of radius $r_0$, and nearly uniform thickness $z_0$, and of the part of the infinite plane of $xy$ which lies beyond this figure.

The surface-integral over the whole disk gives the charge of electricity on it. It may be found, as in the theory of a circular current in Part IV, Art. 704, to be

$$Q = 4\pi a\sigma' c\left\{\log\frac{8a}{r_0 - a} - 2\right\} + \pi\sigma r_0^2.$$

The charge on an equal area of the plane surface is $\pi\sigma r_0^2$, hence the charge on the disk exceeds that on an equal area of the plane very nearly in the ratio of

$$1 + 8\frac{z_0}{r_0}\log\frac{8\pi r_0}{z_0} \text{ to unity,}$$

where $z_0$ is the thickness and $r_0$ the radius of the disk, $z_0$ being supposed small compared with $r_0$.

### *On Electric Accumulators and the Measurement of Capacity.*

226.] An Accumulator or Condenser is an apparatus consisting of two conducting surfaces separated by an insulating dielectric medium.

A Leyden jar is an accumulator in which an inside coating of tinfoil is separated from the outside coating by the glass of which the jar is made. The original Leyden phial was a glass vessel containing water which was separated by the glass from the hand which held it.

The outer surface of any insulated conductor may be considered as one of the surfaces of an accumulator, the other being the earth or the walls of the room in which it is placed, and the intervening air being the dielectric medium.

The capacity of an accumulator is measured by the quantity of electricity with which the inner surface must be charged to make the difference between the potentials of the surfaces unity.

Since every electrical potential is the sum of a number of parts found by dividing each electrical element by its distance from a point, the ratio of a quantity of electricity to a potential must have the dimensions of a line. Hence electrostatic capacity is a linear quantity, or we may measure it in feet or metres without ambiguity.

In electrical researches accumulators are used for two principal purposes, for receiving and retaining large quantities of electricity in as small a compass as possible, and for measuring definite quantities of electricity by means of the potential to which they raise the accumulator.

For the retention of electrical charges nothing has been devised more perfect than the Leyden jar. The principal part of the loss arises from the electricity creeping along the damp uncoated surface of the glass from the one coating to the other. This may be checked in a great degree by artificially drying the air within the jar, and by varnishing the surface of the glass where it is exposed to the atmosphere. In Sir W. Thomson's electroscopes there is a very small percentage of loss from day to day, and I believe that none of this loss can be traced to direct conduction either through air or through glass when the glass is good, but that it arises chiefly from superficial conduction along the various insulating stems and glass surfaces of the instrument.

In fact, the same electrician has communicated a charge to sulphuric acid in a large bulb with a long neck, and has then hermetically sealed the neck by fusing it, so that the charge was completely surrounded by glass, and after some years the charge was found still to be retained.

It is only, however, when cold, that glass insulates in this way, for the charge escapes at once if the glass is heated to a temperature below 100°C.

When it is desired to obtain great capacity in small compass, accumulators in which the dielectric is sheet caoutchouc, mica, or paper impregnated with paraffin are convenient.

227.] For accumulators of the second class, intended for the measurement of quantities of electricity, all solid dielectrics must be employed with great caution on account of the property which they possess called Electric Absorption.

The only safe dielectric for such accumulators is air, which has this inconvenience, that if any dust or dirt gets into the narrow space between the opposed surfaces, which ought to be occupied only by air, it not only alters the thickness of the stratum of air, but may establish a connexion between the opposed surfaces, in which case the accumulator will not hold a charge.

To determine in absolute measure, that is to say in feet or metres, the capacity of an accumulator, we must either first ascertain its form and size, and then solve the problem of the distribution of electricity on its opposed surfaces, or we must compare its capacity with that of another accumulator, for which this problem has been solved.

As the problem is a very difficult one, it is best to begin with an accumulator constructed of a form for which the solution is known. Thus the capacity of an insulated sphere in an unlimited space is known to be measured by the radius of the sphere.

A sphere suspended in a room was actually used by MM. Kohlrausch and Weber, as an absolute standard with which they compared the capacity of other accumulators.

The capacity, however, of a sphere of moderate size is so small when compared with the capacities of the accumulators in common use that the sphere is not a convenient standard measure.

Its capacity might be greatly increased by surrounding the sphere with a hollow concentric spherical surface of somewhat greater radius. The capacity of the inner surface is then a fourth proportional to the thickness of the stratum of air and the radii of the two surfaces.

Sir W. Thomson has employed this arrangement as a standard of capacity, {it has also been used by Prof. Rowland and Mr. Rosa in their determinations of the ratio of the electromagnetic to the electrostatic unit of electricity, *Phil. Mag.* ser. v. 28, pp. 304, 315,} but the difficulties of working the surfaces truly spherical, of making them truly concentric, and of measuring their distance and their radii with sufficient accuracy, are considerable.

We are therefore led to prefer for an absolute measure of capacity a form in which the opposed surfaces are parallel planes.

The accuracy of the surface of the planes can be easily tested, and their distance can be measured by a micrometer screw, and may be made capable of continuous variation, which is a most important property of a measuring instrument.

The only difficulty remaining arises from the fact that the planes must necessarily be bounded, and that the distribution of electricity near the boundaries of the planes has not been rigidly calculated. It is true that if we make them equal circular disks, whose radius is large compared with the distance between them, we may treat the edges of the disks as if they were straight lines, and calculate the distribution of electricity by the method due to Helmholtz, and described in Art. 202. But it will be noticed that in this case part of the electricity is distributed on the back of each disk, and that in the calculation it has been supposed that there are no conductors in the neighbourhood, which is not and cannot be the case with a small instrument.

228.] We therefore prefer the following arrangement, due to Sir W. Thomson, which we may call the Guard-ring arrangement, by means of which the quantity of electricity on an insulated disk may be exactly determined in terms of its potential.

*The Guard-ring Accumulator.*

*Bb* is a cylindrical vessel of conducting material of which the outer surface of the upper face is accurately plane. This upper surface consists of two parts, a disk *A*, and a broad ring *BB* surrounding the disk, separated from it by a very small interval all round, just sufficient to prevent sparks passing. The upper surface of the disk is accurately in the same plane with that of the guard-ring. The disk is supported by pillars of insulating material *GG*. *C* is a metal disk, the under surface of which is accurately plane and parallel to *BB*. The disk *C* is considerably larger than *A*. Its distance from *A* is adjusted and measured by means of a micrometer screw, which is not given in the figure.

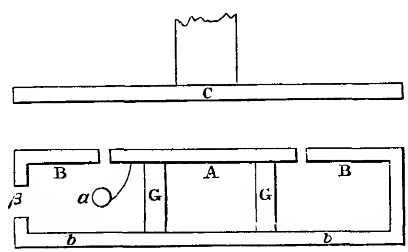


Fig. 21.

This accumulator is used as a measuring instrument as follows:—

Suppose *C* to be at potential zero, and the disk *A* and vessel *Bb* both at potential *V*. Then there will be no electrification on the back of the disk because the vessel is nearly closed and is all at the same potential. There will be very little electrification on the edges of the disk because $BB$ is at the same potential with the disk. On the face of the disk the electrification will be nearly uniform, and therefore the whole charge on the disk will be almost exactly represented by its area multiplied by the surface-density on a plane, as given in Art. 124.

In fact, we learn from the investigation in Art. 201 that the charge on the disk is

$$V\left\{\frac{R^2+R'^2}{8A} - \frac{R'^2-R^2}{8A}\,\frac{a}{A+a}\right\},$$

where $R$ is the radius of the disk, $R'$ that of the hole in the guard-ring, $A$ the distance between $A$ and $C$, and $a$ a quantity which cannot exceed $(R'-R)\dfrac{\log_e 2}{\pi}$.

If the interval between the disk and the guard-ring is small compared with the distance between $A$ and $C$, the second term will be very small, and the charge on the disk will be nearly

$$V\frac{R^2+R'^2}{8A}.$$

{This is very nearly the same as the charge on a disk uniformly electrified with the surface-density $V/4\pi A$, whose radius is the arithmetic mean between those of the original disk and the hole.}

Now let the vessel $Bb$ be put in connexion with the earth. The charge on the disk $A$ will no longer be uniformly distributed, but it will remain the same in quantity, and if we now discharge $A$ we shall obtain a quantity of electricity, the value of which we know in terms of $V$, the original difference of potentials and the measurable quantities $R$, $R'$ and $A$.

### *On the Comparison of the Capacity of Accumulators.*

229.] The form of accumulator which is best fitted to have its capacity determined in absolute measure from the form and dimensions of its parts is not generally the most suitable for electrical experiments. It is desirable that the measures of capacity in actual use should be accumulators having only two conducting surfaces, one of which is as nearly as possible surrounded by the other. The guard-ring accumulator, on the other hand, has three independent conducting portions which must be charged and discharged in a certain order. Hence it is desirable to be able to compare the capacities of two accumulators by an electrical process, so as to test accumulators which may afterwards serve as secondary standards.

I shall first shew how to test the equality of the capacity of two guard-ring accumulators.

Let $A$ be the disk, $B$ the guard-ring with the rest of the conducting vessel attached to it, and $C$ the large disk of one of these accumulators, and let $A'$, $B'$, and $C'$ be the corresponding parts of the other.

If either of these accumulators is of the more simple kind, having only two conductors, we have only to suppress $B$ or $B'$, and to suppose $A$ to be the inner and $C$ the outer conducting surface, $C$ in this case being understood to surround $A$.

Let the following connexions be made.

Let $B$ be kept always connected with $C'$, and $B'$ with $C$, that is, let each guard-ring be connected with the large disk of the other condenser.

(1) Let $A$ be connected with $B$ and $C'$ and with $J$, the electrode of a Leyden jar with a positive charge, and let $A'$ be connected with $B'$ and $C$ and with the earth.

(2) Let $A$, $B$, and $C'$ be insulated from $J$.

(3) Let $A$ be insulated from $B$ and $C'$, and $A'$ from $B'$ and $C$.

(4) Let $B$ and $C'$ be connected with $B'$ and $C$ and with the earth.

(5) Let $A$ be connected with $A'$.

(6) Let $A$ and $A'$ be connected with an electroscope $E$.

We may express these connexions as follows:—

$$
\begin{array}{lllll}
(1) & 0 = C = B' = A' & | & A = B = C' = J. \\
(2) & 0 = C = B' = A' & | & A = B = C' \mid J. \\
(3) & 0 = C = B' \mid A' & | & A \mid B = C'. \\
(4) & 0 = C = B' \mid A' & | & A \mid B = C' = 0. \\
(5) & 0 = C = B' \mid A' & = & A \mid B = C' = 0. \\
(6) & 0 = C = B' \mid A' = E = A & & \mid B = C' = 0.
\end{array}
$$

Here the sign of equality expresses electrical connexion, and the vertical stroke expresses insulation.

In (1) the two accumulators are charged oppositely, so that $A$ is positive and $A'$ negative, the charges on $A$ and $A'$ being uniformly distributed on the upper surface opposed to the large disk of each accumulator.

In (2) the jar is removed, and in (3) the charges on $A$ and $A'$ are insulated.

In (4) the guard-rings are connected with the large disks, so that the charges on $A$ and $A'$, though unaltered in magnitude, are now distributed over their whole surfaces.

In (5) $A$ is connected with $A'$. If the charges are equal and of opposite signs, the electrification will be entirely destroyed, and in (6) this is tested by means of the electroscope $E$.

The electroscope $E$ will indicate positive or negative electrification according as $A$ or $A'$ has the greater capacity.

By means of a key of proper construction*, the whole of these operations can be performed in due succession in a very small fraction of a second, and the capacities adjusted till no electrification can be detected by the electroscope, and in this way the capacity of an accumulator may be adjusted to be equal to that of any other, or to the sum of the capacities of several accumulators, so that a system of accumulators may be formed, each of which has its capacity determined in absolute measure, i.e. in feet or in metres, while at the same time it is of the construction most suitable for electrical experiments.

This method of comparison will probably be found useful in determining the specific capacity for electrostatic induction of different dielectrics in the form of plates or disks. If a disk of the dielectric is interposed between $A$ and $C$, the disk being considerably larger than $A$, then the capacity of the accumulator will be altered and made equal to that of the same accumulator when $A$ and $C$ are nearer together. If the accumulator with the dielectric plate, and with $A$ and $C$ at distance $x$, is of the same capacity as the same accumulator without the dielectric, and with $A$ and $C$ at distance $x'$, then, if $a$ is the thickness of the plate, and $K$ its specific dielectric inductive capacity referred to air as a standard,

$$K = \frac{a}{a + x' - x}.$$

The combination of three cylinders, described in Art. 127, has been employed by Sir W. Thomson as an accumulator whose

* {Such a key is described in Dr. Hopkinson's paper on the Electrostatic Capacity of Glass and of Liquids, *Phil. Trans.*, 1881, Part II, p. 360.}

capacity may be increased or diminished by measurable quantities.

The experiments of MM. Gibson and Barclay with this apparatus are described in the *Proceedings of the Royal Society*, Feb. 2, 1871, and *Phil. Trans.*, 1871, p. 573. They found the specific inductive capacity of solid paraffin to be 1.975, that of air being unity.

# PART II.

## ELECTROKINEMATICS.

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## CHAPTER I.

### THE ELECTRIC CURRENT.

230.] We have seen, in Art. 45, that when a conductor is in electrical equilibrium the potential at every point of the conductor must be the same.

If two conductors $A$ and $B$ are charged with electricity so that the potential of $A$ is higher than that of $B$, then, if they are put in communication by means of a metallic wire $C$ touching both of them, part of the charge of $A$ will be transferred to $B$, and the potentials of $A$ and $B$ will become in a very short time equalized.

231.] During this process certain phenomena are observed in the wire $C$, which are called the phenomena of the electric conflict or current.

The first of these phenomena is the transference of positive electrification from $A$ to $B$ and of negative electrification from $B$ to $A$. This transference may be also effected in a slower manner by bringing a small insulated body into contact with $A$ and $B$ alternately. By this process, which we may call electrical convection, successive small portions of the electrification of each body are transferred to the other. In either case a certain quantity of electricity, or of the state of electrification, passes from one place to another along a certain path in the space between the bodies.

Whatever therefore may be our opinion of the nature of electricity, we must admit that the process which we have described constitutes a current of electricity. This current may be described as a current of positive electricity from $A$ to $B$, or a current of negative electricity from $B$ to $A$, or as a combination of these two currents.

According to Fechner's and Weber's theory it is a combination of a current of positive electricity with an exactly equal current of negative electricity in the opposite direction through the same substance. It is necessary to remember this exceedingly artificial hypothesis regarding the constitution of the current in order to understand the statement of some of Weber's most valuable experimental results.

If, as in Art. 36, we suppose $P$ units of positive electricity transferred from $A$ to $B$, and $N$ units of negative electricity transferred from $B$ to $A$ in unit of time, then, according to Weber's theory, $P = N$, and $P$ or $N$ is to be taken as the numerical measure of the current.

We, on the contrary, make no assumption as to the relation between $P$ and $N$, but attend only to the result of the current, namely, the transference of $P+N$ units of positive electrification from $A$ to $B$, and we shall consider $P+N$ the true measure of the current. The current, therefore, which Weber would call 1 we shall call 2.

*On Steady Currents.*

232.] In the case of the current between two insulated conductors at different potentials the operation is soon brought to an end by the equalization of the potentials of the two bodies, and the current is therefore essentially a Transient Current.

But there are methods by which the difference of potentials of the conductors may be maintained constant, in which case the current will continue to flow with uniform strength as a Steady Current.

*The Voltaic Battery.*

The most convenient method of producing a steady current is by means of the Voltaic Battery.

For the sake of distinctness we shall describe Daniell's Constant Battery :—

A solution of sulphate of zinc is placed in a cell of porous earthenware, and this cell is placed in a vessel containing a saturated solution of sulphate of copper. A piece of zinc is dipped into the sulphate of zinc, and a piece of copper is dipped into the sulphate of copper. Wires are soldered to the zinc and to the copper above the surfaces of the liquids. This combination is called a cell or element of Daniell's battery. See Art. 272.

233.] If the cell is insulated by being placed on a non-conducting stand, and if the wire connected with the copper is put in contact with an insulated conductor $A$, and the wire connected with the zinc is put in contact with $B$, another insulated conductor of the same metal as $A$, then it may be shewn by means of a delicate electrometer that the potential of $A$ exceeds that of $B$ by a certain quantity. This difference of potentials is called the Electromotive Force of the Daniell's Cell.

If $A$ and $B$ are now disconnected from the cell and put in communication by means of a wire, a transient current passes through the wire from $A$ to $B$, and the potentials of $A$ and $B$ become equal. $A$ and $B$ may then be charged again by the cell, and the process repeated as long as the cell will work. But if $A$ and $B$ be connected by means of the wire $C$, and at the same time connected with the battery as before, then the cell will maintain a constant current through $C$, and also a constant difference of potentials between $A$ and $B$. This difference will not, as we shall see, be equal to the whole electromotive force of the cell, for part of this force is spent in maintaining the current through the cell itself.

A number of cells placed in series so that the zinc of the first cell is connected by metal with the copper of the second and so on, is called a Voltaic Battery. The electromotive force of such a battery is the sum of the electromotive forces of the cells of which it is composed. If the battery is insulated it may be charged with electricity as a whole, but the potential of the copper end will always exceed that of the zinc end by the electromotive force of the battery, whatever the absolute value of either of these potentials may be. The cells of the battery may be of very various construction, containing different chemical substances and different metals, provided they are such that chemical action does not go on when no current passes.

234.] Let us now consider a voltaic battery with its ends insulated from each other. The copper end will be positively or vitreously electrified, and the zinc end will be negatively or resinously electrified.

Let the two ends of the battery be now connected by means of a wire. An electric current will commence, and will in a very short time attain a constant value. It is then said to be a Steady Current.

*Properties of the Current.*

235.] The current forms a closed circuit in the direction from copper to zinc through the wires, and from zinc to copper through the solutions.

If the circuit be broken by cutting any of the wires which connect the copper of one cell with the zinc of the next in order, the current will be stopped, and the potential of the end of the wire in connexion with the copper will be found to exceed that of the end of the wire in connexion with the zinc by a constant quantity, namely, the total electromotive force of the circuit.

*Electrolytic Action of the Current.*

236.] As long as the circuit is broken no chemical action goes on in the cells, but as soon as the circuit is completed, zinc is dissolved from the zinc in each of the Daniell's cells, and copper is deposited on the copper.

The quantity of sulphate of zinc increases, and the quantity of sulphate of copper diminishes unless more is constantly supplied.

The quantity of zinc dissolved, and also that of copper deposited, is the same in each of the Daniell's cells throughout the circuit, whatever the size of the plates of the cell, and if any one of the cells be of a different construction, the amount of chemical action in it bears a constant proportion to the action in the Daniell's cell. For instance, if one of the cells consists of two platinum plates dipped into sulphuric acid diluted with water, oxygen will be given off at the surface of the plate where the current enters the liquid, namely, the plate in metallic connexion with the copper of Daniell's cell, and hydrogen at the surface of the plate where the current leaves the liquid, namely, the plate connected with the zinc of Daniell's cell.

The volume of the hydrogen is exactly twice the volume of the oxygen given off in the same time, and the weight of the oxygen is exactly eight times the weight of the hydrogen.

In every cell of the circuit the weight of each substance dissolved, deposited, or decomposed is equal to a certain quantity called the electrochemical equivalent of that substance, multiplied by the strength of the current and by the time during which it has been flowing.

For the experiments which established this principle, see the seventh and eighth series of Faraday's *Experimental Researches*; and for an investigation of the apparent exceptions to the rule, see Miller's *Chemical Physics* and Wiedemann's *Galvanismus*.

237.] Substances which are decomposed in this way are called Electrolytes. The process is called Electrolysis. The places where the current enters and leaves the electrolyte are called Electrodes. Of these the electrode by which the current enters is called the Anode, and that by which it leaves the electrolyte is called the Cathode. The components into which the electrolyte is resolved are called Ions: that which appears at the anode is called the Anion, and that which appears at the cathode is called the Cation.

Of these terms, which were, I believe, invented by Faraday with the help of Dr. Whewell, the first three, namely, electrode, electrolysis, and electrolyte have been generally adopted, and the mode of conduction of the current in which this kind of decomposition and transfer of the components takes place is called Electrolytic Conduction.

If a homogeneous electrolyte is placed in a tube of variable section, and if the electrodes are placed at the ends of this tube, it is found that when the current passes, the anion appears at the anode and the cation at the cathode, the quantities of these ions being electrochemically equivalent, and such as to be together equivalent to a certain quantity of the electrolyte. In the other parts of the tube, whether the section be large or small, uniform or varying, the composition of the electrolyte remains unaltered. Hence the amount of electrolysis which takes place across every section of the tube is the same. Where the section is small the action must therefore be more intense than where the section is large, but the total amount of each ion which crosses any complete section of the electrolyte in a given time is the same for all sections.

The strength of the current may therefore be measured by the amount of electrolysis in a given time. An instrument by which the quantity of the electrolytic products can be readily measured is called a Voltameter.

The strength of the current, as thus measured, is the same at every part of the circuit, and the total quantity of the electrolytic products in the voltameter after any given time is proportional to the amount of electricity which passes any section in the same time.

238.] If we introduce a voltameter at one part of the circuit of a voltaic battery, and break the circuit at another part, we may suppose the measurement of the current to be conducted thus. Let the ends of the broken circuit be $A$ and $B$, and let $A$ be the anode and $B$ the cathode. Let an insulated ball be made to touch $A$ and $B$ alternately, it will carry from $A$ to $B$ a certain measurable quantity of electricity at each journey. This quantity may be measured by an electrometer, or it may be calculated by multiplying the electromotive force of the circuit by the electrostatic capacity of the ball. Electricity is thus carried from $A$ to $B$ on the insulated ball by a process which may be called Convection. At the same time electrolysis goes on in the voltameter and in the cells of the battery, and the amount of electrolysis in each cell may be compared with the amount of electricity carried across by the insulated ball. The quantity of a substance which is electrolysed by one unit of electricity is called an Electrochemical equivalent of that substance.

This experiment would be an extremely tedious and troublesome one if conducted in this way with a ball of ordinary magnitude and a manageable battery, for an enormous number of journeys would have to be made before an appreciable quantity of the electrolyte was decomposed. The experiment must therefore be considered as a mere illustration, the actual measurements of electrochemical equivalents being conducted in a different way. But the experiment may be considered as an illustration of the process of electrolysis itself, for if we regard electrolytic conduction as a species of convection in which an electrochemical equivalent of the anion travels with negative electricity in the direction of the anode, while an equivalent of the cation travels with positive electricity in the direction of the cathode, the whole amount of transfer of electricity being one unit, we shall have an idea of the process of electrolysis, which, so far as I know, is not inconsistent with known facts, though, on account of our ignorance of the nature of electricity and of chemical compounds, it may be a very imperfect representation of what really takes place.

*Magnetic Action of the Current.*

239.] Oersted discovered that a magnet placed near a straight electric current tends to place itself at right angles to the plane passing through the magnet and the current. See Art. 475.

If a man were to place his body in the line of the current so that the current from copper through the wire to zinc should flow from his head to his feet, and if he were to direct his face towards the centre of the magnet, then that end of the magnet which tends to point to the north would, when the current flows, tend to point towards the man's right hand.

The nature and laws of this electromagnetic action will be discussed when we come to the fourth part of this treatise. What we are concerned with at present is the fact that the electric current has a magnetic action which is exerted outside the current, and by which its existence can be ascertained and its intensity measured without breaking the circuit or introducing anything into the current itself.

The amount of the magnetic action has been ascertained to be strictly proportional to the strength of the current as measured by the products of electrolysis in the voltameter, and to be quite independent of the nature of the conductor in which the current is flowing, whether it be a metal or an electrolyte.

240.] An instrument which indicates the strength of an electric current by its magnetic effects is called a Galvanometer.

Galvanometers in general consist of one or more coils of silk-covered wire within which a magnet is suspended with its axis horizontal. When a current is passed through the wire the magnet tends to set itself with its axis perpendicular to the plane of the coils. If we suppose the plane of the coils to be placed parallel to the plane of the earth's equator, and the current to flow round the coil from east to west in the direction of the apparent motion of the sun, then the magnet within will tend to set itself with its magnetization in the same direction as that of the earth considered as a great magnet, the north pole of the earth being similar to that end of the compass needle which points south.

The galvanometer is the most convenient instrument for measuring the strength of electric currents. We shall therefore assume the possibility of constructing such an instrument in studying the laws of these currents, reserving the discussion of the principles of the instrument for our fourth part. When therefore we say that an electric current is of a certain strength we suppose that the measurement is effected by the galvanometer.

# CHAPTER II.

## CONDUCTION AND RESISTANCE.

241.] If by means of an electrometer we determine the electric potential at different points of a circuit in which a constant electric current is maintained, we shall find that in any portion of the circuit consisting of a single metal of uniform temperature throughout, the potential at any point exceeds that at any other point farther on in the direction of the current by a quantity depending on the strength of the current and on the nature and dimensions of the intervening portion of the circuit. The difference of the potentials at the extremities of this portion of the circuit is called the External electromotive force acting on it. If the portion of the circuit under consideration is not homogeneous, but contains transitions from one substance to another, from metals to electrolytes, or from hotter to colder parts, there may be, besides the external electromotive force, Internal electromotive forces which must be taken into account.

The relations between Electromotive Force, Current, and Resistance were first investigated by Dr. G. S. Ohm, in a work published in 1827, entitled *Die Galvanische Kette Mathematisch Bearbeitet*, translated in Taylor's *Scientific Memoirs*. The result of these investigations in the case of homogeneous conductors is commonly called 'Ohm's Law.'

### *Ohm's Law.*

*The electromotive force acting between the extremities of any part of a circuit is the product of the strength of the current and the resistance of that part of the circuit.*

Here a new term is introduced, the Resistance of a conductor, which is defined to be the ratio of the electromotive force to the strength of the current which it produces. The introduction of this term would have been of no scientific value unless Ohm had shewn, as he did experimentally, that it corresponds to a real physical quantity, that is, that it has a definite value which is altered only when the nature of the conductor is altered.

In the first place, then, the resistance of a conductor is independent of the strength of the current flowing through it.

In the second place the resistance is independent of the electric potential at which the conductor is maintained, and of the density of the distribution of electricity on the surface of the conductor.

It depends entirely on the nature of the material of which the conductor is composed, the state of aggregation of its parts, and its temperature.

The resistance of a conductor may be measured to within one ten thousandth or even one hundred thousandth part of its value, and so many conductors have been tested that our assurance of the truth of Ohm's Law is now very high*. In the sixth chapter we shall trace its applications and consequences.

*Generation of Heat by the Current.*

242.] We have seen that when an electromotive force causes a current to flow through a conductor, electricity is transferred from a place of higher to a place of lower potential. If the transfer had been made by convection, that is, by carrying successive charges on a ball from the one place to the other, work would have been done by the electrical forces on the ball, and this might have been turned to account. It is actually turned to account in a partial manner in those dry pile circuits where the electrodes have the form of bells, and the carrier ball is made to swing like a pendulum between the two bells and strike them alternately. In this way the electrical action is made to keep up the swinging of the pendulum and to propagate the sound of the bells to a distance. In the case of the conducting wire we have the same transfer of electricity from a place of high to a place of low potential without any external work being done. The principle of the Conservation of Energy

* {For the verification of Ohm's Law for metallic conductors see Chrystal, B. A. Report 1866, p. 36, who shews that the resistance of a wire for infinitely weak currents does not differ from its resistance for very strong ones by $10^{-10}$ per cent.; for the verification of the law for electrolytes see Fitzgerald and Trouton, B. A. Report, 1886.}

therefore leads us to look for internal work in the conductor. In an electrolyte this internal work consists partly of the separation of its components. In other conductors it is entirely converted into heat.

The energy converted into heat is in this case the product of the electromotive force into the quantity of electricity which passes. But the electromotive force is the product of the current into the resistance, and the quantity of electricity is the product of the current into the time. Hence the quantity of heat multiplied by the mechanical equivalent of unit of heat is equal to the square of the strength of the current multiplied into the resistance and into the time.

The heat developed by electric currents in overcoming the resistance of conductors has been determined by Dr. Joule, who first established that the heat produced in a given time is proportional to the square of the current, and afterwards by careful absolute measurements of all the quantities concerned, verified the equation

$$JH = C^2 R t,$$

where $J$ is Joule's dynamical equivalent of heat, $H$ the number of units of heat, $C$ the strength of the current, $R$ the resistance of the conductor, and $t$ the time during which the current flows. These relations between electromotive force, work, and heat, were first fully explained by Sir. W. Thomson in a paper on the application of the principle of mechanical effect to the measurement of electromotive forces*.

**243.**] The analogy between the theory of the conduction of electricity and that of the conduction of heat is at first sight almost complete. If we take two systems geometrically similar, and such that the conductivity for heat at any part of the first is proportional to the conductivity for electricity at the corresponding part of the second, and if we also make the temperature at any part of the first proportional to the electric potential at the corresponding point of the second, then the flow of heat across any area of the first will be proportional to the flow of electricity across the corresponding area of the second.

Thus, in the illustration we have given, in which flow of electricity corresponds to flow of heat, and electric potential to temperature, electricity tends to flow from places of high to

* *Phil. Mag.*, Dec. 1851.

places of low potential, exactly as heat tends to flow from places of high to places of low temperature.

244.] The theory of electric potential and that of temperature may therefore be made to illustrate one another; there is, however, one remarkable difference between the phenomena of electricity and those of heat.

Suspend a conducting body within a closed conducting vessel by a silk thread, and charge the vessel with electricity. The potential of the vessel and of all within it will be instantly raised, but however long and however powerfully the vessel be electrified, and whether the body within be allowed to come in contact with the vessel or not, no signs of electrification will appear within the vessel, nor will the body within shew any electrical effect when taken out.

But if the vessel is raised to a high temperature, the body within will rise to the same temperature, but only after a considerable time, and if it is then taken out it will be found hot, and will remain so till it has continued to emit heat for some time.

The difference between the phenomena consists in the fact that bodies are capable of absorbing and emitting heat, whereas they have no corresponding property with respect to electricity. A body cannot be made hot without a certain amount of heat being supplied to it, depending on the mass and specific heat of the body, but the electric potential of a body may be raised to any extent in the way already described without communicating any electricity to the body.

245.] Again, suppose a body first heated and then placed inside the closed vessel. The outside of the vessel will be at first at the temperature of surrounding bodies, but it will soon get hot, and will remain hot till the heat of the interior body has escaped.

It is impossible to perform a corresponding electrical experiment. It is impossible so to electrify a body, and so to place it in a hollow vessel, that the outside of the vessel shall at first shew no signs of electrification but shall afterwards become electrified. It was for some phenomenon of this kind that Faraday sought in vain under the name of an absolute charge of electricity.

Heat may be hidden in the interior of a body so as to have no external action, but it is impossible to isolate a quantity of electricity so as to prevent it from being constantly in inductive relation with an equal quantity of electricity of the opposite kind.

There is nothing therefore among electric phenomena which corresponds to the capacity of a body for heat. This follows at once from the doctrine which is asserted in this treatise, that electricity obeys the same condition of continuity as an incompressible fluid. It is therefore impossible to give a bodily charge of electricity to any substance by forcing an additional quantity of electricity into it. See Arts. 61, 111, 329, 334.

# CHAPTER III.

## ELECTROMOTIVE FORCE BETWEEN BODIES IN CONTACT.

### *The Potentials of Different Substances in Contact.*

246.] If we define the potential of a hollow conducting vessel as the potential of the air inside the vessel, we may ascertain this potential by means of an electrometer as described in Part I, Art. 221.

If we now take two hollow vessels of different metals, say copper and zinc, and put them in metallic contact with each other, and then test the potential of the air inside each vessel, the potential of the air inside the zinc vessel will be positive as compared with that inside the copper vessel. The difference of potentials depends on the nature of the surface of the insides of the vessels, being greatest when the zinc is bright and when the copper is coated with oxide.

It appears from this that when two different metals are in contact there is in general an electromotive force acting from the one to the other, so as to make the potential of the one exceed that of the other by a certain quantity. This is Volta's theory of Contact Electricity.

If we take a certain metal, say copper, as the standard, then if the potential of iron in contact with copper at the zero potential is $I$, and that of zinc in contact with copper at zero is $Z$, then the potential of zinc in contact with iron at zero will be $Z-I$, if the medium surrounding the metals remains the same.

It appears from this result, which is true of any three metals, that the difference of potential of any two metals at the same temperature in contact is equal to the difference of their potentials when in contact with a third metal, so that if a circuit be formed of any number of metals at the same temperature there will be electrical equilibrium as soon as they have acquired their proper potentials, and there will be no current kept up in the circuit.

247.] If, however, the circuit consist of two metals and an electrolyte, the electrolyte, according to Volta's theory, tends to reduce the potentials of the metals in contact with it to equality, so that the electromotive force at the metallic junction is no longer balanced, and a continuous current is kept up. The energy of this current is supplied by the chemical action which takes place between the electrolyte and the metals.

248.] The electric effect may, however, be produced without chemical action if by any other means we can produce an equalization of the potentials of two metals in contact. Thus, in an experiment due to Sir W. Thomson*, a copper funnel is placed in contact with a vertical zinc cylinder, so that when copper filings are allowed to pass through the funnel, they separate from each other and from the funnel near the middle of the zinc cylinder, and then fall into an insulated receiver placed below. The receiver is then found to be charged negatively, and the charge increases as the filings continue to pour into it. At the same time the zinc cylinder with the copper funnel in it becomes charged more and more positively.

If now the zinc cylinder were connected with the receiver by a wire, there would be a positive current in the wire from the cylinder to the receiver. The stream of copper filings, each filing charged negatively by induction, constitutes a negative current from the funnel to the receiver, or, in other words, a positive current from the receiver to the copper funnel. The positive current, therefore, passes through the air (by the filings) from zinc to copper, and through the metallic junction from copper to zinc, just as in the ordinary voltaic arrangement, but in this case the force which keeps up the current is not chemical action but gravity, which causes the filings to fall, in spite of the electrical attraction between the positively charged funnel and the negatively charged filings.

249.] A remarkable confirmation of the theory of contact electricity is supplied by the discovery of Peltier, that, when a current of electricity crosses the junction of two metals, the

* *North British Review*, 1864, p. 353; and *Proc. R. S.*, June 20, 1867.

junction is heated when the current is in one direction, and cooled when it is in the other direction. It must be remembered that a current in its passage through a metal always produces heat, because it meets with resistance, so that the cooling effect on the whole conductor must always be less than the heating effect. We must therefore distinguish between the generation of heat in each metal, due to ordinary resistance, and the generation or absorption of heat at the junction of two metals. We shall call the first the frictional generation of heat by the current, and, as we have seen, it is proportional to the square of the current, and is the same whether the current be in the positive or the negative direction. The second we may call the Peltier effect, which changes its sign with that of the current.

The total heat generated in a portion of a compound conductor consisting of two metals may be expressed by

$$H = \frac{R}{J} C^2 t - \Pi C t,$$

where $H$ is the quantity of heat, $J$ the mechanical equivalent of unit of heat, $R$ the resistance of the conductor, $C$ the current, and $t$ the time; $\Pi$ being the coefficient of the Peltier effect, that is, the heat absorbed at the junction by unit of current in unit of time.

Now the heat generated is mechanically equivalent to the work done against electrical forces in the conductor, that is, it is equal to the product of the current into the electromotive force producing it. Hence, if $E$ is the external electromotive force which causes the current to flow through the conductor,

$$JH = CEt = RC^2 t - J\Pi C t,$$

whence $$E = RC - J\Pi.$$

It appears from this equation that the external electromotive force required to drive the current through the compound conductor is less than that due to its resistance alone by the electromotive force $J\Pi$. Hence $J\Pi$ represents the electromotive contact force at the junction acting in the positive direction.

This application, due to Sir W. Thomson*, of the dynamical theory of heat to the determination of a local electromotive force is of great scientific importance, since the ordinary method of connecting two points of the compound conductor with the

* *Proc. R. S. Edin.*, Dec. 15, 1851; and *Trans. R. S. Edin.*, 1854.

electrodes of a galvanometer or electroscope by wires would be useless, owing to the contact forces at the junctions of the wires with the materials of the compound conductor. In the thermal method, on the other hand, we know that the only source of energy is the current of electricity, and that no work is done by the current in a certain portion of the circuit except in heating that portion of the conductor. If, therefore, we can measure the amount of the current and the amount of heat produced or absorbed, we can determine the electromotive force required to urge the current through that portion of the conductor, and this measurement is entirely independent of the effect of contact forces in other parts of the circuit.

The electromotive force at the junction of two metals, as determined by this method, does not account for Volta's electromotive force as described in Art. 246. The latter is in general far greater than that of this Article, and is sometimes of opposite sign. Hence the assumption that the potential of a metal is to be measured by that of the air in contact with it must be erroneous, and the greater part of Volta's electromotive force must be sought for, not at the junction of the two metals, but at one or both of the surfaces which separate the metals from the air or other medium which forms the third element of the circuit.

250.] The discovery by Seebeck of thermoelectric currents in circuits of different metals with their junctions at different temperatures, shews that these contact forces do not always balance each other in a complete circuit. It is manifest, however, that in a complete circuit of different metals at uniform temperature the contact forces must balance each other. For if this were not the case there would be a current formed in the circuit, and this current might be employed to work a machine or to generate heat in the circuit, that is, to do work, while at the same time there is no expenditure of energy, as the circuit is all at the same temperature, and no chemical or other change takes place. Hence, if the Peltier effect at the junction of two metals $a$ and $b$ be represented by $\Pi_{ab}$ when the current flows from $a$ to $b$, then for a circuit of two metals at the same temperature we must have

$$\Pi_{ab} + \Pi_{ba} = 0,$$

and for a circuit of three metals $a$, $b$, $c$, we must have

$$\Pi_{bc} + \Pi_{ca} + \Pi_{ab} = 0.$$

It follows from this equation that the three Peltier effects are not independent, but that one of them can be deduced from the other two. For instance, if we suppose $c$ to be a standard metal, and if we write $P_a = J\Pi_{ac}$ and $P_b = J\Pi_{bc}$, then

$$J\Pi_{ab} = P_a - P_b.$$

The quantity $P_a$ is a function of the temperature, and depends on the nature of the metal $a$.

251.] It has also been shewn by Magnus that if a circuit is formed of a single metal no current will be formed in it, however the section of the conductor and the temperature may vary in different parts*.

Since in this case there is conduction of heat and consequent dissipation of energy, we cannot, as in the former case, consider this result as self-evident. The electromotive force, for instance, between two portions of a circuit might have depended on whether the current was passing from a thick portion of the conductor to a thin one, or the reverse, as well as on its passing rapidly or slowly from a hot portion to a cold one, or the reverse, and this would have made a current possible in an unequally heated circuit of one metal.

Hence, by the same reasoning as in the case of Peltier's phenomenon, we find that if the passage of a current through a conductor of one metal produces any thermal effect which is reversed when the current is reversed, this can only take place when the current flows from places of high to places of low temperature, or the reverse, and if the heat generated in a conductor of one metal in flowing from a place where the temperature is $x$ to a place where it is $y$, is $H$, then

$$JH = RC^2t - S_{xy}Ct,$$

and the electromotive force tending to maintain the current will be $S_{xy}$.

If $x$, $y$, $z$ be the temperatures at three points of a homogeneous circuit, we must have

$$S_{yz} + S_{zx} + S_{xy} = 0,$$

according to the result of Magnus. Hence, if we suppose $z$ to be the zero temperature, and if we put

$$Q_x = S_{xz} \text{ and } Q_y = S_{yz},$$

* {Le Roux has shewn that this does not hold when there are such sudden changes in the section that the temperature changes by a finite amount in a distance comparable with molecular distances.}

we find
$$S_{xy} = Q_x - Q_y,$$
where $Q_x$ is a function of the temperature $x$, the form of the function depending on the nature of the metal.

If we now consider a circuit of two metals $a$ and $b$ in which the temperature is $x$ where the current passes from $a$ to $b$, and $y$ where it passes from $b$ to $a$, the electromotive force will be
$$F = P_{ax} - P_{bx} + Q_{bx} - Q_{by} + P_{by} - P_{ay} + Q_{ay} - Q_{ax},$$
where $P_{ax}$ signifies the value of $P$ for the metal $a$ at the temperature $x$, or
$$F = P_{ax} - Q_{ax} - (P_{ay} - Q_{ay}) - (P_{bx} - Q_{bx}) + P_{by} - Q_{by}.$$

Since in unequally heated circuits of different metals there are in general thermoelectric currents, it follows that $P$ and $Q$ are in general different for the same metal and same temperature.

252]. The existence of the quantity $Q$ was first demonstrated by Sir. W. Thomson, in the memoir we have referred to, as a deduction from the phenomenon of thermoelectric inversion discovered by Cumming*, who found that the order of certain metals in the thermoelectric scale is different at high and at low temperatures, so that for a certain temperature two metals may be neutral to each other. Thus, in a circuit of copper and iron if one junction be kept at the ordinary temperature while the temperature of the other is raised, a current sets from copper to iron through the hot junction, and the electromotive force continues to increase till the hot junction has reached a temperature $T$, which, according to Thomson, is about 284°C. When the temperature of the hot junction is raised still further the electromotive force is reduced, and at last, if the temperature be raised high enough, the current is reversed. The reversal of the current may be obtained more easily by raising the temperature of the colder junction. If the temperature of both junctions is above $T$ the current sets from iron to copper through the hotter junction, that is, in the reverse direction to that observed when both junctions are below $T$.

Hence, if one of the junctions is at the neutral temperature $T$ and the other is either hotter or colder, the current will set from copper to iron through the junction at the neutral temperature.

253.] From this fact Thomson reasoned as follows:—

Suppose the other junction at a temperature lower than $T$.

* *Cambridge Transactions*, 1823.

The current may be made to work an engine or to generate heat in a wire, and this expenditure of energy must be kept up by the transformation of heat into electric energy, that is to say, heat must disappear somewhere in the circuit. Now at the temperature $T$ iron and copper are neutral to each other, so that no reversible thermal effect is produced at the hot junction, and at the cold junction there is, by Peltier's principle, an evolution of heat by the current. Hence the only place where the heat can disappear is in the copper or iron portions of the circuit, so that either a current in iron from hot to cold must cool the iron, or a current in copper from cold to hot must cool the copper, or both these effects may take place. {This reasoning assumes that the thermoelectric junction acts merely as a heat engine, and that there is no alteration (such as would occur in a battery) in the energy of the substance forming the junction when electricity passes across it.} By an elaborate series of ingenious experiments Thomson succeeded in detecting the reversible thermal action of the current in passing between parts of different temperatures, and he found that the current produced opposite effects in copper and in iron*.

When a stream of a material fluid passes along a tube from a hot part to a cold part it heats the tube, and when it passes from cold to hot it cools the tube, and these effects depend on the specific capacity for heat of the fluid. If we supposed electricity, whether positive or negative, to be a material fluid, we might measure its specific heat by the thermal effect on an unequally heated conductor. Now Thomson's experiments shew that positive electricity in copper and negative electricity in iron carry heat with them from hot to cold. Hence, if we supposed either positive or negative electricity to be a fluid, capable of being heated and cooled, and of communicating heat to other bodies, we should find the supposition contradicted by iron for positive electricity and by copper for negative electricity, so that we should have to abandon both hypotheses.

This scientific prediction of the reversible effect of an electric current upon an unequally heated conductor of one metal is another instructive example of the application of the theory of Conservation of Energy to indicate new directions of scientific research. Thomson has also applied the Second Law of Thermo-

* 'On the Electrodynamic Qualities of Metals.' *Phil. Trans.*, Part III, 1856.

dynamics to indicate relations between the quantities which we have denoted by $P$ and $Q$, and has investigated the possible thermoelectric properties of bodies whose structure is different in different directions. He has also investigated experimentally the conditions under which these properties are developed by pressure, magnetization, &c.

254.] Professor Tait* has recently investigated the electromotive force of thermoelectric circuits of different metals, having their junctions at different temperatures. He finds that the electromotive force of a circuit may be expressed very accurately by the formula

$$E = a(t_1 - t_2)\left[t_0 - \tfrac{1}{2}(t_1 + t_2)\right],$$

where $t_1$ is the absolute temperature of the hot junction, $t_2$ that of the cold junction, and $t_0$ the temperature at which the two metals are neutral to each other. The factor $a$ is a coefficient depending on the nature of the two metals composing the circuit. This law has been verified through considerable ranges of temperature by Professor Tait and his students, and he hopes to make the thermoelectric circuit available as a thermometric instrument in his experiments on the conduction of heat, and in other cases in which the mercurial thermometer is not convenient or has not a sufficient range.

According to Tait's theory, the quantity which Thomson calls the specific heat of electricity is proportional to the absolute temperature in each pure metal, though its magnitude and even its sign vary in different metals. From this he has deduced by thermodynamic principles the following results. Let $k_a t$, $k_b t$, $k_c t$ be the specific heats of electricity in three metals $a$, $b$, $c$, and let $T_{bc}$, $T_{ca}$, $T_{ab}$ be the temperatures at which pairs of these metals are neutral to each other, then the equations

$$(k_b - k_c)\,T_{bc} + (k_c - k_a)\,T_{ca} + (k_a - k_b)T_{ab} = 0,$$
$$J\Pi_{ab} = (k_a - k_b)\,t\,(T_{ab} - t),$$
$$E_{ab} = (k_a - k_b)\,(t_1 - t_2)\left[T_{ab} - \tfrac{1}{2}(t_1 + t_2)\right]$$

express the relation of the neutral temperatures, the value of the Peltier effect, and the electromotive force of a thermoelectric circuit.

* *Proc. R. S. Edin.*, Session 1870–71, p. 308, also Dec. 18, 1871.

# CHAPTER IV.

## ELECTROLYSIS.

### *Electrolytic Conduction.*

255.] I HAVE already stated that when an electric current in any part of its circuit passes through certain compound substances called Electrolytes, the passage of the current is accompanied by a certain chemical process called Electrolysis, in which the substance is resolved into two components called Ions, of which one, called the Anion, or the electronegative component, appears at the Anode, or place where the current enters the electrolyte, and the other, called the Cation, appears at the Cathode, or the place where the current leaves the electrolyte.

The complete investigation of Electrolysis belongs quite as much to Chemistry as to Electricity. We shall consider it from an electrical point of view, without discussing its application to the theory of the constitution of chemical compounds.

Of all electrical phenomena electrolysis appears the most likely to furnish us with a real insight into the true nature of the electric current, because we find currents of ordinary matter and currents of electricity forming essential parts of the same phenomenon.

It is probably for this very reason that, in the present imperfectly formed state of our ideas about electricity, the theories of electrolysis are so unsatisfactory.

The fundamental law of electrolysis, which was established by Faraday, and confirmed by the experiments of Beetz, Hittorf, and others down to the present time, is as follows:—

The number of electrochemical equivalents of an electrolyte which are decomposed by the passage of an electric current during a given time is equal to the number of units of electricity which are transferred by the current in the same time.

The electrochemical equivalent of a substance is that quantity of the substance which is electrolysed by a unit current passing through the substance for a unit of time, or, in other words, by the passage of a unit of electricity. When the unit of electricity is defined in absolute measure the absolute value of the electrochemical equivalent of each substance can be determined in grains or in grammes.

The electrochemical equivalents of different substances are proportional to their ordinary chemical equivalents. The ordinary chemical equivalents, however, are the mere numerical ratios in which the substances combine, whereas the electrochemical equivalents are quantities of matter of a determinate magnitude, depending on the definition of the unit of electricity.

Every electrolyte consists of two components, which, during the electrolysis, appear where the current enters and leaves the electrolyte, and nowhere else. Hence, if we conceive a surface described within the substance of the electrolyte, the amount of electrolysis which takes place through this surface, as measured by the electrochemical equivalents of the components transferred across it in opposite directions, will be proportional to the total electric current through the surface.

The actual transfer of the ions through the substance of the electrolyte in opposite directions is therefore part of the phenomenon of the conduction of an electric current through an electrolyte. At every point of the electrolyte through which an electric current is passing there are also two opposite material currents of the anion and the cation, which have the same lines of flow with the electric current, and are proportional to it in magnitude.

It is therefore extremely natural to suppose that the currents of the ions are convection currents of electricity, and, in particular, that every molecule of the cation is charged with a certain fixed quantity of positive electricity, which is the same for the molecules of all cations, and that every molecule of the anion is charged with an equal quantity of negative electricty.

The opposite motion of the ions through the electrolyte would then be a complete physical representation of the electric current. We may compare this motion of the ions with the motion of gases and liquids through each other during the process of diffusion, there being this difference between the two processes, that, in diffusion, the different substances are only mixed together and the mixture is not homogeneous, whereas in electrolysis they are chemically combined and the electrolyte is homogeneous. In diffusion the determining cause of the motion of a substance in a given direction is a diminution of the quantity of that substance per unit of volume in that direction, whereas in electrolysis the motion of each ion is due to the electromotive force acting on the charged molecules.

256.] Clausius *, who has bestowed much study on the theory of the molecular agitation of bodies, supposes that the molecules of all bodies are in a state of constant agitation, but that in solid bodies each molecule never passes beyond a certain distance from its original position, whereas in fluids a molecule, after moving a certain distance from its original position, is just as likely to move still farther from it as to move back again. Hence the molecules of a fluid apparently at rest are continually changing their positions, and passing irregularly from one part of the fluid to another. In a compound fluid he supposes that not only do the compound molecules travel about in this way, but that, in the collisions which occur between the compound molecules, the molecules of which they are composed are often separated and change partners, so that the same individual atom is at one time associated with one atom of the opposite kind, and at another time with another. This process Clausius supposes to go on in the liquid at all times, but when an electromotive force acts on the liquid the motions of the molecules, which before were indifferently in all directions, are now influenced by the electromotive force, so that the positively charged molecules have a greater tendency towards the cathode than towards the anode, and the negatively charged molecules have a greater tendency to move in the opposite direction. Hence the molecules of the cation will during their intervals of freedom struggle towards the cathode, but will continually be checked in their course by pairing for a time with molecules of the anion, which are also struggling through the crowd, but in the opposite direction.

257.] This theory of Clausius enables us to understand how it is, that whereas the actual decomposition of an electrolyte requires an electromotive force of finite magnitude, the conduction of the current in the electrolyte obeys the law of Ohm,

* Pogg. *Ann.* ci. p. 338 (1857).

so that every electromotive force within the electrolyte, even the feeblest, produces a current of proportionate magnitude.

According to the theory of Clausius, the decomposition and recomposition of the electrolyte is continually going on even when there is no current, and the very feeblest electromotive force is sufficient to give this process a certain degree of direction, and so to produce the currents of the ions and the electric current, which is part of the same phenomenon. Within the electrolyte, however, the ions are never set free in finite quantity, and it is this liberation of the ions which requires a finite electromotive force. At the electrodes the ions accumulate, for the successive portions of the ions, as they arrive at the electrodes, instead of finding molecules of the opposite ion ready to combine with them, are forced into company with molecules of their own kind, with which they cannot combine. The electromotive force required to produce this effect is of finite magnitude, and forms an opposing electromotive force which produces a reversed current when other electromotive forces are removed. When this reversed electromotive force, owing to the accumulation of the ions at the electrode, is observed, the electrodes are said to be Polarized.

258.] One of the best methods of determining whether a body is or is not an electrolyte is to place it between platinum electrodes and to pass a current through it for some time, and then, disengaging the electrodes from the voltaic battery, and connecting them with a galvanometer, to observe whether a reverse current, due to polarization of the electrodes, passes through the galvanometer. Such a current, being due to accumulation of different substances on the two electrodes, is a proof that the substance has been electrolytically decomposed by the original current from the battery. This method can often be applied where it is difficult, by direct chemical methods, to detect the presence of the products of decomposition at the electrodes. See Art. 271.

259.] So far as we have gone the theory of electrolysis appears very satisfactory. It explains the electric current, the nature of which we do not understand, by means of the currents of the material components of the electrolyte, the motion of which, though not visible to the eye, is easily demonstrated. It gives a clear explanation, as Faraday has shewn, why an electrolyte which conducts in the liquid state is a non-conductor when solidified, for unless the molecules can pass from one part to another no electrolytic conduction can take place, so that the substance must be in a liquid state, either by fusion or by solution, in order to be a conductor.

But if we go on, and assume that the molecules of the ions within the electrolyte are actually charged with certain definite quantities of electricity, positive and negative, so that the electrolytic current is simply a current of convection, we find that this tempting hypothesis leads us into very difficult ground.

In the first place, we must assume that in every electrolyte each molecule of the cation, as it is liberated at the cathode, communicates to the cathode a charge of positive electricity, the amount of which is the same for every molecule, not only of that cation but of all other cations. In the same way each molecule of the anion when liberated, communicates to the anode a charge of negative electricity, the numerical magnitude of which is the same as that of the positive charge due to a molecule of a cation, but with sign reversed.

If, instead of a single molecule, we consider an assemblage of molecules constituting an electrochemical equivalent of the ion, then the total charge of all the molecules is, as we have seen, one unit of electricity, positive or negative.

260.] We do not as yet know how many molecules there are in an electrochemical equivalent of any substance, but the molecular theory of chemistry, which is corroborated by many physical considerations, supposes that the number of molecules in an electrochemical equivalent is the same for all substances. We may therefore, in molecular speculations, assume that the number of molecules in an electrochemical equivalent is $N$, a number unknown at present, but which we may hereafter find means to determine *.

Each molecule, therefore, on being liberated from the state of combination, parts with a charge whose magnitude is $\frac{1}{N}$, and is positive for the cation and negative for the anion. This definite quantity of electricity we shall call the molecular charge. If it were known it would be the most natural unit of electricity.

Hitherto we have only increased the precision of our ideas by

* See note to Art. 5.

exercising our imagination in tracing the electrification of molecules and the discharge of that electrification.

The liberation of the ions and the passage of positive electricity from the anode and into the cathode are simultaneous facts. The ions, when liberated, are not charged with electricity, hence, when they are in combination, they have the molecular charges as above described.

The electrification of a molecule, however, though easily spoken of, is not so easily conceived.

We know that if two metals are brought into contact at any point, the rest of their surfaces will be electrified, and if the metals are in the form of two plates separated by a narrow interval of air, the charge on each plate may become of considerable magnitude. Something like this may be supposed to occur when the two components of an electrolyte are in combination. Each pair of molecules may be supposed to touch at one point, and to have the rest of their surface charged with electricity due to the electromotive force of contact.

But to explain the phenomenon, we ought to shew why the charge thus produced on each molecule is of a fixed amount, and why, when a molecule of chlorine is combined with a molecule of zinc, the molecular charges are the same as when a molecule of chlorine is combined with a molecule of copper, although the electromotive force between chlorine and zinc is much greater than that between chlorine and copper. If the charging of the molecules is the effect of the electromotive force of contact, why should electromotive forces of different intensities produce exactly equal charges?

Suppose, however, that we leap over this difficulty by simply asserting the fact of the constant value of the molecular charge, and that we call this constant molecular charge, for convenience in description, *one molecule of electricity*.

This phrase, gross as it is, and out of harmony with the rest of this treatise, will enable us at least to state clearly what is known about electrolysis, and to appreciate the outstanding difficulties.

Every electrolyte must be considered as a binary compound of its anion and its cation. The anion or the cation or both may be compound bodies, so that a molecule of the anion or the cation may be formed by a number of molecules of simple bodies. A molecule of the anion and a molecule of the cation combined together form one molecule of the electrolyte.

In order to act as an anion in an electrolyte, the molecule which so acts must be charged with what we have called one molecule of negative electricity, and in order to act as a cation the molecule must be charged with one molecule of positive electricity.

These charges are connected with the molecules only when they are combined as anion and cation in the electrolyte.

When the molecules are electrolysed, they part with their charges to the electrodes, and appear as unelectrified bodies when set free from combination.

If the same molecule is capable of acting as a cation in one electrolyte and as an anion in another, and also of entering into compound bodies which are not electrolytes, then we must suppose that it receives a positive charge of electricity when it acts as a cation, a negative charge when it acts as an anion, and that it is without charge when it is not in an electrolyte.

Iodine, for instance, acts as an anion in the iodides of the metals and in hydriodic acid, but is said to act as a cation in the bromide of iodine.

This theory of molecular charges may serve as a method by which we may remember a good many facts about electrolysis. It is extremely improbable however that when we come to understand the true nature of electrolysis we shall retain in any form the theory of molecular charges, for then we shall have obtained a secure basis on which to form a true theory of electric currents, and so become independent of these provisional theories.

261.] One of the most important steps in our knowledge of electrolysis has been the recognition of the secondary chemical processes which arise from the evolution of the ions at the electrodes.

In many cases the substances which are found at the electrodes are not the actual ions of the electrolysis, but the products of the action of these ions on the electrolyte.

Thus, when a solution of sulphate of soda is electrolysed by a current which also passes through dilute sulphuric acid, equal quantities of oxygen are given off at the anodes, both in the sulphate of soda and in the dilute acid, and equal quantities of hydrogen at the cathodes.

But if the electrolysis is conducted in suitable vessels, such as

U-shaped tubes or vessels with a porous diaphragm, so that the substance surrounding each electrode can be examined separately, it is found that at the anode of the sulphate of soda there is an equivalent of sulphuric acid as well as an equivalent of oxygen, and at the cathode there is an equivalent of soda as well as an equivalent of hydrogen.

It would at first sight seem as if, according to the old theory of the constitution of salts, the sulphate of soda were electrolysed into its constituents sulphuric acid and soda, while the water of the solution is electrolysed at the same time into oxygen and hydrogen. But this explanation would involve the admission that the same current which passing through dilute sulphuric acid electrolyses one equivalent of water, when it passes through a solution of sulphate of soda electrolyses one equivalent of the salt as well as one equivalent of the water, and this would be contrary to the law of electrochemical equivalents.

But if we suppose that the components of sulphate of soda are not $SO_3$ and $Na_2O$ but $SO_4$ and $Na_2$,—not sulphuric acid and soda but sulphion and sodium—then the sulphion travels to the anode and is set free, but being unable to exist in a free state it breaks up into sulphuric acid and oxygen, one equivalent of each. At the same time the sodium is set free at the cathode, and there decomposes the water of the solution, forming one equivalent of soda and one of hydrogen.

In the dilute sulphuric acid the gases collected at the electrodes are the constituents of water, namely one volume of oxygen and two volumes of hydrogen. There is also an increase of sulphuric acid at the anode, but its amount is not equal to an equivalent.

It is doubtful whether pure water is an electrolyte or not. The greater the purity of the water, the greater the resistance to electrolytic conduction. The minutest traces of foreign matter are sufficient to produce a great diminution of the electrical resistance of water. The electric resistance of water as determined by different observers has values so different that we cannot consider it as a determined quantity. The purer the water the greater its resistance, and if we could obtain really pure water it is doubtful whether it would conduct at all *.

* {See F. Kohlrausch, 'Die Elektrische Leitungsfähigkeit des im Vacuum destillirten Wassers.' Wied. *Ann.* 24, p. 48. Bleekrode Wied. *Ann.* 3, p. 161, has shewn that pure hydrochloric acid is a non-conductor.}

As long as water was considered an electrolyte, and was, indeed, taken as the type of electrolytes, there was a strong reason for maintaining that it is a binary compound, and that two volumes of hydrogen are chemically equivalent to one volume of oxygen. If, however, we admit that water is not an electrolyte, we are free to suppose that equal volumes of oxygen and of hydrogen are chemically equivalent.

The dynamical theory of gases leads us to suppose that in perfect gases equal volumes always contain an equal number of molecules, and that the principal part of the specific heat, that, namely, which depends on the motion of agitation of the molecules among each other, is the same for equal numbers of molecules of all gases. Hence we are led to prefer a chemical system in which equal volumes of oxygen and of hydrogen are regarded as equivalent, and in which water is regarded as a compound of two equivalents of hydrogen and one of oxygen, and therefore probably not capable of direct electrolysis.

While electrolysis fully establishes the close relationship between electrical phenomena and those of chemical combination, the fact that every chemical compound is not an electrolyte shews that chemical combination is a process of a higher order of complexity than any purely electrical phenomenon. Thus the combinations of the metals with each other, though they are good conductors, and their components stand at different points of the scale of electrification by contact, are not, even when in a fluid state, decomposed by the current*. Most of the combinations of the substances which act as anions are not conductors, and therefore are not electrolytes. Besides these we have many compounds, containing the same components as electrolytes, but not in equivalent proportions, and these are also non-conductors, and therefore not electrolytes.

### *On the Conservation of Energy in Electrolysis.*

262.] Consider any voltaic circuit consisting partly of a battery, partly of a wire, and partly of an electrolytic cell.

During the passage of unit of electricity through any section of the circuit, one electrochemical equivalent of each of the substances in the cells, whether voltaic or electrolytic, is electrolysed.

* {See Roberts-Austen, B. A. Report, 1887.}

The amount of mechanical energy equivalent to any given chemical process can be ascertained by converting the whole energy due to the process into heat, and then expressing the heat in dynamical measure by multiplying the number of thermal units by Joule's mechanical equivalent of heat.

Where this direct method is not applicable, if we can estimate the heat given out by the substances taken first in the state before the process and then in the state after the process during their reduction to a final state, which is the same in both cases, then the thermal equivalent of the process is the difference of the two quantities of heat.

In the case in which the chemical action maintains a voltaic circuit, Joule found that the heat developed in the voltaic cells is less than that due to the chemical process within the cell, and that the remainder of the heat is developed in the connecting wire, or, when there is an electromagnetic engine in the circuit, part of the heat may be accounted for by the mechanical work of the engine.

For instance, if the electrodes of the voltaic cell are first connected by a short thick wire, and afterwards by a long thin wire, the heat developed in the cell for each grain of zinc dissolved is greater in the first case than in the second, but the heat developed in the wire is greater in the second case than in the first. The sum of the heat developed in the cell and in the wire for each grain of zinc dissolved is the same in both cases. This has been established by Joule by direct experiment.

The ratio of the heat generated in the cell to that generated in the wire is that of the resistance of the cell to that of the wire, so that if the wire were made of sufficient resistance nearly the whole of the heat would be generated in the wire, and if it were made of sufficient conducting power nearly the whole of the heat would be generated in the cell.

Let the wire be made so as to have great resistance, then the heat generated in it is equal in dynamical measure to the product of the quantity of electricity which is transmitted, multiplied by the electromotive force under which it is made to pass through the wire.

263.] Now during the time in which an electrochemical equivalent of the substance in the cell undergoes the chemical process which gives rise to the current, one unit of electricity passes through the wire. Hence, the heat developed by the passage of one unit of electricity is in this case measured by the electromotive force. But this heat is that which one electrochemical equivalent of the substance generates, whether in the cell or in the wire, while undergoing the given chemical process.

Hence the following important theorem, first proved by Thomson (*Phil. Mag.*, Dec. 1851):—

'The electromotive force of an electrochemical apparatus is in absolute measure equal to the mechanical equivalent of the chemical action on one electrochemical equivalent of the substance *.'

The thermal equivalents of many chemical actions have been determined by Andrews, Hess, Favre and Silbermann, Thomsen, &c., and from these their mechanical equivalents can be deduced by multiplication by the mechanical equivalent of heat.

This theorem not only enables us to calculate from purely thermal data the electromotive forces of different voltaic arrangements, and the electromotive forces required to effect electrolysis in different cases, but affords the means of actually measuring chemical affinity.

It has long been known that chemical affinity, or the tendency which exists towards the going on of a certain chemical change, is stronger in some cases than in others, but no proper measure of this tendency could be made till it was shewn that this tendency in certain cases is exactly equivalent to a certain electromotive force, and can therefore be measured according to the very same principles used in the measurement of electromotive forces.

Chemical affinity being therefore, in certain cases, reduced to the form of a measurable quantity, the whole theory of chemical processes, of the rate at which they go on, of the displacement of one substance by another, &c., becomes much more intelligible than when chemical affinity was regarded as a quality *sui generis*, and irreducible to numerical measurement.

* {This theorem only applies when there are no reversible thermal effects in the cell; when these exist the relation between the electromotive force $p$ and the mechanical equivalent of the chemical action, $\omega$, is expressed by the relation

$$p - \theta \frac{dp}{d\theta} = \omega,$$

where $\theta$ is the absolute temperature of the cell. v. Helmholtz, 'Die Thermodynamik chemischer Vorgänge.' *Wissenschaftliche Abhandlungen*, ii. p. 958.}

When the volume of the products of electrolysis is greater than that of the electrolyte, work is done during the electrolysis in overcoming the pressure. If the volume of an electrochemical equivalent of the electrolyte is increased by a volume $v$ when electrolysed under a pressure $p$, then the work done during the passage of a unit of electricity in overcoming pressure is $vp$, and the electromotive force required for electrolysis must include a part equal to $vp$, which is spent in performing this mechanical work.

If the products of electrolysis are gases which, like oxygen and hydrogen, are much rarer than the electrolyte, and fulfil Boyle's law very exactly, $vp$ will be very nearly constant for the same temperature; and the electromotive force required for electrolysis will not depend in any sensible degree on the pressure *. Hence it has been found impossible to check the electrolytic decomposition of dilute sulphuric acid by confining the decomposed gases in a small space.

When the products of electrolysis are liquid or solid the quantity $vp$ will increase as the pressure increases, so that if $v$ is positive an increase of pressure will increase the electromotive force required for electrolysis.

In the same way, any other kind of work done during electrolysis will have an effect on the value of the electromotive force, as, for instance, if a vertical current passes between two zinc electrodes in a solution of sulphate of zinc a greater electromotive force will be required when the current in the solution flows upwards than when it flows downwards, for, in the first case, it carries zinc from the lower to the upper electrode, and in the second from the upper to the lower. The electromotive force required for this purpose is less than the millionth part of that of a Daniell's cell per foot.

* {This result is inconsistent with the Second Law of Thermodynamics; according to this Law an increase in the pressure increases the Electromotive force required for Electrolysis. See J. J. Thomson's 'Applications of Dynamics to Physics and Chemistry,' p. 86. v. Helmholtz, 'Weitere Untersuchungen die Electrolyse des Wassers betreffend.' *Wied. Ann.* 34, p. 737.}

# CHAPTER V.

## ELECTROLYTIC POLARIZATION.

264.] WHEN an electric current is passed through an electrolyte bounded by metal electrodes, the accumulation of the ions at the electrodes produces the phenomenon called Polarization, which consists in an electromotive force acting in the opposite direction to the current, and producing an apparent increase of the resistance.

When a continuous current is employed, the resistance appears to increase rapidly from the commencement of the current, and at last reaches a value nearly constant. If the form of the vessel in which the electrolyte is contained is changed, the resistance is altered in the same way as a similar change of form of a metallic conductor would alter its resistance, but an additional apparent resistance, depending on the nature of the electrodes, has always to be added to the true resistance of the electrolyte.

265.] These phenomena have led some to suppose that there is a finite electromotive force required for a current to pass through an electrolyte. It has been shewn, however, by the researches of Lenz, Neumann, Beetz, Wiedemann *, Paalzow †, and recently by those of MM. F. Kohlrausch and W. A. Nippoldt ‡, Fitzgerald and Trouton §, that the conduction in the electrolyte itself obeys Ohm's Law with the same precision as in metallic conductors, and that the apparent resistance at the bounding surface of the electrolyte and the electrodes is entirely due to polarization.

266.] The phenomenon called polarization manifests itself in the case of a continuous current by a diminution in the current, indicating a force opposed to the current. Resistance is also

* Elektricität, i. 568, bd. i.

† *Berlin. Monatsbericht*, July, 1868.

‡ Pogg. *Ann.* bd. cxxxviii. s. 286 (October, 1869).

§ B. A. Report, 1887.

perceived as a force opposed to the current, but we can distinguish between the two phenomena by instantaneously removing or reversing the electromotive force.

The resisting force is always opposite in direction to the current, and the external electromotive force required to overcome it is proportional to the strength of the current, and changes its direction when the direction of the current is changed. If the external electromotive force becomes zero the current simply stops.

The electromotive force due to polarization, on the other hand, is in a fixed direction, opposed to the current which produced it. If the electromotive force which produced the current is removed, the polarization produces a current in the opposite direction.

The difference between the two phenomena may be compared with the difference between forcing a current of water through a long capillary tube, and forcing water through a tube of moderate bore up into a cistern. In the first case if we remove the pressure which produces the flow the current will simply stop. In the second case, if we remove the pressure the water will begin to flow down again from the cistern.

To make the mechanical illustration more complete, we have only to suppose that the cistern is of moderate depth, so that when a certain amount of water is raised into it, it begins to overflow. This will represent the fact that the total electromotive force due to polarization has a maximum limit.

267.] The cause of polarization appears to be the existence at the electrodes of the products of the electrolytic decomposition of the fluid between them. The surfaces of the electrodes are thus rendered electrically different, and an electromotive force between them is called into action, the direction of which is opposite to that of the current which caused the polarization.

The ions, which by their presence at the electrodes produce the phenomena of polarization, are not in a perfectly free state, but are in a condition in which they adhere to the surface of the electrodes with considerable force.

The electromotive force due to polarization depends upon the density with which the electrode is covered with the ion, but it is not proportional to this density, for the electromotive force does not increase so rapidly as this density.

This deposit of the ion is constantly tending to become free, and either to diffuse into the liquid, to escape as a gas, or to be precipitated as a solid.

The rate of this dissipation of the polarization is exceedingly small for slight degrees of polarization, and exceedingly rapid near the limiting value of polarization.

268.] We have seen, Art. 262, that the electromotive force acting in any electrolytic process is numerically equal to the mechanical equivalent of the result of that process on one electrochemical equivalent of the substance. If the process involves a diminution of the intrinsic energy of the substances which take part in it, as in the voltaic cell, then the electromotive force is in the direction of the current. If the process involves an increase of the intrinsic energy of the substances, as in the case of the electrolytic cell, the electromotive force is in the direction opposite to that of the current, and this electromotive force is called polarization.

In the case of a steady current in which electrolysis goes on continuously, and the ions are separated in a free state at the electrodes, we have only by a suitable process to measure the intrinsic energy of the separated ions, and compare it with that of the electrolyte in order to calculate the electromotive force required for the electrolysis. This will give the maximum polarization.

But during the first instants of the process of electrolysis the ions when deposited at the electrodes are not in a free state, and their intrinsic energy is less than their energy in a free state, though greater than their energy when combined in the electrolyte. In fact, the ion in contact with the electrode is in a state which when the deposit is very thin may be compared with that of chemical combination with the electrode, but as the deposit increases in density, the succeeding portions are no longer so intimately combined with the electrode, but simply adhere to it, and at last the deposit, if gaseous, escapes in bubbles, if liquid, diffuses through the electrolyte, and if solid, forms a precipitate.

In studying polarization we have therefore to consider

(1) The superficial density of the deposit, which we may call $\sigma$. This quantity $\sigma$ represents the number of electrochemical equivalents of the ion deposited on unit of area. Since each electrochemical equivalent deposited corresponds to one unit of

electricity transmitted by the current, we may consider $\sigma$ as representing either a surface-density of matter or a surface-density of electricity.

(2) The electromotive force of polarization, which we may call $p$. This quantity $p$ is the difference between the electric potentials of the two electrodes when the current through the electrolyte is so feeble that the proper resistance of the electrolyte makes no sensible difference between these potentials.

The electromotive force $p$ at any instant is numerically equal to the mechanical equivalent of the electrolytic process going on at that instant which corresponds to one electrochemical equivalent of the electrolyte. This electrolytic process, it must be remembered, consists in the deposit of the ions on the electrodes, and the state in which they are deposited depends on the actual state of the surfaces of the electrodes, which may be modified by previous deposits.

Hence the electromotive force at any instant depends on the previous history of the electrodes. It is, speaking very roughly, a function of $\sigma$, the density of the deposit, such that $p = 0$ when $\sigma = 0$, but $p$ approaches a limiting value much sooner than $\sigma$ does. The statement, however, that $p$ is a function of $\sigma$ cannot be considered accurate. It would be more correct to say that $p$ is a function of the chemical state of the superficial layer of the deposit, and that this state depends on the density of the deposit according to some law involving the time.

269.] (3) The third thing we must take into account is the dissipation of the polarization. The polarization when left to itself diminishes at a rate depending partly on the intensity of the polarization or the density of the deposit, and partly on the nature of the surrounding medium, and the chemical, mechanical, or thermal action to which the surface of the electrode is exposed.

If we determine a time $T$ such that at the rate at which the deposit is dissipated, the whole deposit would be removed in the time $T$, we may call $T$ the modulus of the time of dissipation. When the density of the deposit is very small, $T$ is very large, and may be reckoned by days or months. When the density of the deposit approaches its limiting value $T$ diminishes very rapidly, and is probably a minute fraction of a second. In fact, the rate of dissipation increases so rapidly that when the strength of the current is maintained constant, the separated gas, instead of contributing to increase the density of the deposit, escapes in bubbles as fast as it is formed.

270.] There is therefore a great difference between the state of polarization of the electrodes of an electrolytic cell when the polarization is feeble, and when it is at its maximum value. For instance, if a number of electrolytic cells of dilute sulphuric acid with platinum electrodes are arranged in series, and if a small electromotive force, such as that of one Daniell's cell, be made to act on the circuit, the electromotive force will produce a current of exceedingly short duration, for after a very short time the electromotive force arising from the polarization of the cells will balance that of the Daniell's cell.

The dissipation will be very small in the case of so feeble a state of polarization, and it will take place by a very slow absorption of the gases and diffusion through the liquid. The rate of this dissipation is indicated by the exceedingly feeble current which still continues to flow without any visible separation of gases.

If we neglect this dissipation for the short time during which the state of polarization is set up, and if we call $Q$ the total quantity of electricity which is transmitted by the current during this time, then if $A$ is the area of one of the electrodes, and $\sigma$ the density of the deposit, supposed uniform,

$$Q = A\,\sigma.$$

If we now disconnect the electrodes of the electrolytic apparatus from the Daniell's cell, and connect them with a galvanometer capable of measuring the whole discharge through it, a quantity of electricity nearly equal to $Q$ will be discharged as the polarization disappears.

271.] Hence we may compare the action of this apparatus, which is a form of Ritter's Secondary Pile, with that of a Leyden jar.

Both the secondary pile and the Leyden jar are capable of being charged with a certain amount of electricity, and of being afterwards discharged. During the discharge a quantity of electricity nearly equal to the charge passes in the opposite direction. The difference between the charge and the discharge arises partly from dissipation, a process which in the case of small charges is very slow, but which, when the charge exceeds a certain limit, becomes exceedingly rapid. Another part of the difference between the charge and the discharge arises from the fact that after the electrodes have been connected for a time sufficient to produce an apparently complete discharge, so that the current has completely disappeared, if we separate the electrodes for a time, and afterwards connect them, we obtain a second discharge in the same direction as the original discharge. This is called the residual discharge, and is a phenomenon of the Leyden jar as well as of the secondary pile.

The secondary pile may therefore be compared in several respects to a Leyden jar. There are, however, certain important differences. The charge of a Leyden jar is very exactly proportional to the electromotive force of the charge, that is, to the difference of potentials of the two surfaces, and the charge corresponding to unit of electromotive force is called the capacity of the jar, a constant quantity. The corresponding quantity, which may be called the capacity of the secondary pile, increases when the electromotive force increases.

The capacity of the jar depends on the area of the opposed surfaces, on the distance between them, and on the nature of the substance between them, but not on the nature of the metallic surfaces themselves. The capacity of the secondary pile depends on the area of the surfaces of the electrodes, but not on the distance between them, and it depends on the nature of the surface of the electrodes, as well as on that of the fluid between them. The maximum difference of the potentials of the electrodes in each element of a secondary pile is very small compared with the maximum difference of the potentials of those of a charged Leyden jar, so that in order to obtain much electromotive force a pile of many elements must be used.

On the other hand, the superficial density of the charge in the secondary pile is immensely greater than the utmost superficial density of the charge which can be accumulated on the surfaces of a Leyden jar, insomuch that Mr. C. F. Varley *, in describing the construction of a condenser of great capacity, recommends a series of gold or platinum plates immersed in dilute acid as preferable in point of cheapness to induction plates of tinfoil separated by insulating material.

The form in which the energy of a Leyden jar is stored up is the state of constraint of the dielectric between the conducting

* Specification of C. F. Varley, 'Electric Telegraphs, &c.,' Jan. 1860.

surfaces, a state which I have already described under the name of electric polarization, pointing out those phenomena attending this state which are at present known, and indicating the imperfect state of our knowledge of what really takes place. See Arts. 62, 111.

The form in which the energy of the secondary pile is stored up is the chemical condition of the material stratum at the surface of the electrodes, consisting of the ions of the electrolyte and the substance of the electrodes in a relation varying from chemical combination to superficial condensation, mechanical adherence, or simple juxtaposition.

The seat of this energy is close to the surfaces of the electrodes, and not throughout the substance of the electrolyte, and the form in which it exists may be called electrolytic polarization.

After studying the secondary pile in connexion with the Leyden jar, the student should again compare the voltaic battery with some form of the electrical machine, such as that described in Art. 211.

Mr. Varley has lately * found that the capacity of one square inch is from 175 to 542 microfarads and upwards for platinum plates in dilute sulphuric acid, and that the capacity increases with the electromotive force, being about 175 for 0.02 of a Daniell's cell, and 542 for 1.6 Daniell's cells.

But the comparison between the Leyden jar and the secondary pile may be carried still farther, as in the following experiment, due to Buff †. It is only when the glass of the jar is cold that it is capable of retaining a charge. At a temperature below 100°C the glass becomes a conductor. If a test-tube containing mercury is placed in a vessel of mercury, and if a pair of electrodes are connected, one with the inner and the other with the outer portion of mercury, the arrangement constitutes a Leyden jar which will hold a charge at ordinary temperatures. If the electrodes are connected with those of a voltaic battery, no current will pass as long as the glass is cold, but if the apparatus is gradually heated a current will begin to pass, and will increase rapidly in intensity as the temperature rises, though the glass remains apparently as hard as ever.

* *Proc. R. S.*, Jan. 12, 1871. For an account of other investigations on this subject, see Wiedemanns *Elektricität*, bd. ii. pp. 744–771.

† *Annalen der Chemie und Pharmacie*, bd. xc. 257 (1854).

This current is manifestly electrolytic, for if the electrodes are disconnected from the battery, and connected with a galvanometer, a considerable reverse current passes, due to polarization of the surfaces of the glass.

If, while the battery is in action the apparatus is cooled, the current is stopped by the cold glass as before, but the polarization of the surface remains. The mercury may be removed, the surfaces may be washed with nitric acid and with water, and fresh mercury introduced. If the apparatus is then heated, the current of polarization appears as soon as the glass is sufficiently warm to conduct it.

We may therefore regard glass at 100°C, though apparently a solid body, as an electrolyte, and there is considerable reason to believe that in most instances in which a dielectric has a slight degree of conductivity the conduction is electrolytic. The existence of polarization may be regarded as conclusive evidence of electrolysis, and if the conductivity of a substance increases as the temperature rises, we have good grounds for suspecting that the conduction is electrolytic.

### *On Constant Voltaic Elements.*

272.] When a series of experiments is made with a voltaic battery in which polarization occurs, the polarization diminishes during the time the current is not flowing, so that when it begins to flow again the current is stronger than after it has flowed for some time. If, on the other hand, the resistance of the circuit is diminished by allowing the current to flow through a short shunt, then, when the current is again made to flow through the ordinary circuit, it is at first weaker than its normal strength on account of the great polarization produced by the use of the short circuit.

To get rid of these irregularities in the current, which are exceedingly troublesome in experiments involving exact measurements, it is necessary to get rid of the polarization, or at least to reduce it as much as possible.

It does not appear that there is much polarization at the surface of the zinc plate when immersed in a solution of sulphate of zinc or in dilute sulphuric acid. The principal seat of polarization is at the surface of the negative metal. When the fluid in which the negative metal is immersed is dilute sulphuric acid, it is seen to become covered with bubbles of hydrogen gas, arising from the electrolytic decomposition of the fluid. Of course these bubbles, by preventing the fluid from touching the metal, diminish the surface of contact and increase the resistance of the circuit. But besides the visible bubbles it is certain that there is a thin coating of hydrogen, probably not in a free state, adhering to the metal, and as we have seen that this coating is able to produce an electromotive force in the reverse direction, it must necessarily diminish the electromotive force of the battery.

Various plans have been adopted to get rid of this coating of hydrogen. It may be diminished to some extent by mechanical means, such as stirring the liquid, or rubbing the surface of the negative plate. In Smee's battery the negative plates are vertical, and covered with finely divided platinum from which the bubbles of hydrogen easily escape, and in their ascent produce a current of liquid which helps to brush off other bubbles as they are formed.

A far more efficacious method, however, is to employ chemical means. These are of two kinds. In the batteries of Grove and Bunsen the negative plate is immersed in a fluid rich in oxygen, and the hydrogen, instead of forming a coating on the plate, combines with this substance. In Grove's battery the plate is of platinum immersed in strong nitric acid. In Bunsen's first battery it is of carbon in the same acid. Chromic acid is also used for the same purpose, and has the advantage of being free from the acid fumes produced by the reduction of nitric acid.

A different mode of getting rid of the hydrogen is by using copper as the negative metal, and covering the surface with a coat of oxide. This, however, rapidly disappears when it is used as the negative electrode. To renew it Joule has proposed to make the copper plates in the form of disks, half immersed in the liquid, and to rotate them slowly, so that the air may act on the parts exposed to it in turn.

The other method is by using as the liquid an electrolyte, the cation of which is a metal highly negative to zinc.

In Daniell's battery a copper plate is immersed in a saturated solution of sulphate of copper. When the current flows through the solution from the zinc to the copper no hydrogen appears on the copper plate, but copper is deposited on it. When the solution is saturated, and the current is not too strong, the copper appears to act as a true cation, the anion $SO_4$ travelling towards the zinc.

When these conditions are not fulfilled hydrogen is evolved at the cathode, but immediately acts on the solution, throwing down copper, and uniting with $SO_4$ to form oil of vitriol. When this is the case, the sulphate of copper next the copper plate is replaced by oil of vitriol, the liquid becomes colourless, and polarization by hydrogen gas again takes place. The copper deposited in this way is of a looser and more friable structure than that deposited by true electrolysis.

To ensure that the liquid in contact with the copper shall be saturated with sulphate of copper, crystals of this substance must be placed in the liquid close to the copper, so that when the solution is made weak by the deposition of the copper, more of the crystals may be dissolved.

We have seen that it is necessary that the liquid next the copper should be saturated with sulphate of copper. It is still more necessary that the liquid in which the zinc is immersed should be free from sulphate of copper. If any of this salt makes its way to the surface of the zinc it is reduced, and copper is deposited on the zinc. The zinc, copper, and fluid then form a little circuit in which rapid electrolytic action goes on, and the zinc is eaten away by an action which contributes nothing to the useful effect of the battery.

To prevent this, the zinc is immersed either in dilute sulphuric acid or in a solution of sulphate of zinc, and to prevent the solution of sulphate of copper from mixing with this liquid, the two liquids are separated by a division consisting of bladder or porous earthenware, which allows electrolysis to take place through it, but effectually prevents mixture of the fluids by visible currents.

In some batteries sawdust is used to prevent currents. The experiments of Graham, however, shew that the process of diffusion goes on nearly as rapidly when two liquids are separated by a division of this kind as when they are in direct contact, provided there are no visible currents, and it is probable that if a septum is employed which diminishes the diffusion, it will increase in exactly the same ratio the resistance of the element, because electrolytic conduction is a process the mathematical laws of which have the same form as those of diffusion, and whatever interferes with one must interfere equally with the other. The only difference is that diffusion is always going on, whereas the current flows only when the battery is in action.

In all forms of Daniell's battery the final result is that the sulphate of copper finds its way to the zinc and spoils the battery. To retard this result indefinitely, Sir W. Thomson * has constructed Daniell's battery in the following form.

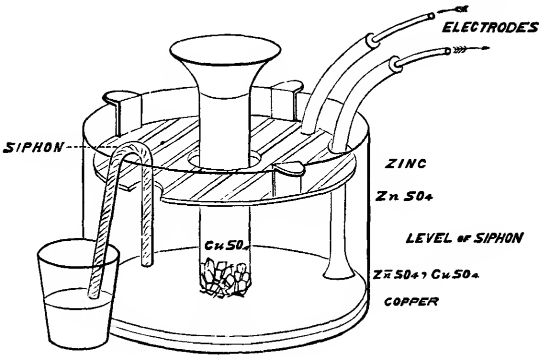


Fig. 22.

In each cell the copper plate is placed horizontally at the bottom and a saturated solution of sulphate of zinc is poured over it. The zinc is in the form of a grating and is placed horizontally near the surface of the solution. A glass tube is placed vertically in the solution with its lower end just above the surface of the copper plate. Crystals of sulphate of copper are dropped down this tube, and, dissolving in the liquid, form a solution of greater density than that of sulphate of zinc alone, so that it cannot get to the zinc except by diffusion. To retard this process of diffusion, a siphon, consisting of a glass tube stuffed with cotton wick, is placed with one extremity midway between the zinc and copper, and the other in a vessel outside the cell, so that the liquid is very slowly drawn off near the middle of its depth. To supply its place, water, or a weak solution of sulphate of zinc, is added above when required. In this way the greater part of the sulphate of copper rising through the liquid by diffusion is drawn off by the siphon before it reaches the zinc, and the zinc is surrounded by liquid nearly free

* *Proc. R. S.*, Jan. 19, 1871.

from sulphate of copper, and having a very slow downward motion in the cell, which still further retards the upward motion of the sulphate of copper. During the action of the battery copper is deposited on the copper plate, and $SO_4$ travels slowly through the liquid to the zinc with which it combines, forming sulphate of zinc. Thus the liquid at the bottom becomes less dense by the deposition of the copper, and the liquid at the top becomes more dense by the addition of the zinc. To prevent this action from changing the order of density of the strata, and so producing instability and visible currents in the vessel, care must be taken to keep the tube well supplied with crystals of sulphate of copper, and to feed the cell above with a solution of sulphate of zinc sufficiently dilute to be lighter than any other stratum of the liquid in the cell.

Daniell's battery is by no means the most powerful in common use. The electromotive force of Grove's cell is 192,000,000, of Daniell's 107,900,000 and that of Bunsen's 188,000,000.

The resistance of Daniell's cell is in general greater than that of Grove's or Bunsen's of the same size.

These defects, however, are more than counterbalanced in all cases where exact measurements are required, by the fact that Daniell's cell exceeds every other known arrangement in constancy of electromotive force *. It has also the advantage of continuing in working order for a long time, and of emitting no gas.

* {When a standard Electromotive foree is required a Clark's cell is now most frequently used. For the precautions which must be taken in the construction and use of such cells, see Lord Rayleigh's paper on 'The Clark Cell as a Standard of Electromotive Force.' *Phil. Trans.* part ii. 1885.}

# CHAPTER VI.

## LINEAR ELECTRIC CURRENTS.

### *On Systems of Linear Conductors.*

273.] ANY conductor may be treated as a linear conductor if it is arranged so that the current must always pass in the same manner between two portions of its surface which are called its electrodes. For instance, a mass of metal of any form the surface of which is entirely covered with insulating material except at two places, at which the exposed surface of the conductor is in metallic contact with electrodes formed of a perfectly conducting material, may be treated as a linear conductor. For if the current be made to enter at one of these electrodes and escape at the other the lines of flow will be determinate, and the relation between electromotive force, current and resistance will be expressed by Ohm's Law, for the current in every part of the mass will be a linear function of $E$. But if there be more possible electrodes than two, the conductor may have more than one independent current through it, and these may not be conjugate to each other. See Arts. 282 *a* and 282 *b*.

### *Ohm's Law.*

274.] Let $E$ be the electromotive force in a linear conductor from the electrode $A_1$ to the electrode $A_2$. (See Art. 69.) Let $C$ be the strength of the electric current along the conductor, that is to say, let $C$ units of electricity pass across every section in the direction $A_1\,A_2$ in unit of time, and let $R$ be the resistance of the conductor, then the expression of Ohm's Law is

$$E = CR. \tag{1}$$

### *Linear Conductors arranged in Series.*

275.] Let $A_1$, $A_2$ be the electrodes of the first conductor and let the second conductor be placed with one of its electrodes in contact with $A_2$, so that the second conductor has for its electrodes $A_2$, $A_3$. The electrodes of the third conductor may be denoted by $A_3$ and $A_4$.

Let the electromotive forces along these conductors be denoted by $E_{12}, E_{23}, E_{34}$, and so on for the other conductors.

Let the resistances of the conductors be

$$R_{12}, \quad R_{23}, \quad R_{34}, \text{ \&c.}$$

Then, since the conductors are arranged in series so that the same current $C$ flows through each, we have by Ohm's Law,

$$E_{12} = CR_{12}, \quad E_{23} = CR_{23}, \quad E_{34} = CR_{34}, \text{ \&c.} \tag{2}$$

If $E$ is the resultant electromotive force, and $R$ the resultant resistance of the system, we must have by Ohm's Law,

$$E = CR. \tag{3}$$

Now $$E = E_{12} + E_{23} + E_{34} + \text{\&c.}, \tag{4}$$

the sum of the separate electromotive forces,

$$= C(R_{12} + R_{23} + R_{34} + \text{\&c.}) \text{ by equations (2).}$$

Comparing this result with (3), we find

$$R = R_{12} + R_{23} + R_{34} + \text{\&c.} \tag{5}$$

Or, *the resistance of a series of conductors is the sum of the resistances of the conductors taken separately.*

*Potential at any Point of the Series.*

Let $A$ and $C$ be the electrodes of the series, $B$ a point between them, $a$, $c$, and $b$ the potentials of these points respectively. Let $R_1$ be the resistance of the part from $A$ to $B$, $R_2$ that of the part from $B$ to $C$, and $R$ that of the whole from $A$ to $C$, then, since

$$a - b = R_1 C, \quad b - c = R_2 C, \quad \text{and} \quad a - c = RC,$$

the potential at $B$ is

$$b = \frac{R_2 a + R_1 c}{R}, \tag{6}$$

which determines the potential at $B$ when the potentials at $A$ and $C$ are given.

*Resistance of a Multiple Conductor.*

276.] Let a number of conductors $ABZ$, $ACZ$, $ADZ$ be arranged side by side with their extremities in contact with the same two points $A$ and $Z$. They are then said to be arranged in multiple arc.

Let the resistances of these conductors be $R_1$, $R_2$, $R_3$ respectively, and the currents $C_1$, $C_2$, $C_3$, and let the resistance of the multiple conductor be $R$, and the total current $C$. Then, since the potentials at $A$ and $Z$ are the same for all the conductors, they have the same difference, which we may call $E$. We then have

$$E = C_1 R_1 = C_2 R_2 = C_3 R_3 = CR,$$

but

$$C = C_1 + C_2 + C_3,$$

whence

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}. \tag{7}$$

Or, *the reciprocal of the resistance of a multiple conductor is the sum of the reciprocals of the component conductors.*

If we call the reciprocal of the resistance of a conductor the conductivity of the conductor, then we may say that *the conductivity of a multiple conductor is the sum of the conductivities of the component conductors.*

*Current in any Branch of a Multiple Conductor.*

From the equations of the preceding article, it appears that if $C_1$ is the current in any branch of the multiple conductor, and $R_1$ the resistance of that branch,

$$C_1 = C\frac{R}{R_1}, \tag{8}$$

where $C$ is the total current, and $R$ is the resistance of the multiple conductor as previously determined.

*Longitudinal Resistance of Conductors of Uniform Section.*

277.] Let the resistance of a cube of a given material to a current parallel to one of its edges be $\rho$, the side of the cube being unit of length, $\rho$ is called the 'specific resistance of that material for unit of volume.'

Consider next a prismatic conductor of the same material whose length is $l$, and whose section is unity. This is equivalent to $l$ cubes arranged in series. The resistance of the conductor is therefore $l\rho$.

Finally, consider a conductor of length $l$ and uniform section $s$. This is equivalent to $s$ conductors similar to the last arranged in multiple arc. The resistance of this conductor is therefore

$$R = \frac{l\rho}{s}.$$

When we know the resistance of a uniform wire we can determine the specific resistance of the material of which it is made if we can measure its length and its section.

The sectional area of small wires is most accurately determined by calculation from the length, weight, and specific gravity of the specimen. The determination of the specific gravity is sometimes inconvenient, and in such cases the resistance of a wire of unit length and unit mass is used as the 'specific resistance per unit of weight.'

If $r$ is this resistance, $l$ the length, and $m$ the mass of a wire, then

$$R = \frac{l^2 r}{m}.$$

*On the Dimensions of the Quantities involved in these Equations.*

278.] The resistance of a conductor is the ratio of the electromotive force acting on it to the current produced. The conductivity of the conductor is the reciprocal of this quantity, or in other words, the ratio of the current to the electromotive force producing it.

Now we know that in the electrostatic system of measurement the ratio of a quantity of electricity to the potential of the conductor on which it is spread is the capacity of the conductor, and is measured by a line. If the conductor is a sphere placed in an unlimited field, this line is the radius of the sphere. The ratio of a quantity of electricity to an electromotive force is therefore a line, but the ratio of a quantity of electricity to a current is the time during which the current flows to transmit that quantity. Hence the ratio of a current to an electromotive force is that of a line to a time, or in other words, it is a velocity.

The fact that the conductivity of a conductor is expressed in the electrostatic system of measurement by a velocity may be verified by supposing a sphere of radius $r$ charged to potential $V$, and then connected with the earth by the given conductor. Let the sphere contract, so that as the electricity escapes through the conductor the potential of the sphere is always kept equal to $V$. Then the charge on the sphere is $rV$ at any instant, and the current is $-\frac{d}{dt}(rV)$, but, since $V$ is constant, the current is $-\frac{dr}{dt}V$, and the electromotive force through the conductor is $V$.

The conductivity of the conductor is the ratio of the current to the electromotive force, or $-\frac{dr}{dt}$, that is, the *velocity* with which the radius of the sphere must diminish in order to maintain the potential constant when the charge is allowed to pass to earth through the conductor.

In the electrostatic system, therefore, the conductivity of a conductor is a velocity, and so of the dimensions $[LT^{-1}]$.

The resistance of the conductor is therefore of the dimensions $[L^{-1}T]$.

The specific resistance per unit of volume is of the dimension of $[T]$, and the specific conductivity per unit of volume is of the dimension of $[T^{-1}]$.

The numerical magnitude of these coefficients depends only on the unit of time, which is the same in different countries.

The specific resistance per unit of weight is of the dimensions $[L^{-3}MT]$.

279.] We shall afterwards find that in the electromagnetic system of measurement the resistance of a conductor is expressed by a velocity, so that in this system the dimensions of the resistance of a conductor are $[LT^{-1}]$.

The conductivity of the conductor is of course the reciprocal of this.

The specific resistance per unit of volume in this system is of the dimensions $[L^2T^{-1}]$, and the specific resistance per unit of weight is of the dimensions $[L^{-1}T^{-1}M]$.

### *On Linear Systems of Conductors in general.*

280.] The most general case of a linear system is that of $n$ points, $A_1, A_2, \ldots A_n$, connected together in pairs by $\frac{1}{2}n(n-1)$ linear conductors. Let the conductivity (or reciprocal of the resistance) of that conductor which connects any pair of points, say $A_p$ and $A_q$, be called $K_{pq}$, and let the current from $A_p$ to $A_q$ be $C_{pq}$. Let $P_p$ and $P_q$ be the electric potentials at the points $A_p$ and $A_q$ respectively, and let the internal electromotive force, if there be any, along the conductor from $A_p$ to $A_q$ be $E_{pq}$.

The current from $A_p$ to $A_q$ is, by Ohm's Law,

$$C_{pq} = K_{pq}(P_p - P_q + E_{pq}). \tag{1}$$

Among these quantities we have the following sets of relations:

The conductivity of a conductor is the same in either direction, or

$$K_{pq} = K_{qp}. \tag{2}$$

The electromotive force and the current are directed quantities, so that

$$E_{pq} = -E_{qp}, \quad \text{and} \quad C_{pq} = -C_{qp}. \tag{3}$$

Let $P_1, P_2, \ldots P_n$ be the potentials at $A_1, A_2, \ldots A_n$ respectively, and let $Q_1, Q_2, \ldots Q_n$ be the quantities of electricity which enter the system in unit of time at each of these points respectively. These are necessarily subject to the condition of 'continuity'

$$Q_1 + Q_2 \ldots + Q_n = 0, \tag{4}$$

since electricity can neither be indefinitely accumulated nor produced within the system.

The condition of 'continuity' at any point $A_p$ is

$$Q_p = C_{p1} + C_{p2} + \&c. + C_{pn}. \tag{5}$$

Substituting the values of the currents in terms of equation (1), this becomes

$$Q_p = (K_{p1} + K_{p2} + \&c. + K_{pn})P_p - (K_{p1}P_1 + K_{p2}P_2 + \&c. + K_{pn}P_n) \\ + (K_{p1}E_{p1} + \&c. + K_{pn}E_{pn}). \tag{6}$$

The symbol $K_{pp}$ does not occur in this equation. Let us therefore give it the value

$$K_{pp} = -(K_{p1} + K_{p2} + \&c. + K_{pn}); \tag{7}$$

that is, let $K_{pp}$ be a quantity equal and opposite to the sum of all the conductivities of the conductors which meet in $A_p$. We may then write the condition of continuity for the point $A_p$,

$$K_{p1}P_1 + K_{p2}P_2 + \&c. + K_{pp}P_p + \&c. + K_{pn}P_n \\ = K_{p1}E_{p1} + \&c. + K_{pn}E_{pn} - Q_p. \tag{8}$$

By substituting 1, 2, &c. $n$ for $p$ in this equation we shall obtain $n$ equations of the same kind from which to determine the $n$ potentials $P_1$, $P_2$, &c., $P_n$.

Since, however, if we add the system of equations (8) the result is identically zero by (3), (4) and (7), there will be only $n-1$ independent equations. These will be sufficient to determine the differences of the potentials of the points, but not to determine the absolute potential of any. This, however, is not required to calculate the currents in the system.

If we denote by $D$ the determinant

$$\begin{vmatrix} K_{11}, & K_{12}, & \ldots\ldots K_{1(n-1)}, \\ K_{21}, & K_{22}, & \ldots\ldots K_{2(n-1)}, \\ - - - & - - - & - - - - - \\ K_{(n-1)1}, & K_{(n-1)2}, & \ldots\ldots K_{(n-1)(-1n)}, \end{vmatrix} \qquad (9)$$

and by $D_{pq}$, the minor of $K_{pq}$, we find for the value of $P_p - P_n$,

$$(P_p - P_n)D = (K_{12}E_{12} + \&c. - Q_1)D_{p1} + (K_{21}E_{21} + \&c. - Q_2)D_{p2} + \&c. + (K_{q1}E_{q1} + \&c. + K_{qn}E_{qn} - Q_q)D_{pq} + \&c. \qquad (10)$$

In the same way the excess of the potential of any other point, say $A_q$, over that of $A_n$ may be determined. We may then determine the current between $A_p$ and $A_q$ from equation (1), and so solve the problem completely.

281.] We shall now demonstrate a reciprocal property of any two conductors of the system, answering to the reciprocal property we have already demonstrated for statical electricity in Art. 86.

The coefficient of $Q_q$ in the expression for $P_p$ is $-\frac{D_{pq}}{D}$. That of $Q_p$ in the expression for $P_q$ is $-\frac{D_{qp}}{D}$.

Now $D_{pq}$ differs from $D_{qp}$ only by the substitution of the symbols such as $K_{qp}$ for $K_{pq}$. But by equation (2), these two symbols are equal, since the conductivity of a conductor is the same both ways. Hence $D_{pq} = D_{qp}$. (11)

It follows from this that the part of the potential at $A_p$ arising from the introduction of a unit current at $A_q$ is equal to the part of the potential at $A_q$ arising from the introduction of a unit current at $A_p$.

We may deduce from this a proposition of a more practical form.

Let $A, B, C, D$ be any four points of the system, and let the effect of a current $Q$, made to enter the system at $A$ and leave it at $B$, be to make the potential at $C$ exceed that at $D$ by $P$. Then, if an equal current $Q$ be made to enter the system at $C$ and leave it at $D$, the potential at $A$ will exceed that at $B$ by the same quantity $P$.

If an electromotive force $E$ be introduced, acting in the conductor from $A$ to $B$, and if this causes a current $C$ from $X$ to $Y$, then the same electromotive force $E$ introduced into the conductor from $X$ to $Y$ will cause an equal current $C$ from $A$ to $B$.

The electromotive force $E$ may be that of a voltaic battery introduced between the points named, care being taken that the resistance of the conductor is the same before and after the introduction of the battery.

282 *a.*] If an electromotive force $E_{pq}$ act along the conductor $A_p A_q$, the current produced along another conductor of the system $A_r A_s$ is easily found to be

$$K_{rs} K_{pq} E_{pq} (D_{rp} + D_{sq} - D_{rq} - D_{sp}) \div D.$$

There will be no current if

$$D_{rp} + D_{sq} - D_{rq} - D_{sp} = 0. \tag{12}$$

But, by (11), the same equation holds if, when the electromotive force acts along $A_r A_s$, there is no current in $A_p A_q$. On account of this reciprocal relation the two conductors referred to are said to be *conjugate.*

The theory of conjugate conductors has been investigated by Kirchhoff, who has stated the conditions of a linear system in the following manner, in which the consideration of the potential is avoided.

(1) (Condition of 'continuity.') At any point of the system the sum of all the currents which flow towards that point is zero.

(2) In any complete circuit formed by the conductors the sum of the electromotive forces taken round the circuit is equal to the sum of the products of the current in each conductor multiplied by the resistance of that conductor.

We obtain this result by adding equations of the form (1) for the complete circuit, when the potentials necessarily disappear.

*282 *b.*] If the conducting wires form a simple network and if we suppose that a current circulates round each mesh, then the actual current in the wire which forms a thread of each of two neighbouring meshes will be the difference between the two currents circulating in the two meshes, the currents being reckoned positive when they circulate in a direction opposite to the motion of the hands of a watch. It is easy to establish in this case the following proposition :—Let $x$ be the current, $E$ the electromotive force, and $R$ the total resistance in any mesh; let also $y, z, \ldots$ be currents circulating in neighbouring meshes

* [Extracted from notes of Professor Maxwell's lectures by Mr. J. A. Fleming, B.A., St. John's College. See also a paper by Mr. Fleming in the *Phil. Mag.*, xx. p. 221, 1885.]

which have threads in common with that in which $x$ circulates, the resistances of those parts being $s, t, \ldots$; then

$$Rx - sy - tz - \&c. = E.$$

To illustrate the use of this rule we will take the arrangement known as Wheatstone's Bridge, adopting the figure and notation of Art. 347. We have then the three following equations representing the application of the rule in the case of the three circuits $OBC$, $OCA$, $OAB$ in which the currents $x$, $y$, $z$ respectively circulate, viz.

$$\begin{aligned}
(a+\beta+\gamma)\,x \qquad & -\gamma\, y & -\beta\, z &= E,\\
-\gamma x + (b+\gamma+a) y & & -a\, z &= 0,\\
-\beta\, x \qquad & -a\, y + (c+a+\beta)\, z & &= 0.
\end{aligned}$$

From these equations we may now determine the value of $z-y$ the galvanometer current in the branch $OA$, but the reader is referred to Art. 347 et seq. where this and other questions connected with Wheatstone's Bridge are discussed.

### *Heat Generated in the System.*

283.] The mechanical equivalent of the quantity of heat generated in a conductor whose resistance is $R$ by a current $C$ in unit of time is, by Art. 242,

$$JH = RC^2. \tag{13}$$

We have therefore to determine the sum of such quantities as $RC^2$ for all the conductors of the system.

For the conductor from $A_p$ to $A_q$ the conductivity is $K_{pq}$, and the resistance $R_{pq}$, where

$$K_{pq} \,.\, R_{pq} = 1. \tag{14}$$

The current in this conductor is, according to Ohm's Law,

$$C_{pq} = K_{pq}(P_p - P_q). \tag{15}$$

We shall suppose, however, that the value of the current is not that given by Ohm's Law, but $X_{pq}$, where

$$X_{pq} = C_{pq} + Y_{pq}. \tag{16}$$

To determine the heat generated in the system we have to find the sum of all the quantities of the form

$$R_{pq} X^2{}_{pq},$$

or
$$JH = \Sigma\,\{R_{pq} C^2{}_{pq} + 2 R_{pq} C_{pq} Y_{pq} + R_{pq} Y^2{}_{pq}\}. \tag{17}$$

Giving $C_{pq}$ its value, and remembering the relation between $K_{pq}$ and $R_{pq}$, this becomes

$$\Sigma\,[(P_p - P_q)(C_{pq} + 2\, Y_{pq}) + R_{pq}\, Y^2{}_{pq}]. \tag{18}$$

Now since both $C$ and $X$ must satisfy the condition of continuity at $A_p$, we have

$$Q_p = C_{p1} + C_{p2} + \&c. + C_{pn}, \tag{19}$$
$$Q_p = X_{p1} + X_{p2} + \&c. + X_{pn}, \tag{20}$$

therefore
$$0 = Y_{p1} + Y_{p2} + \&c. + Y_{pn}. \tag{21}$$

Adding together therefore all the terms of (18), we find

$$\Sigma (R_{pq} X^2_{pq}) = \Sigma P_p Q_p + \Sigma R_{pq} Y^2_{pq}. \tag{22}$$

Now since $R$ is always positive and $Y^2$ is essentially positive, the last term of this equation must be essentially positive. Hence the first term is a minimum when $Y$ is zero in every conductor, that is, when the current in every conductor is that given by Ohm's Law *.

Hence the following theorem :

284.] In any system of conductors in which there are no internal electromotive forces the heat generated by currents distributed in accordance with Ohm's Law is less than if the currents had been distributed in any other manner consistent with the actual conditions of supply and outflow of the current.

The heat actually generated when Ohm's Law is fulfilled is mechanically equivalent to $\Sigma P_p Q_p$, that is, to the sum of the products of the quantities of electricity supplied at the different external electrodes, each multiplied by the potential at which it is supplied.

* {We can prove in a similar way that when there are electromotive forces in the different branches the currents adjust themselves so that $\Sigma RC^2 - 2\Sigma EC$ is a minimum, where $E$ is the electromotive force in the branch when the current is $C$. If we express this quantity, which we shall call $F$, in terms of the independent currents flowing round the circuits, the distribution of current $x, y, z, \ldots$ among the conductors may be found from the equations

$$\frac{dF}{dx} = 0, \quad \frac{dF}{dy} = 0.$$

Thus in the case of Wheatstone's Bridge considered in Art. 382,

$$F = ax^2 + by^2 + cz^2 + \beta(x-z)^2 + \gamma(y-x)^2 + a(z-y)^2 - 2Ex,$$

and the equations in that Art. are identical with

$$\frac{dF}{dx} = 0, \quad \frac{dF}{dy} = 0, \quad \frac{dF}{dz} = 0.$$

This is often the most convenient way of finding the distribution of current among the conductors. The reciprocal properties of Art. 281 can be deduced by it with great ease.}

# APPENDIX TO CHAPTER VI.

THE laws of the distribution of currents which are investigated in Art. 280 may be expressed by the following rules, which are easily remembered.

Let us take the potential of one of the points, say $A_n$, as the zero potential, then if a quantity of electricity $Q_s$ flows into $A_s$ the potential of a point $A_p$ is shewn in the text to be

$$-\frac{D_{ps}}{D} Q_s.$$

The quantities $D$ and $D_{ps}$ may be got by the following rules.—$D$ is the sum of the products of the conductivities taken $(n-1)$ at a time, omitting all those terms which contain the products of the conductivities of branches which form closed circuits. $D_{ps}$ is the sum of the products of the conductivities taken $(n-2)$ at a time, omitting all those terms which contain the conductivities of the branches $A_p\,A_n$ or $A_s\,A_n$, or which contain products of conductivities of branches which form closed circuits either by themselves or with the aid of $A_p\,A_n$ or $A_s\,A_n$.

We see from equation (10) that the effect of an electromotive force $E_{qr}$ acting in the branch $A_q\,A_r$ is the same as the effect due to a sink of strength $K_{qr}\,E_{qr}$ at $Q$ and a source of the same strength at $R$, so that the preceding rule will include this case. The result of the application of this rule can however be stated more simply as follows. If an electromotive force $E_{pq}$ act along the conductor $A_p\,A_q$, the current produced along another conductor $A_r\,A_s$ is

$$K_{rs}\,K_{pq}\frac{\Delta}{D}E_{pq},$$

where $D$ is got by the rule given above, and $\Delta = \Delta_1 - \Delta_2$. Where $\Delta_1$ is got by selecting from the sum of the products of the conductivities taken $(n-2)$ at a time those products which contain the conductivities of both $A_p\,A_r$ (or the product of the conductivities of branches making a closed circuit with $A_p\,A_r$) and $A_q\,A_s$ (or the product of the conductivities of branches making a closed circuit with $A_s\,A_q$), omitting from the terms thus selected all those which contain the conductivities of $A_r\,A_s$, or $A_p\,A_q$, or the product of the conductivities of branches making closed circuits by themselves or with the help of $A_r\,A_s$ or $A_p\,A_q$; $\Delta_2$ corresponds to $\Delta_1$, the branches $A_p\,A_s$, $A_q\,A_r$ being taken instead of $A_p\,A_r$ and $A_s\,A_q$ respectively.

If a current enters at $P$ and leaves at $Q$, the ratio of the current to the difference of potential between $A_p$ and $A_q$ is $\frac{D}{\Delta'}$.

Where $\Delta'$ is the sum of the products of the conductivities taken $n-2$ at a time, omitting all those terms which contain the conductivity of $A_p\, A_q$ or the products of the conductivities of branches forming a closed circuit with it.

In these expressions all the terms which contain the product of the conductivities of branches forming a closed circuit are omitted.

We may illustrate these rules by applying them to a very important case, that of 4 points connected by 6 conductors. Let us call the points 1, 2, 3, 4.

Then $D =$ the sum of the product of the conductivities taken 3 at a time, leaving out, however, the 4 products $K_{12}\, K_{23}\, K_{31}$, $K_{12}\, K_{24}\, K_{41}$, $K_{13}\, K_{34}\, K_{41}$, $K_{23}\, K_{34}\, K_{42}$; as these correspond to the four closed circuits (123), (124), (134), (234).

Thus

$$D = (K_{14}+K_{24}+K_{34})(K_{12}K_{13}+K_{12}K_{23}+K_{13}K_{23})+K_{14}K_{24}\,(K_{13}+K_{23})$$
$$+K_{14}K_{34}\,(K_{12}+K_{23})+K_{34}K_{24}\,(K_{12}+K_{13})+K_{14}K_{24}K_{34}.$$

Let us suppose that an electromotive force $E$ acts along (23), the current through the branch (14)

$$= \frac{\Delta_1-\Delta_2}{D}\, EK_{14}\, K_{23},$$

$$\Delta_1 = K_{13}\, K_{24} \text{ (by definition)},$$
$$\Delta_2 = K_{12}\, K_{43}.$$

Hence if no current passes through (14), $K_{13}K_{24}-K_{12}K_{43}=0$, this is the condition that (23) and (14) may be conjugate.

The current through (13)

$$= \frac{K_{12}(K_{14}+K_{24}+K_{34})+K_{14}K_{24}}{D}.\, EK_{14}K_{23}.$$

The conductivity of the net work when a current enters at (2) and leaves at (3)

$$= \frac{D}{(K_{14}+K_{24}+K_{34})\,(K_{12}+K_{13})+K_{14}\,(K_{24}+K_{34})}.$$

If we have 5 points, the condition that (23) and (14) are conjugate is

$$K_{12}K_{34}\,(K_{15}+K_{25}+K_{35}+K_{45})+K_{12}K_{35}K_{45}+K_{34}K_{51}K_{52}$$
$$= K_{13}K_{24}\,(K_{15}+K_{25}+K_{35}+K_{45})+K_{13}K_{52}K_{54}+K_{24}K_{51}K_{53}.$$

# CHAPTER VII.

## CONDUCTION IN THREE DIMENSIONS.

### *Notation of Electric Currents.*

285.] At any point let an element of area $dS$ be taken normal to the axis of $x$, and let $Q$ units of electricity pass across this area from the negative to the positive side in unit of time, then, if $\frac{Q}{dS}$ becomes ultimately equal to $u$ when $dS$ is indefinitely diminished, $u$ is said to be the Component of the electric current in the direction of $x$ at the given point.

In the same way we may determine $v$ and $w$, the components of the current in the directions of $y$ and $z$ respectively.

286.] To determine the component of the current in any other direction $OR$ through the given point $O$, let $l$, $m$, $n$ be the direction-cosines of $OR$; then if we cut off from the axes of $x$, $y$, $z$ portions equal to

$$\frac{r}{l}, \quad \frac{r}{m}, \quad \text{and} \quad \frac{r}{n}$$

respectively at $A$, $B$ and $C$, the triangle $ABC$ will be normal to $OR$.

The area of this triangle $ABC$ will be

$$dS = \tfrac{1}{2}\frac{r^2}{lmn},$$

and by diminishing $r$ this area may be diminished without limit.

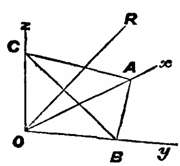

Fig. 23.

The quantity of electricity which leaves the tetrahedron $ABCO$ by the triangle $ABC$ must be equal to that which enters it through the three triangles $OBC$, $OCA$, and $OAB$.

The area of the triangle $OBC$ is $\frac{1}{2}\frac{r^2}{mn}$, and the component of the current normal to its plane is $u$, so that the quantity which enters through this triangle in unit time is $\frac{1}{2} r^2 \frac{u}{mn}$.

The quantities which enter through the triangles $OCA$ and $OAB$ respectively in unit time are

$$\tfrac{1}{2} r^2 \frac{v}{nl}, \quad \text{and} \quad \tfrac{1}{2} r^2 \frac{w}{lm}.$$

If $\gamma$ is the component of the current in the direction $OR$, then the quantity which leaves the tetrahedron in unit time through $ABC$ is

$$\tfrac{1}{2} r^2 \frac{\gamma}{lmn}.$$

Since this is equal to the quantity which enters through the three other triangles,

$$\tfrac{1}{2} \frac{r^2 \gamma}{lmn} = \tfrac{1}{2} r^2 \left\{ \frac{u}{mn} + \frac{v}{nl} + \frac{w}{lm} \right\};$$

multiplying by $\frac{2\,lmn}{r^2}$, we get

$$\gamma = lu + mv + nw. \tag{1}$$

If we put $\qquad u^2 + v^2 + w^2 = \Gamma^2,$

and make $l'$, $m'$, $n'$ such that

$$u = l'\Gamma, \quad v = m'\Gamma, \quad \text{and} \quad w = n'\Gamma;$$

then
$$\gamma = \Gamma(ll' + mm' + nn'). \tag{2}$$

Hence, if we define the resultant current as a vector whose magnitude is $\Gamma$, and whose direction-cosines are $l'$, $m'$, $n'$, and if $\gamma$ denotes the current resolved in a direction making an angle $\theta$ with that of the resultant current, then

$$\gamma = \Gamma \cos\theta; \tag{3}$$

shewing that the law of resolution of currents is the same as that of velocities, forces, and all other vectors.

287.] To determine the condition that a given surface may be a surface of flow, let

$$F(x, y, z) = \lambda \tag{4}$$

be the equation of a family of surfaces any one of which is given by making $\lambda$ constant; then, if we make

$$\overline{\frac{d\lambda}{dx}}\Big|^2 + \overline{\frac{d\lambda}{dy}}\Big|^2 + \overline{\frac{d\lambda}{dz}}\Big|^2 = \frac{1}{N^2}, \tag{5}$$

the direction-cosines of the normal, reckoned in the direction in which $\lambda$ increases, are

$$l = N\frac{d\lambda}{dx}, \qquad m = N\frac{d\lambda}{dy}, \qquad n = N\frac{d\lambda}{dz}. \tag{6}$$

Hence, if $\gamma$ is the component of the current normal to the surface,

$$\gamma = N\left\{u\frac{d\lambda}{dx} + v\frac{d\lambda}{dy} + w\frac{d\lambda}{dz}\right\}. \tag{7}$$

If $\gamma = 0$ there will be no current through the surface, and the surface may be called a Surface of Flow, because the lines of flow are in the surface.

288.] The equation of a surface of flow is therefore

$$u\frac{d\lambda}{dx} + v\frac{d\lambda}{dy} + w\frac{d\lambda}{dz} = 0. \tag{8}$$

If this equation is true for all values of $\lambda$, all the surfaces of the family will be surfaces of flow.

289.] Let there be another family of surfaces, whose parameter is $\lambda'$, then, if these are also surfaces of flow, we shall have

$$u\frac{d\lambda'}{dx} + v\frac{d\lambda'}{dy} + w\frac{d\lambda'}{dz} = 0. \tag{9}$$

If there is a third family of surfaces of flow, whose parameter is $\lambda''$, then

$$u\frac{d\lambda''}{dx} + v\frac{d\lambda''}{dy} + w\frac{d\lambda''}{dz} = 0. \tag{10}$$

If we eliminate $u$, $v$, and $w$ between these three equations, we find

$$\begin{vmatrix} \frac{d\lambda}{dx}, & \frac{d\lambda}{dy}, & \frac{d\lambda}{dz} \\ \frac{d\lambda'}{dx}, & \frac{d\lambda'}{dy}, & \frac{d\lambda'}{dz} \\ \frac{d\lambda''}{dx}, & \frac{d\lambda''}{dy}, & \frac{d\lambda''}{dz} \end{vmatrix} = 0; \tag{11}$$

$$\text{or} \quad \lambda'' = \phi(\lambda, \lambda'); \tag{12}$$

that is, $\lambda''$ is some function of $\lambda$ and $\lambda'$.

290.] Now consider the four surfaces whose parameters are $\lambda$, $\lambda + \delta\lambda$, $\lambda'$, and $\lambda' + \delta\lambda'$. These four surfaces enclose a quadrilateral tube, which we may call the tube $\delta\lambda . \delta\lambda'$. Since this tube is bounded by surfaces across which there is no flow, we may call it a Tube of Flow. If we take any two sections across the tube, the quantity which enters the tube at one section must be equal to the quantity which leaves it at the other, and since this quantity is therefore the same for every section of the tube, let us call it $L\,\delta\lambda . \delta\lambda'$, where $L$ is a function of $\lambda$ and $\lambda'$, the parameters which determine the particular tube.

291.] If $\delta S$ denotes the section of a tube of flow by a plane normal to $x$, we have by the theory of the change of the independent variables,

$$\delta\lambda \, . \, \delta\lambda' = \delta S \left(\frac{d\lambda}{dy}\frac{d\lambda'}{dz} - \frac{d\lambda}{dz}\frac{d\lambda'}{dy}\right), \tag{13}$$

and by the definition of the components of the current

$$u dS = L \delta\lambda \, . \, \delta\lambda'. \tag{14}$$

$$\left.\begin{aligned} \text{Hence} \qquad u &= L\left(\frac{d\lambda}{dy}\frac{d\lambda'}{dz} - \frac{d\lambda}{dz}\frac{d\lambda'}{dy}\right). \\ \text{Similarly} \qquad v &= L\left(\frac{d\lambda}{dz}\frac{d\lambda'}{dx} - \frac{d\lambda}{dx}\frac{d\lambda'}{dz}\right), \\ w &= L\left(\frac{d\lambda}{dx}\frac{d\lambda'}{dy} - \frac{d\lambda}{dy}\frac{d\lambda}{dx}\right). \end{aligned}\right\} \tag{15}$$

292.] It is always possible when one of the functions $\lambda$ or $\lambda'$ is known, to determine the other so that $L$ may be equal to unity. For instance, let us take the plane of $yz$, and draw upon it a series of equidistant lines parallel to $y$, to represent the sections of the family $\lambda'$ by this plane. In other words, let the function $\lambda'$ be determined by the condition that when $x = 0$ $\lambda' = z$. If we then make $L = 1$, and therefore (when $x = 0$)

$$\lambda = \int u dy,$$

then in the plane ($x = 0$) the amount of electricity which passes through any portion will be

$$\iint u \, dy \, dz = \iint d\lambda \, d\lambda'. \tag{16}$$

The nature of the sections of the surfaces of flow by the plane of $yz$ being determined, the form of the surfaces elsewhere is determined by the conditions (8) and (9). The two functions $\lambda$ and $\lambda'$ thus determined are sufficient to determine the current at every point by equations (15), unity being substituted for $L$.

*On Lines of Flow.*

293.] Let a series of values of $\lambda$ and of $\lambda'$ be chosen, the successive differences in each series being unity. The two series of surfaces defined by these values will divide space into a system of quadrilateral tubes through each of which there will be a unit current. By assuming the unit sufficiently small, the details of the current may be expressed by these tubes with any desired amount of minuteness. Then if any surface be drawn cutting the system of tubes, the quantity of the current which passes through this surface will be expressed by the *number* of tubes which cut it, since each tube carries a unit current.

The actual intersections of the surfaces may be called Lines of Flow. When the unit is taken sufficiently small, the number of lines of flow which cut a surface is approximately equal to the number of tubes of flow which cut it, so that we may consider the lines of flow as expressing not only the *direction* of the current but also its *strength*, since each line of flow through a given section corresponds to a unit current.

### *On Current-Sheets and Current-Functions.*

294.] A stratum of a conductor contained between two consecutive surfaces of flow of one system, say that of $\lambda'$, is called a Current-Sheet. The tubes of flow within this sheet are determined by the function $\lambda$. If $\lambda_A$ and $\lambda_P$ denote the values of $\lambda$ at the points $A$ and $P$ respectively, then the current from right to left across any line drawn on the sheet from $A$ to $P$ is $\lambda_P - \lambda_A$*. If $AP$ be an element, $ds$, of a curve drawn on the sheet, the current which crosses this element from right to left is

$$\frac{d\lambda}{ds}\,ds.$$

This function $\lambda$, from which the distribution of the current in the sheet can be completely determined, is called the Current-Function.

Any thin sheet of metal or conducting matter bounded on both sides by air or some other non-conducting medium may be treated as a current-sheet, in which the distribution of the current may be expressed by means of a current-function. See Art. 647.

### *Equation of 'Continuity.'*

295.] If we differentiate the three equations (15) with respect to $x$, $y$, $z$ respectively, remembering that $L$ is a function of $\lambda$ and $\lambda'$, we find

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0. \tag{17}$$

* {By the 'current across $AP$' is meant the current through the tube of flow bounded by the surfaces $\lambda_A$, $\lambda_P$, $\lambda'$ and $\lambda'+1$.}

The corresponding equation in Hydrodynamics is called the Equation of 'Continuity.' The continuity which it expresses is the continuity of existence, that is, the fact that a material substance cannot leave one part of space and arrive at another, without going through the space between. It cannot simply vanish in the one place and appear in the other, but it must travel along a continuous path, so that if a closed surface be drawn, including the one place and excluding the other, a material substance in passing from the one place to the other must go through the closed surface. The most general form of the equation in hydrodynamics is

$$\frac{d(\rho u)}{dx} + \frac{d(\rho v)}{dy} + \frac{d(\rho w)}{dz} + \frac{d\rho}{dt} = 0; \tag{18}$$

where $\rho$ signifies the ratio of the quantity of the substance to the volume it occupies, that volume being in this case the differential element of volume, and $(\rho u)$, $(\rho v)$, and $(\rho w)$ signify the ratio of the quantity of the substance which crosses an element of area in unit of time to that area, these areas being normal to the axes of $x$, $y$, and $z$ respectively. Thus understood, the equation is applicable to any material substance, solid or fluid, whether the motion be continuous or discontinuous, provided the existence of the parts of that substance is continuous. If anything, though not a substance, is subject to the condition of continuous existence in time and space, the equation will express this condition. In other parts of Physical Science, as, for instance, in the theory of electric and magnetic quantities, equations of a similar form occur. We shall call such equations 'equations of continuity' to indicate their form, though we may not attribute to these quantities the properties of matter, or even continuous existence in time and space.

The equation (17), which we have arrived at in the case of electric currents, is identical with (18) if we make $\rho = 1$, that is, if we suppose the substance homogeneous and incompressible. The equation, in the case of fluids, may also be established by either of the modes of proof given in treatises on Hydrodynamics. In one of these we trace the course and the deformation of a certain element of the fluid as it moves along. In the other, we fix our attention on an element of space, and take account of all that enters or leaves it. The former of these methods cannot be applied to electric currents, as we do not know the velocity with which the electricity passes through the body, or even whether it moves in the positive or the negative direction of the current. All that we know is the algebraical value of the quantity which crosses unit of area in unit of time, a quantity corresponding to $(\rho u)$ in the equation (18). We have no means of ascertaining the value of either of the factors $\rho$ or $u$, and therefore we cannot follow a particular portion of electricity in its course through the body. The other method of investigation, in which we consider what passes through the walls of an element of volume, is applicable to electric currents, and is perhaps preferable in point of form to that which we have given, but as it may be found in any treatise on Hydrodynamics we need not repeat it here.

*Quantity of Electricity which passes through a given Surface.*

296.] Let $\Gamma$ be the resultant current at any point of the surface. Let $dS$ be an element of the surface, and let $\epsilon$ be the angle between $\Gamma$ and the normal to the surface drawn outwards, then the total current through the surface will be

$$\iint \Gamma \cos \epsilon \, dS,$$

the integration being extended over the surface.

As in Art. 21, we may transform this integral into the form

$$\iint \Gamma \cos \epsilon \, dS = \iiint \left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) dx \, dy \, dz \tag{19}$$

in the case of any closed surface, the limits of the triple integration being those included by the surface. This is the expression for the total efflux from the closed surface. Since in all cases of steady currents this must be zero whatever the limits of the integration, the quantity under the integral sign must vanish, and we obtain in this way the equation of continuity (17).

# CHAPTER VIII.

## RESISTANCE AND CONDUCTIVITY IN THREE DIMENSIONS.

### *On the most General Relations between Current and Electromotive Force.*

297.] LET the components of the current at any point be $u$, $v$, $w$.

Let the components of the electromotive intensity be $X$, $Y$, $Z$.

The electromotive intensity at any point is the resultant force on a unit of positive electricity placed at that point. It may arise (1) from electrostatic action, in which case if $V$ is the potential,

$$X = -\frac{dV}{dx}, \quad Y = -\frac{dV}{dy}, \quad Z = -\frac{dV}{dz}; \tag{1}$$

or (2) from electromagnetic induction, the laws of which we shall afterwards examine; or (3) from thermoelectric or electrochemical action at the point itself, tending to produce a current in a given direction.

We shall in general suppose that $X$, $Y$, $Z$ represent the components of the actual electromotive intensity at the point, whatever be the origin of the force, but we shall occasionally examine the result of supposing it entirely due to variation of potential.

By Ohm's Law the current is proportional to the electromotive intensity. Hence $X$, $Y$, $Z$ must be linear functions of $u$, $v$, $w$. We may therefore assume as the equations of Resistance.

$$\left.\begin{aligned} X &= R_1 u + Q_3 v + P_2 w, \\ Y &= P_3 u + R_2 v + Q_1 w, \\ Z &= Q_2 u + P_1 v + R_3 w. \end{aligned}\right\} \tag{2}$$

We may call the coefficients $R$ the coefficients of longitudinal resistance in the directions of the axes of coordinates.

The coefficients $P$ and $Q$ may be called the coefficients of transverse resistance. They indicate the electromotive intensity in one direction required to produce a current in a different direction.

If we were at liberty to assume that a solid body may be treated as a system of linear conductors, then, from the reciprocal property (Art. 281) of any two conductors of a linear system, we might shew that the electromotive force along $z$ required to produce a unit current parallel to $y$ must be equal to the electromotive force along $y$ required to produce a unit current parallel to $z$. This would shew that $P_1 = Q_1$, and similarly we should find $P_2 = Q_2$, and $P_3 = Q_3$. When these conditions are satisfied the system of coefficients is said to be Symmetrical. When they are not satisfied it is called a Skew system.

We have great reason to believe that in every actual case the system is symmetrical*, but we shall examine some of the consequences of admitting the possibility of a skew system.

298.] The quantities $u$, $v$, $w$ may be expressed as linear functions of $X$, $Y$, $Z$ by a system of equations, which we may call Equations of Conductivity,

$$\left.\begin{aligned} u &= r_1 X + p_3 Y + q_2 Z, \\ v &= q_3 X + r_2 Y + p_1 Z, \\ w &= p_2 X + q_1 Y + r_3 Z; \end{aligned}\right\} \tag{3}$$

we may call the coefficients $r$ the coefficients of Longitudinal conductivity, and $p$ and $q$ those of Transverse conductivity.

The coefficients of resistance are inverse to those of conductivity. This relation may be defined as follows:

Let $[PQR]$ be the determinant of the coefficients of resistance, and $[pqr]$ that of the coefficients of conductivity, then

$$[PQR] = P_1 P_2 P_3 + Q_1 Q_2 Q_3 + R_1 R_2 R_3 - P_1 Q_1 R_1 - P_2 Q_2 R_2 - P_3 Q_3 R_3, \tag{4}$$

$$[pqr] = p_1 p_2 p_3 + q_1 q_2 q_3 + r_1 r_2 r_3 - p_1 q_1 r_1 - p_2 q_2 r_2 - p_3 q_3 r_3, \tag{5}$$

$$[PQR]\,[pqr] = 1, \tag{6}$$

$$[PQR]\, p_1 = (P_2 P_3 - Q_1 R_1), \quad [pqr]\, P_1 = (p_2 p_3 - q_1 r_1), \tag{7}$$

&c. &c.

The other equations may be formed by altering the symbols, $P$, $Q$, $R$, $p$, $q$, $r$, and the suffixes 1, 2, 3 in cyclical order.

## *Rate of Generation of Heat.*

299.] To find the work done by the current in unit of time in overcoming resistance, and so generating heat, we multiply the components of the current by the corresponding components

* {See note to Art. 303.}

of the electromotive intensity. We thus obtain the following expressions for $W$, the quantity of work expended in unit of time:

$$W = Xu + Yv + Zw; \tag{8}$$

$$= R_1 u^2 + R_2 v^2 + R_3 w^2 + (P_1 + Q_1)vw + (P_2 + Q_2)wu + (P_3 + Q_3)uv; \tag{9}$$

$$= r_1 X^2 + r_2 Y^2 + r_3 Z^2 + (p_1 + q_1)YZ + (p_2 + q_2)ZX + (p_3 + q_3)XY. \tag{10}$$

By a proper choice of axes, (9) may be deprived of the terms involving the products of $u$, $v$, $w$ or else (10) of those involving the products of $X$, $Y$, $Z$. The system of axes, however, which reduces $W$ to the form

$$R_1 u^2 + R_2 v^2 + R_3 w^2$$

is not in general the same as that which reduces it to the form

$$r_1 X^2 + r_2 Y^2 + r_3 Z^2.$$

It is only when the coefficients $P_1, P_2, P_3$ are equal respectively to $Q_1, Q_2, Q_3$ that the two systems of axes coincide.

If with Thomson * we write

$$\left.\begin{aligned} P &= S + T, & Q &= S - T; \\ \text{and} \quad p &= s + t, & q &= s - t; \end{aligned}\right\} \tag{11}$$

then we have

$$\left.\begin{aligned}[PQR] = {} & R_1 R_2 R_3 + 2\, S_1 S_2 S_3 - S_1^2 R_1 - S_2^2 R_2 - S_3^2 R_3 \\ & + 2\,(S_1 T_2 T_3 + S_2 T_3 T_1 + S_3 T_1 T_2) + R_1 T_1^2 + R_2 T_2^2 + R_3 T_3^2; \end{aligned}\right\} \tag{12}$$

and

$$\left.\begin{aligned} [PQR]\, r_1 &= R_2 R_3 - S_1^2 + T_1^2, \\ [PQR]\, s_1 &= T_2 T_3 + S_2 S_3 - R_1 S_1, \\ [PQR]\, t_1 &= R_1 T_1 + S_2 T_3 + S_3 T_2. \end{aligned}\right\} \tag{13}$$

If therefore we cause $S_1$, $S_2$, $S_3$ to disappear, the coefficients $s$ will not also disappear unless the coefficients $T$ are zero.

## *Condition of Stability.*

300.] Since the equilibrium of electricity is stable, the work spent in maintaining the current must always be positive. The conditions that $W$ must be positive are that the three coefficients $R_1$, $R_2$, $R_3$, and the three expressions

$$\left.\begin{aligned} &4\, R_2 R_3 - (P_1 + Q_1)^2, \\ &4\, R_3 R_1 - (P_2 + Q_2)^2, \\ &4\, R_1 R_2 - (P_3 + Q_3)^2, \end{aligned}\right\} \tag{14}$$

must all be positive.

There are similar conditions for the coefficients of conductivity.

* *Trans. R. S. Edin.*, 1853-4, p. 165.

*Equation of Continuity in a Homogeneous Medium.*

301.] If we express the components of the electromotive force as the derivatives of the potential $V$, the equation of continuity

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0 \tag{15}$$

becomes in a homogeneous medium

$$r_1 \frac{d^2V}{dx^2} + r_2 \frac{d^2V}{dy^2} + r_3 \frac{d^2V}{dz^2} + 2s_1 \frac{d^2V}{dy\,dz} + 2s_2 \frac{d^2V}{dz\,dx} + 2s_3 \frac{d^2V}{dx\,dy} = 0. \tag{16}$$

If the medium is not homogeneous there will be terms arising from the variation of the coefficients of conductivity in passing from one point to another.

This equation corresponds to Laplace's equation in a non-isotropic medium.

302.] If we put

$$[rs] = r_1 r_2 r_3 + 2\, s_1 s_2 s_3 - r_1 s_1^2 - r_2 s_2^2 - r_3 s_3^2, \tag{17}$$

and $$[AB] = A_1 A_2 A_3 + 2\, B_1 B_2 B_3 - A_1 B_1^2 - A_2 B_2^2 - A_3 B_3^2, \tag{18}$$

where

$$\left.\begin{aligned} [rs]\, A_1 &= r_2 r_3 - s_1^2, \\ [rs]\, B_1 &= s_2 s_3 - r_1 s_1, \\ &\text{- - - - - -} \end{aligned}\right\} \tag{19}$$

and so on, the system $A, B$ will be inverse to the system $r, s$, and if we make

$$A_1 x^2 + A_2 y^2 + A_3 z^2 + 2\, B_1 yz + 2\, B_2 zx + 2\, B_3 xy = [AB]\rho^2, \tag{20}$$

we shall find that

$$V = \frac{C}{4\pi} \frac{1}{\rho} \tag{21}$$

is a solution of the equation *.

* {Suppose that by the transformation

$$\begin{aligned} x &= a\ X + b\ Y + c\ Z, \\ y &= a' X + b' Y + c' Z, \\ z &= a'' X + b'' Y + c'' Z, \end{aligned} \tag{1}$$

the left-hand side of (16) becomes

$$\frac{d^2V}{dX^2} + \frac{d^2V}{dY^2} + \frac{d^2V}{dZ^2}. \tag{2}$$

For this to be the case, we see that

$$r_1 \xi^2 + r_2 \eta^2 + r_3 \zeta^2 + 2\, s_1 \eta \zeta + \dots$$

must be identical with

$$(a\xi + a'\eta + a''\zeta)^2 + (b\xi + b'\eta + b''\zeta)^2 + (c\xi + c'\eta + c''\zeta)^2,$$

which we shall call $U$.

In the case in which the coefficients $T$ are zero, the coefficients $A$ and $B$ become identical with the coefficients $R$ and $S$ of Art. 299. When $T$ exists this is not the case.

In the case therefore of electricity flowing out from a centre in an infinite, homogeneous, but not isotropic, medium, the equipotential surfaces are ellipsoids, for each of which $\rho$ is constant. The axes of these ellipsoids are in the directions of the principal axes of conductivity, and these do not coincide with the principal axes of resistance unless the system is symmetrical.

By a transformation of the equation (16) we may take for the axes of $x, y, z$ the principal axes of conductivity. The coefficients of the forms $s$ and $B$ will then be reduced to zero, and each coefficient of the form $A$ will be the reciprocal of the corresponding coefficient of the form $r$. The expression for $\rho$ will be

$$\frac{x^2}{r_1}+\frac{y^2}{r_2}+\frac{z^2}{r_3}=\frac{\rho^2}{r_1 r_2 r_3}. \tag{22}$$

303.] The theory of the complete system of equations of resistance and of conductivity is that of linear functions of three variables, and it is exemplified in the theory of Strains *, and in other parts of physics. The most appropriate method of treating it is that by which Hamilton and Tait treat a linear and vector function of a vector. We shall not, however, expressly introduce Quaternion notation.

The coefficients $T_1$, $T_2$, $T_3$ may be regarded as the rectangular components of a vector $T$, the absolute magnitude and direction

If we eliminate $\xi$, $\eta$, $\zeta$ by the equations

$$x=\tfrac{1}{2}\frac{dU}{d\xi},\quad y=\tfrac{1}{2}\frac{dU}{d\eta},\quad z=\tfrac{1}{2}\frac{dU}{d\zeta},$$

or

$$\left.\begin{aligned} x &= a\ (a\xi+a'\eta+a''\zeta)+b\ \ (b\xi+b'\eta+b''\zeta)+c\ \ (c\xi+c'\eta+c''\zeta),\\ y &= a'\ (a\xi+a'\eta+a''\zeta)+b'\ (b\xi+b'\eta+b''\zeta)+c'\ (c\xi+c'\eta+c''\zeta),\\ z &= a''(a\xi+a'\eta+a''\zeta)+b''(b\xi+b'\eta+b''\zeta)+c''(c\xi+c'\eta+c''\zeta),\end{aligned}\right\} \tag{3}$$

we get, since the system $AB$ is inverse to the system $rs$,

$$U=A_1x^2+A_2y^2+A_3z^2+2B_1yz+\dots.$$

But from equations (1) and (3) we see that

$$X=a\xi+a'\eta+a''\zeta,$$
$$Y=b\xi+b'\eta+b''\zeta,$$
$$Z=c\xi+c'\eta+c''\zeta;$$

hence $$U=X^2+Y^2+Z^2.$$

But by (2) $V=\dfrac{1}{\sqrt{X^2+Y^2+Z^2}}$ satisfies the differential equation, hence $1/\sqrt{U}$ must satisfy it.

* See Thomson and Tait's *Natural Philosophy*, § 154.

of which are fixed in the body, and independent of the direction of the axes of reference. The same is true of $t_1$, $t_2$, $t_3$, which are the components of another vector $t$.

The vectors $T$ and $t$ do not in general coincide in direction.

Let us now take the axis of $z$ so as to coincide with the vector $T$, and transform the equations of resistance accordingly. They will then have the form

$$\left.\begin{aligned} X &= R_1 u + S_3 v + S_2 w - Tv, \\ Y &= S_3 u + R_2 v + S_1 w + Tu, \\ Z &= S_2 u + S_1 v + R_3 w. \end{aligned}\right\} \tag{23}$$

It appears from these equations that we may consider the electromotive intensity as the resultant of two forces, one of them depending only on the coefficients $R$ and $S$, and the other depending on $T$ alone. The part depending on $R$ and $S$ is related to the current in the same way that the perpendicular on the tangent plane of an ellipsoid is related to the radius vector. The other part, depending on $T$, is equal to the product of $T$ into the resolved part of the current perpendicular to the axis of $T$, and its direction is perpendicular to $T$ and to the current, being always in the direction in which the resolved part of the current would lie if turned 90° in the positive direction round $T$.

If we consider the current and $T$ as vectors, the part of the electromotive intensity due to $T$ is the vector part of the product, $T \times$ current.

The coefficient $T$ may be called the Rotatory coefficient. We have reason to believe that it does not exist in any known substance. It should be found, if anywhere, in magnets, which have a polarization in one direction, probably due to a rotational phenomenon in the substance *.

304.] Assuming then that there is no rotatory coefficient, we shall shew how Thomson's Theorem given in Arts. 100 $a$–100 $e$ may be extended to prove that the heat generated by the currents in the system in a given time is a unique minimum.

To simplify the algebraical work let the axes of coordinates be chosen so as to reduce expression (9), and therefore also in this

* {Mr. Hall's discovery of the action of magnetism on a permanent electric current (*Phil. Mag.* ix. p. 225; x. p. 301, 1880) may be described by saying that a conductor placed in a magnetic field has a rotatory coefficient. See Hopkinson (*Phil. Mag.* x. p. 430, 1880.)}

case expression (10), to three terms; and let us consider the general characteristic equation (16) which then reduces to

$$r_1 \frac{d^2V}{dx^2} + r_2 \frac{d^2V}{dy^2} + r_3 \frac{d^2V}{dz^2} = 0. \tag{24}$$

Also, let $a$, $b$, $c$ be three functions of $x$, $y$, $z$ satisfying the condition

$$\frac{da}{dx} + \frac{db}{dy} + \frac{dc}{dz} = 0; \tag{25}$$

and let

$$\left.\begin{aligned} a &= -r_1 \frac{dV}{dx} + u, \\ b &= -r_2 \frac{dV}{dy} + v, \\ c &= -r_3 \frac{dV}{dz} + w. \end{aligned}\right\} \tag{26}$$

Finally, let the triple-integral

$$W = \iiint (R_1 a^2 + R_2 b^2 + R_3 c^2)\, dx\, dy\, dz \tag{27}$$

be extended over spaces bounded as in the enunciation of Art. 100 $a$; such viz. that over certain portions $V$ is constant or else the normal component of the vector $a$, $b$, $c$ is given, the former condition being accompanied by the further restriction that the integral of this component over the whole bounding surface must be zero: then $W$ will be a minimum when

$$u = 0, \qquad v = 0, \qquad w = 0.$$

For we have in this case

$$r_1 R_1 = 1, \qquad r_2 R_2 = 1, \qquad r_3 R_3 = 1;$$

and therefore, by (26),

$$\begin{aligned} W = &\iiint \left( r_1 \overline{\frac{dV}{dx}}\Big|^2 + r_2 \overline{\frac{dV}{dy}}\Big|^2 + r_3 \overline{\frac{dV}{dz}}\Big|^2 \right) dx\, dy\, dz \\ &+ \iiint (R_1 u^2 + R_2 v^2 + R_3 w^2)\, dx\, dy\, dz \\ &- 2 \iiint \left( u \frac{dV}{dx} + v \frac{dV}{dy} + w \frac{dV}{dz} \right) dx\, dy\, dz. \end{aligned} \tag{28}$$

But since

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0, \tag{29}$$

the third term vanishes by virtue of the conditions at the limits.

The first term of (28) is therefore the unique minimum value of $W$.

305.] As this proposition is of great importance in the theory of electricity, it may be useful to present the following proof of the most general case in a form free from analytical operations.

Let us consider the propagation of electricity through a conductor of any form, homogeneous or heterogeneous.

Then we know that

(1) If we draw a line along the path and in the direction of the electric current, the line must pass from places of high potential to places of low potential.

(2) If the potential at every point of the system be altered in a given uniform ratio, the current will be altered in the same ratio, according to Ohm's Law.

(3) If a certain distribution of potential gives rise to a certain distribution of currents, and a second distribution of potential gives rise to a second distribution of currents, then a third distribution in which the potential is the sum or difference of those in the first and second will give rise to a third distribution of currents, such that the total current passing through a given finite surface in the third case is the sum or difference of the currents passing through it in the first and second cases. For, by Ohm's Law, the additional current due to an alteration of potentials is independent of the original current due to the original distribution of potentials.

(4) If the potential is constant over the whole of a closed surface, and if there are no electrodes or intrinsic electromotive forces within it, then there will be no currents within the closed surface, and the potential at any point within it will be equal to that at the surface.

If there are currents within the closed surface they must either form closed curves, or they must begin and end either within the closed surface or at the surface itself.

But since the current must pass from places of high to places of low potential, it cannot flow in a closed curve.

Since there are no electrodes within the surface the current cannot begin or end within the closed surface, and since the potential at all points of the surface is the same, there can be no current along lines passing from one point of the surface to another.

Hence there are no currents within the surface, and therefore there can be no difference of potential, as such a difference would produce currents, and therefore the potential within the closed surface is everywhere the same as at the surface.

(5) If there is no electric current through any part of a closed surface, and no electrodes or intrinsic electromotive forces within the surface, there will be no currents within the surface, and the potential will be uniform.

We have seen that the currents cannot form closed curves, or begin or terminate within the surface, and since by the hypothesis they do not pass through the surface, there can be no currents, and therefore the potential is constant.

(6) If the potential is uniform over part of a closed surface, and if there is no current through the remainder of the surface, the potential within the surface will be uniform for the same reasons.

(7) If over part of the surface of a body the potential of every point is known, and if over the rest of the surface of the body the current passing through the surface at each point is known, then only one distribution of potential at points within the body can exist.

For if there were two different values of the potential at any point within the body, let these be $V_1$ in the first case and $V_2$ in the second case, and let us imagine a third case in which the potential of every point of the body is the excess of potential in the first case over that in the second. Then on that part of the surface for which the potential is known the potential in the third case will be zero, and on that part of the surface through which the currents are known the currents in the third case will be zero, so that by (6) the potential everywhere within the surface will be zero, or there is no excess of $V_1$ over $V_2$, or the reverse. Hence there is only one possible distribution of potentials. This proposition is true whether the solid be bounded by one closed surface or by several.

### *On the Approximate Calculation of the Resistance of a Conductor of a given Form.*

306.] The conductor here considered has its surface divided into three portions. Over one of these portions the potential is maintained at a constant value. Over a second portion the potential has a constant value different from the first. The whole of the remainder of the surface is impervious to electricity.

We may suppose the conditions of the first and second portions to be fulfilled by applying to the conductor two electrodes of perfectly conducting material, and that of the remainder of the surface by coating it with perfectly non-conducting material.

Under these circumstances the current in every part of the conductor is simply proportional to the difference between the potentials of the electrodes. Calling this difference the electromotive force, the total current from the one electrode to the other is the product of the electromotive force by the conductivity of the conductor as a whole, and the resistance of the conductor is the reciprocal of the conductivity.

It is only when a conductor is approximately in the circumstances above defined that it can be said to have a definite resistance or conductivity as a whole. A resistance coil, consisting of a thin wire terminating in large masses of copper, approximately satisfies these conditions, for the potential in the massive electrodes is nearly constant, and any differences of potential in different points of the same electrode may be neglected in comparison with the difference of the potentials of the two electrodes.

A very useful method of calculating the resistance of such conductors has been given, so far as I know, for the first time, by Lord Rayleigh, in a paper 'On the Theory of Resonance'*.

It is founded on the following considerations.

If the specific resistance of any portion of the conductor be changed, that of the remainder being unchanged, the resistance of the whole conductor will be increased if that of the portion is increased, and diminished if that of the portion is diminished.

This principle may be regarded as self-evident, but it may easily be shewn that the value of the expression for the resistance of a system of conductors between two points selected as electrodes, increases as the resistance of each member of the system increases.

It follows from this that if a surface of any form be described in the substance of the conductor, and if we further suppose this surface to be an infinitely thin sheet of a perfectly conducting substance, the resistance of the conductor as a whole will be diminished unless the surface is one of the equipotential surfaces in the natural state of the conductor, in which case no effect will

* *Phil. Trans.*, 1871, p. 77. See Art. 102 *a*.

be produced by making it a perfect conductor, as it is already in electrical equilibrium.

If therefore we draw within the conductor a series of surfaces, the first of which coincides with the first electrode, and the last with the second, while the intermediate surfaces are bounded by the non-conducting surface and do not intersect each other, and if we suppose each of these surfaces to be an infinitely thin sheet of perfectly conducting matter, we shall have obtained a system the resistance of which is certainly not greater than that of the original conductor, and is equal to it only when the surfaces we have chosen are the natural equipotential surfaces.

To calculate the resistance of the artificial system is an operation of much less difficulty than the original problem. For the resistance of the whole is the sum of the resistances of all the strata contained between the consecutive surfaces, and the resistance of each stratum can be found thus :

Let $dS$ be an element of the surface of the stratum, $\nu$ the thickness of the stratum perpendicular to the element, $\rho$ the specific resistance, $E$ the difference of potential of the perfectly conducting surfaces, and $dC$ the current through $dS$, then

$$dC = E\frac{1}{\rho\nu}dS, \tag{1}$$

and the whole current through the stratum is

$$C = E\iint\frac{1}{\rho\nu}dS, \tag{2}$$

the integration being extended over the whole stratum bounded by the non-conducting surface of the conductor.

Hence the conductivity of the stratum is

$$\frac{C}{E} = \iint\frac{1}{\rho\nu}dS, \tag{3}$$

and the resistance of the stratum is the reciprocal of this quantity.

If the stratum be that bounded by the two surfaces for which the function $F$ has the values $F$ and $F+dF$ respectively, then

$$\frac{dF}{\nu} = \nabla F = \left[\left(\frac{dF}{dx}\right)^2 + \left(\frac{dF}{dy}\right)^2 + \left(\frac{dF}{dz}\right)^2\right]^{\frac{1}{2}}, \tag{4}$$

and the resistance of the stratum is

$$\frac{dF}{\iint\frac{1}{\rho}\nabla F dS}. \tag{5}$$

To find the resistance of the whole artificial conductor, we have only to integrate with respect to $F$, and we find

$$R_1 = \int \frac{dF}{\iint \frac{1}{\rho} \nabla F dS}. \tag{6}$$

The resistance $R$ of the conductor in its natural state is greater than the value thus obtained, unless all the surfaces we have chosen are the natural equipotential surfaces. Also, since the true value of $R$ is the absolute maximum of the values of $R_1$ which can thus be obtained, a small deviation of the chosen surfaces from the true equipotential surfaces will produce an error of $R$ which is comparatively small.

This method of determining a lower limit of the value of the resistance is evidently perfectly general, and may be applied to conductors of any form, even when $\rho$, the specific resistance, varies in any manner within the conductor.

The most familiar example is the ordinary method of determining the resistance of a straight wire of variable section. In this case the surfaces chosen are planes perpendicular to the axis of the wire, the strata have parallel faces, and the resistance of a stratum of section $S$ and thickness $ds$ is

$$dR_1 = \frac{\rho ds}{S}, \tag{7}$$

and that of the whole wire of length $s$ is

$$R_1 = \int \frac{\rho ds}{S}, \tag{8}$$

where $S$ is the transverse section and is a function of $s$.

This method in the case of wires whose section varies slowly with the length gives a result very near the truth, but it is really only a lower limit, for the true resistance is always greater than this, except in the case where the section is perfectly uniform.

307.] To find the higher limit of the resistance, let us suppose a surface drawn in the conductor to be rendered impermeable to electricity. The effect of this must be to increase the resistance of the conductor unless the surface is one of the natural surfaces of flow. By means of two systems of surfaces we can form a set of tubes which will completely regulate the flow, and the effect, if there is any, of this system of impermeable surfaces must be to increase the resistance above its natural value.

The resistance of each of the tubes may be calculated by the method already given for a fine wire, and the resistance of the whole conductor is the reciprocal of the sum of the reciprocals of the resistances of all the tubes. The resistance thus found is greater than the natural resistance, except when the tubes follow the natural lines of flow.

In the case already considered, where the conductor is in the form of an elongated solid of revolution, let us measure $x$ along the axis, and let the radius of the section at any point be $b$. Let one set of impermeable surfaces be the planes through the axis for each of which $\phi$ is constant, and let the other set be surfaces of revolution for which

$$y^2 = \psi b^2, \tag{9}$$

where $\psi$ is a numerical quantity between 0 and 1.

Let us consider a portion of one of the tubes bounded by the surfaces $\phi$ and $\phi + d\phi$, $\psi$ and $\psi + d\psi$, $x$ and $x + dx$.

The section of the tube taken perpendicular to the axis is

$$y\,dy\,d\phi = \tfrac{1}{2} b^2 d\psi\, d\phi. \tag{10}$$

If $\theta$ be the angle which the tube makes with the axis

$$\tan\theta = \psi^{\frac{1}{2}} \frac{db}{dx}. \tag{11}$$

The true length of the element of the tube is $dx \sec\theta$, and its true section is $\tfrac{1}{2} b^2 d\psi\, d\phi \cos\theta$,

so that its resistance is

$$2\rho \frac{dx}{b^2 d\psi\, d\phi} \sec^2\theta = 2\rho \frac{dx}{b^2 d\psi\, d\phi}\left(1 + \psi \overline{\frac{db}{dx}}\bigg|^2\right). \tag{12}$$

Let $$A = \int \frac{\rho}{b^2} dx, \text{ and } B = \int \frac{\rho}{b^2}\left(\frac{db}{dx}\right)^2 dx, \tag{13}$$

the integration being extended over the whole length, $x$, of the conductor, then the resistance of the tube $d\psi\, d\phi$ is

$$\frac{2}{d\psi\, d\phi}(A + \psi B),$$

and its conductivity is

$$\frac{d\psi\, d\phi}{2(A + \psi B)}.$$

To find the conductivity of the whole conductor, which is the sum of the conductivities of the separate tubes, we must integrate this expression between $\phi = 0$ and $\phi = 2\pi$, and between

$\psi = 0$ and $\psi = 1$. The result is

$$\frac{1}{R'} = \frac{\pi}{B}\log_e\left(1 + \frac{B}{A}\right), \tag{14}$$

which may be less, but cannot be greater, than the true conductivity of the conductor.

When $\frac{db}{dx}$ is always a small quantity $\frac{B}{A}$ will also be small, and we may expand the expression for the conductivity, thus

$$\frac{1}{R'} = \frac{\pi}{A}\left(1 - \tfrac{1}{2}\frac{B}{A} + \tfrac{1}{3}\frac{B^2}{A^2} - \tfrac{1}{4}\frac{B^3}{A^3} + \&c.\right). \tag{15}$$

The first term of this expression, $\frac{\pi}{A}$, is that which we should have found by the former method as the superior limit of the conductivity. Hence the true conductivity is less than the first term but greater than the whole series. The superior value of the resistance is the reciprocal of this, or

$$R' = \frac{A}{\pi}\left(1 + \tfrac{1}{2}\frac{B}{A} - \frac{1}{12}\frac{B^2}{A^2} + \frac{1}{24}\frac{B^3}{A^3} - \&c.\right). \tag{16}$$

If, besides supposing the flow to be guided by the surfaces $\phi$ and $\psi$, we had assumed that the flow through each tube is proportional to $d\psi\, d\phi$, we should have obtained as the value of the resistance under this additional constraint

$$R'' = \frac{1}{\pi}(A + \tfrac{1}{2}B)\,*, \tag{17}$$

which is evidently greater than the former value, as it ought to be, on account of the additional constraint. In Lord Rayleigh's paper this is the supposition made, and the superior limit of the resistance there given has the value (17), which is a little greater than that which we have obtained in (16).

308.] We shall now apply the same method to find the correction which must be applied to the length of a cylindrical conductor of radius $a$ when its extremity is placed in metallic contact with a massive electrode, which we may suppose of a different metal.

For the lower limit of the resistance we shall suppose that an infinitely thin disk of perfectly conducting matter is placed between the end of the cylinder and the massive electrode, so as to bring the end of the cylinder to one and the same potential

* Lord Rayleigh, *Theory of Sound*, ii. p. 171.

throughout. The potential within the cylinder will then be a function of its length only, and if we suppose the surface of the electrode where the cylinder meets it to be approximately plane, and all its dimensions to be large compared with the diameter of the cylinder, the distribution of potential will be that due to a conductor in the form of a disk placed in an infinite medium. See Arts. 151, 177.

If $E$ is the difference of the potential of the disk from that of the distant parts of the electrode, $C$ the current issuing from the surface of the disk into the electrode, and $\rho'$ the specific resistance of the electrode; then if $Q$ is the amount of electricity on the disk, which we assume distributed as in Art. 151, we see that the integral over the disk of the electromotive intensity is

$$\rho' C = \tfrac{1}{2} \cdot 4\pi Q = 2\pi \frac{aE}{\frac{\pi}{2}}, \text{ by Art. 151,}$$

$$= 4aE. \qquad (18)$$

Hence, if the length of the wire from a given point to the electrode is $L$, and its specific resistance $\rho$, the resistance from that point to any point of the electrode not near the junction is

$$R = \rho \frac{L}{\pi a^2} + \frac{\rho'}{4a},$$

and this may be written

$$R = \frac{\rho}{\pi a^2}\left(L + \frac{\rho'}{\rho}\frac{\pi a}{4}\right), \qquad (19)$$

where the second term within brackets is a quantity which must be added to the length of the cylinder or wire in calculating its resistance, and this is certainly too small a correction.

To understand the nature of the outstanding error we may observe, that whereas we have supposed the flow in the wire up to the disk to be uniform throughout the section, the flow from the disk to the electrode is not uniform, but is at any point inversely proportional (Art. 151) to the minimum chord through that point. In the actual case the flow through the disk will not be uniform, but it will not vary so much from point to point as in this supposed case. The potential of the disk in the actual case will not be uniform, but will diminish from the middle to the edge.

309.] We shall next determine a quantity greater than the true resistance by constraining the flow through the disk to be uniform at every point. We may suppose electromotive forces introduced for this purpose acting perpendicular to the surface of the disk.

The resistance within the wire will be the same as before, but in the electrode the rate of generation of heat will be the surface-integral of the product of the flow into the potential. The rate of flow at any point is $\frac{C}{\pi a^2}$, and the potential is the same as that of an electrified surface whose surface-density is $\sigma$, where

$$2\pi\sigma = \frac{C\rho'}{\pi a^2}, \tag{20}$$

$\rho'$ being the specific resistance.

We have therefore to determine the potential energy of the electrification of the disk with the uniform surface-density $\sigma$.

*The potential at the edge of a disk of uniform density $\sigma$ is easily found to be $4a\sigma$. The work done in adding a strip of breadth $da$ at the circumference of the disk is $2\pi a\sigma da \, . \, 4a\sigma$, and the whole potential energy of the disk is the integral of this,

$$\text{or} \qquad P = \frac{8\pi}{3} a^3 \sigma^2. \tag{21}$$

In the case of electrical conduction the rate at which work is done in the electrode whose resistance is $R'$ is $C^2 R'$. But from the general equation of conduction the current across the disk per unit area is of the form

$$-\frac{1}{\rho'}\frac{dV}{d\nu}$$

$$\text{or} \qquad \frac{2\pi}{\rho'}\sigma.$$

The rate at which work is done is, if $V$ is the potential of the disk, and $ds$ an element of its surface,

$$= \frac{C}{\pi a^2}\int V ds$$

$$= \frac{2C}{\pi a^2}\frac{P}{\sigma}, \qquad \text{since } P = \tfrac{1}{2}\int V\sigma ds,$$

$$= \frac{4\pi}{\rho'} P \text{ (by (20))}.$$

We have therefore

$$C^2 R' = \frac{4\pi}{\rho'} P, \tag{22}$$

* See a Paper by Professor Cayley, *London Math. Soc. Proc.* vi. p. 38.

whence, by (20) and (21),

$$R' = \frac{8\rho'}{3\pi^2 a},$$

and the correction to be added to the length of the cylinder is

$$\frac{\rho'}{\rho}\frac{8}{3\pi}a,$$

this correction being greater than the true value. The true correction to be added to the length is therefore $\frac{\rho'}{\rho}an$, where $n$ is a number lying between $\frac{\pi}{4}$ and $\frac{8}{3\pi}$, or between 0·785 and 0·849.

* Lord Rayleigh, by a second approximation, has reduced the superior limit of $n$ to 0·8282.

* *Phil. Mag.* Nov. 1872, p. 344. Lord Rayleigh subsequently obtained ·8242 as the superior limit. See *London Math. Soc. Proc.* vii. p. 74, also *Theory of Sound*, vol. ii. Appendix A. p. 291.

# CHAPTER IX.

## CONDUCTION THROUGH HETEROGENEOUS MEDIA.

### *On the Conditions to be Fulfilled at the Surface of Separation between Two Conducting Media.*

310.] THERE are two conditions which the distribution of currents must fulfil in general, the condition that the potential must be continuous, and the condition of 'continuity' of the electric currents.

At the surface of separation between two media the first of these conditions requires that the potentials at two points on opposite sides of the surface, but infinitely near each other, shall be equal. The potentials are here understood to be measured by an electrometer put in connexion with the given point by means of an electrode of a given metal. If the potentials are measured by the method described in Arts. 222, 246, where the electrode terminates in a cavity of the conductor filled with air, then the potentials at contiguous points of different metals measured in this way will differ by a quantity depending on the temperature and on the nature of the two metals.

The other condition at the surface is that the current through any element of the surface is the same when measured in either medium.

Thus, if $V_1$ and $V_2$ are the potentials in the two media, then at any point in the surface of separation

$$V_1 = V_2, \tag{1}$$

and if $u_1$, $v_1$, $w_1$ and $u_2$, $v_2$, $w_2$ are the components of currents in the two media, and $l$, $m$, $n$ the direction-cosines of the normal to the surface of separation

$$u_1 l + v_1 m + w_1 n = u_2 l + v_2 m + w_2 n. \tag{2}$$

In the most general case the components $u$, $v$, $w$ are linear functions of the derivatives of $V$, the forms of which are given in the equations

$$\left.\begin{aligned} u &= r_1X+p_3Y+q_2Z, \\ v &= q_3X+r_2Y+p_1Z, \\ w &= p_2X+q_1Y+r_3Z, \end{aligned}\right\} \tag{3}$$

where $X, Y, Z$ are the derivatives of $V$ with respect to $x, y, z$ respectively.

Let us take the case of the surface which separates a medium having these coefficients of conduction from an isotropic medium having a coefficient of conduction equal to $r$.

Let $X', Y', Z'$ be the values of $X, Y, Z$ in the isotropic medium, then we have at the surface

$$V = V', \tag{4}$$

or $$X\,dx+Y\,dy+Z\,dz = X'\,dx+Y'\,dy+Z'\,dz, \tag{5}$$

when $$l\,dx+m\,dy+n\,dz = 0. \tag{6}$$

This condition leads to

$$X' = X+4\pi\sigma l, \quad Y' = Y+4\pi\sigma m, \quad Z' = Z+4\pi\sigma n, \tag{7}$$

where $\sigma$ is the surface-density.

We have also in the isotropic medium

$$u' = rX', \quad v' = rY', \quad w' = rZ', \tag{8}$$

and at the boundary the condition of flow is

$$u'l+v'm+w'n = ul+vm+wn, \tag{9}$$

or $\quad r(lX+mY+nZ+4\pi\sigma)$

$$=l(r_1X+p_3Y+q_2Z)+m(q_3X+r_2Y+p_1Z)+n(p_2X+q_1Y+r_3Z), \tag{10}$$

whence

$$4\pi\sigma r = \{l(r_1-r)+mq_3+np_2\}X+\{lp_3+m(r_2-r)+nq_1\}Y \\ +\{lq_2+mp_1+n(r_3-r)\}Z. \tag{11}$$

The quantity $\sigma$ represents the surface-density of the charge on the surface of separation. In crystallized and organized substances it depends on the direction of the surface as well as on the force perpendicular to it. In isotropic substances the coefficients $p$ and $q$ are zero, and the coefficients $r$ are all equal, so that

$$4\pi\sigma = \left(\frac{r_1}{r}-1\right)(lX+mY+nZ), \tag{12}$$

where $r_1$ is the conductivity of the substance, $r$ that of the external medium, and $l, m, n$ the direction-cosines of the normal drawn towards the medium whose conductivity is $r$.

When both media are isotropic the conditions may be greatly simplified, for if $k$ is the specific resistance per unit of volume, then $$u = -\frac{1}{k}\frac{dV}{dx}, \qquad v = -\frac{1}{k}\frac{dV}{dy}, \qquad w = -\frac{1}{k}\frac{dV}{dz}, \tag{13}$$

and if $\nu$ is the normal drawn at any point of the surface of separation from the first medium towards the second, the condition of continuity is

$$\frac{1}{k_1}\frac{dV_1}{d\nu} = \frac{1}{k_2}\frac{dV_2}{d\nu}. \tag{14}$$

If $\theta_1$ and $\theta_2$ are the angles which the lines of flow in the first and second media respectively make with the normal to the surface of separation, then the tangents to these lines of flow are in the same plane with the normal and on opposite sides of it, and

$$k_1 \tan\theta_1 = k_2 \tan\theta_2. \tag{15}$$

This may be called the law of refraction of lines of flow.

311.] As an example of the conditions which must be fulfilled when electricity crosses the surface of separation of two media, let us suppose the surface spherical and of radius $a$, the specific resistance being $k_1$ within and $k_2$ without the surface.

Let the potential, both within and without the surface, be expanded in solid harmonics, and let the part which depends on the surface harmonic $S_i$ be

$$V_1 = (A_1 r^i + B_1 r^{-(i+1)})\, S_i, \tag{1}$$
$$V_2 = (A_2 r^i + B_2 r^{-(i+1)})\, S_i, \tag{2}$$

within and without the sphere respectively.

At the surface of separation where $r = a$ we must have

$$V_1 = V_2, \text{ and } \frac{1}{k_1}\frac{dV_1}{dr} = \frac{1}{k_2}\frac{dV_2}{dr}. \tag{3}$$

From these conditions we get the equations

$$\left.\begin{aligned}(A_1 - A_2)\, a^{2i+1} + B_1 - B_2 = 0,\\ \left(\frac{1}{k_1}A_1 - \frac{1}{k_2}A_2\right) i a^{2i+1} - \left(\frac{1}{k_1}B_1 - \frac{1}{k_2}B_2\right)(i+1) = 0.\end{aligned}\right\} \tag{4}$$

These equations are sufficient, when we know two of the four quantities $A_1$, $A_2$, $B_1$, $B_2$, to deduce the other two.

Let us suppose $A_1$ and $B_1$ known, then we find the following expressions for $A_2$ and $B_2$,

$$\left.\begin{aligned}A_2 &= \frac{\{k_1(i+1) + k_2 i\}\, A_1 + (k_1 - k_2)(i+1) B_1 a^{-(2i+1)}}{k_1(2i+1)},\\ B_2 &= \frac{(k_1 - k_2)\, i A_1 a^{2i+1} + \{k_1 i + k_2(i+1)\}\, B_1}{k_1(2i+1)}.\end{aligned}\right\} \tag{5}$$

In this way we can find the conditions which each term of the harmonic expansion of the potential must satisfy for any number of strata bounded by concentric spherical surfaces.

312.] Let us suppose the radius of the first spherical surface to be $a_1$, and let there be a second spherical surface of radius $a_2$ greater than $a_1$, beyond which the specific resistance is $k_3$. If there are no sources or sinks of electricity within these spheres there will be no infinite values of $V$, and we shall have $B_1 = 0$.

We then find for $A_3$ and $B_3$, the coefficients for the outer medium,

$$\left.\begin{aligned} A_3 k_1 k_2 (2i+1)^2 &= \Big[\{k_1(i+1)+k_2 i\}\{k_2(i+1)+k_3 i\} \\ &\qquad + i(i+1)(k_1-k_2)(k_2-k_3)\left(\frac{a_1}{a_2}\right)^{2i+1}\Big] A_1, \\ B_3 k_1 k_2 (2i+1)^2 &= \Big[i(k_2-k_3)\{k_1(i+1)+k_2 i\}\, a_2^{2i+1} \\ &\qquad + i(k_1-k_2)\{k_2 i + k_3(i+1)\}\, a_1^{2i+1}\Big] A_1. \end{aligned}\right\} \quad (6)$$

The value of the potential in the outer medium depends partly on the external sources of electricity, which produce currents independently of the existence of the sphere of heterogeneous matter within, and partly on the disturbance caused by the introduction of the heterogeneous sphere.

The first part must depend on solid harmonics of positive degrees only, because it cannot have infinite values within the sphere. The second part must depend on harmonics of negative degrees, because it must vanish at an infinite distance from the centre of the sphere.

Hence the potential due to the external electromotive forces must be expanded in a series of solid harmonics of positive degree. Let $A_3$ be the coefficient of one of these, of the form

$$A_3 S_i r^i.$$

Then we can find $A_1$, the corresponding coefficient for the inner sphere by equation (6), and from this deduce $A_2$, $B_2$, and $B_3$. Of these $B_3$ represents the effect on the potential in the outer medium due to the introduction of the heterogeneous sphere.

Let us now suppose $k_3 = k_1$, so that the case is that of a hollow shell for which $k = k_2$, separating an inner from an outer portion of a medium for which $k = k_1$.

If we put

$$C = \frac{1}{(2i+1)^2 k_1 k_2 + i(i+1)(k_2-k_1)^2\left(1-\left(\frac{a_1}{a_2}\right)^{2i+1}\right)},$$

then

$$\left.\begin{aligned} A_1 &= k_1 k_2 (2i+1)^2 C A_3, \\ A_2 &= k_2 (2i+1)(k_1(i+1)+k_2 i) C A_3, \\ B_2 &= k_2 i (2i+1)(k_1-k_2) a_1^{2i+1} C A_3, \\ B_3 &= i(k_2-k_1)(k_1(i+1)+k_2 i)(a_2^{2i+1}-a_1^{2i+1}) C A_3. \end{aligned}\right\} \quad (7)$$

The difference between $A_3$ the undisturbed coefficient, and $A_1$ its value in the hollow within the spherical shell, is

$$A_3 - A_1 = (k_2-k_1)^2 i(i+1)\left(1-\left(\frac{a_1}{a_2}\right)^{2i+1}\right) C A_3. \quad (8)$$

Since this quantity is always of the same sign as $A_3$ whatever be the values of $k_1$ and $k_2$, it follows that, whether the spherical shell conducts better or worse than the rest of the medium, the electrical action in the space occupied by the shell is less than it would otherwise be. If the shell is a better conductor than the rest of the medium it tends to equalize the potential all round the inner sphere. If it is a worse conductor, it tends to prevent the electrical currents from reaching the inner sphere at all.

The case of a solid sphere may be deduced from this by making $a_1 = 0$, or it may be worked out independently.

313.] The most important term in the harmonic expansion is that in which $i = 1$, for which

$$\left.\begin{aligned} C &= \frac{1}{9 k_1 k_2 + 2(k_2-k_1)^2\left(1-\left(\frac{a_1}{a_2}\right)^3\right)}, \\ A_1 &= 9 k_1 k_2 C A_3, \qquad A_2 = 3 k_2 (2k_1+k_2) C A_3, \\ B_2 &= 3k_2(k_1-k_2) a_1^3 C A_3, \; B_3 = (k_2-k_1)(2k_1+k_2)(a_2^3-a_1^3) C A_3. \end{aligned}\right\} \quad (9)$$

The case of a solid sphere of resistance $k_2$ may be deduced from this by making $a_1 = 0$. We then have

$$\left.\begin{aligned} A_2 &= \frac{3k_2}{k_1+2k_2} A_3, \quad B_2 = 0, \\ B_3 &= \frac{k_2-k_1}{k_1+2k_2} a_2^3 A_3. \end{aligned}\right\} \quad (10)$$

It is easy to shew from the general expressions that the value of $B_3$ in the case of a hollow sphere having a nucleus of resistance $k_1$, surrounded by a shell of resistance $k_2$, is the same as that of a uniform solid sphere of the radius of the outer surface, and of resistance $K$, where

$$K = \frac{(2\,k_1 + k_2)\,a_2^3 + (k_1 - k_2)\,a_1^3}{(2\,k_1 + k_2)\,a_2^3 - 2\,(k_1 - k_2)\,a_1^3}\,k_2. \tag{11}$$

314.] If there are $n$ spheres of radius $a_1$ and resistance $k_1$, placed in a medium whose resistance is $k_2$, at such distances from each other that their effects in disturbing the course of the current may be taken as independent of each other, then if these spheres are all contained within a sphere of radius $a_2$, the potential at a great distance $r$ from the centre of this sphere will be of the form

$$V = \left(Ar + nB\frac{1}{r^2}\right)\cos\theta, \tag{12}$$

where the value of $B$ is

$$B = \frac{k_1 - k_2}{2\,k_1 + k_2}\,a_1^3 A. \tag{13}$$

The ratio of the volume of the $n$ small spheres to that of the sphere which contains them is

$$p = \frac{n\,a_1^3}{a_2^3}. \tag{14}$$

The value of the potential at a great distance from the sphere may therefore be written

$$V = A\left(r + pa_2^3\,\frac{k_1 - k_2}{2\,k_1 + k_2}\,\frac{1}{r^2}\right)\cos\theta. \tag{15}$$

Now if the whole sphere of radius $a_2$ had been made of a material of specific resistance $K$, we should have had

$$V = A\left\{r + a_2^3\,\frac{K - k_2}{2\,K + k_2}\,\frac{1}{r^2}\right\}\cos\theta. \tag{16}$$

That the one expression should be equivalent to the other,

$$K = \frac{2\,k_1 + k_2 + p(k_1 - k_2)}{2\,k_1 + k_2 - 2\,p(k_1 - k_2)}\,k_2. \tag{17}$$

This, therefore, is the specific resistance of a compound medium consisting of a substance of specific resistance $k_2$, in which are disseminated small spheres of specific resistance $k_1$, the ratio of the volume of all the small spheres to that of the whole being $p$. In order that the action of these spheres may not produce effects depending on their interference, their radii must be small compared with their distances, and therefore $p$ must be a small fraction.

This result may be obtained in other ways, but that here given involves only the repetition of the result already obtained for a single sphere.

When the distance between the spheres is not great compared with their radii, and when $\frac{k_1 - k_2}{2k_1 + k_2}$ is considerable, then other terms enter into the result, which we shall not now consider. In consequence of these terms certain systems of arrangement of the spheres cause the resistance of the compound medium to be different in different directions.

*Application of the Principle of Images.*

315.] Let us take as an example the case of two media separated by a plane surface, and let us suppose that there is a source $S$ of electricity at a distance $a$ from the plane surface in the first medium, the quantity of electricity flowing from the source in unit of time being $S$.

If the first medium had been infinitely extended the current at any point $P$ would have been in the direction $SP$, and the potential at $P$ would have been $\frac{E}{r_1}$, where $E = \frac{Sk_1}{4\pi}$, and $r_1 = SP$.

In the actual case the conditions may be satisfied by taking a point $I$, the image of $S$ in the second medium, such that $IS$ is normal to the plane of separation and is bisected by it. Let $r_2$ be the distance of any point from $I$, then at the surface of separation

$$r_1 = r_2, \tag{1}$$

$$\frac{dr_1}{d\nu} = -\frac{dr_2}{d\nu}. \tag{2}$$

Let the potential $V_1$ at any point in the first medium be that due to a quantity of electricity $E$ placed at $S$, together with an imaginary quantity $E_2$ at $I$, and let the potential $V_2$ at any point of the second medium be that due to an imaginary quantity $E_1$ at $S$, then if

$$V_1 = \frac{E}{r_1} + \frac{E_2}{r_2} \quad \text{and} \quad V_2 = \frac{E_1}{r_1}, \tag{3}$$

the superficial condition $V_1 = V_2$ gives

$$E + E_2 = E_1, \tag{4}$$

and the condition

$$\frac{1}{k_1}\frac{dV_1}{d\nu} = \frac{1}{k_2}\frac{dV_2}{d\nu} \tag{5}$$

gives
$$\frac{1}{k_1}(E-E_2) = \frac{1}{k_2}E_1, \tag{6}$$

whence
$$E_1 = \frac{2k_2}{k_1+k_2}E, \qquad E_2 = \frac{k_2-k_1}{k_1+k_2}E. \tag{7}$$

The potential in the first medium is therefore the same as would be produced in air by a charge $E$ placed at $S$, and a charge $E_2$ at $I$ on the electrostatic theory, and the potential in the second medium is the same as that which would be produced in air by a charge $E_1$ at $S$.

The current at any point of the first medium is the same as would have been produced by the source $S$ together with a source $\frac{k_2-k_1}{k_1+k_2}S$ placed at $I$ if the first medium had been infinite, and the current at any point of the second medium is the same as would have been produced by a source $\frac{2k_2S}{(k_1+k_2)}$ placed at $S$ if the second medium had been infinite.

We have thus a complete theory of electrical images in the case of two media separated by a plane boundary. Whatever be the nature of the electromotive forces in the first medium, the potential they produce in the first medium may be found by combining their direct effect with the effect of their image.

If we suppose the second medium a perfect conductor, then $k_2 = 0$, and the image at $I$ is equal and opposite to the source at $S$. This is the case of electric images, as in Thomson's theory in electrostatics.

If we suppose the second medium a perfect insulator, then $k_2 = \infty$, and the image at $I$ is equal to the source at $S$ and of the same sign. This is the case of images in hydrokinetics when the fluid is bounded by a rigid plane surface*.

316.] The method of inversion, which is of so much use in electrostatics when the bounding surface is supposed to be that of a perfect conductor, is not applicable to the more general case of the surface separating two conductors of unequal electric resistance. The method of inversion in two dimensions is, how-

* {A similar investigation will give the electric field due to a charge of electricity at $S$ placed in a dielectric whose specific inductive capacity is $K_1$, this dielectric being separated by a plane face from another dielectric whose specific inductive capacity is $K_2$. $V_1$ and $V_2$ will represent the potentials in this case if the charge $= K_1 E$ and if $K_1 k_1 = 1 = K_2 k_2$.}

ever, applicable, as well as the more general method of transformation in two dimensions given in Art. 190 *.

*Conduction through a Plate separating Two Media.*

317.] Let us next consider the effect of a plate of thickness $AB$ of a medium whose resistance is $k_2$, and separating two media whose resistances are $k_1$ and $k_3$, in altering the potential due to a source $S$ in the first medium.

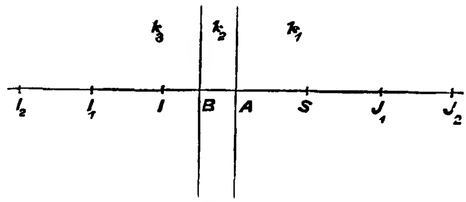


Fig. 24.

The potential will be equal to that due to a system of charges placed in air at certain points along the normal to the plate through $S$.

Make

$AI = SA$, $BI_1 = SB$, $AJ_1 = I_1A$, $BI_2 = J_1B$, $AJ_2 = I_2A$, &c.;

then we have two series of points at distances from each other equal to twice the thickness of the plate.

318.] The potential in the first medium at any point $P$ is

$$\frac{E}{PS} + \frac{I}{PI} + \frac{I_1}{PI_1} + \frac{I_2}{PI_2} + \&c., \tag{8}$$

that at a point $P'$ in the second

$$\frac{E'}{P'S} + \frac{I'}{P'I} + \frac{I_1'}{P'I_1} + \frac{I_2'}{P'I_2} + \&c.$$
$$+ \frac{J_1'}{P'J_1} + \frac{J_2'}{P'J_2} + \&c., \tag{9}$$

and that at a point $P''$ in the third

$$\frac{E''}{P''S} + \frac{J_1}{P''J_1} + \frac{J_2}{P''J_2} + \&c., \tag{10}$$

where $I$, $I'$, &c. represent the imaginary charges placed at the points $I$, &c., and the accents denote that the potential is to be taken within the plate.

* See Kirchhoff, Pogg. *Ann.* lxiv. 497, and lxvii. 344; Quincke, Pogg. xcvii. 382; Smith, *Proc. R. S. Edin.*, 1869–70, p. 79. Holzmüller, *Einführung in die Theorie der isogonalen Verwandschaften*, Leipzig, 1882. Guebhard, *Journal de Physique*, t. i. p. 483, 1882. W. G. Adams, *Phil. Mag.* iv. 50, p. 548, 1876; G. C. Foster and O. J. Lodge, *Phil. Mag.* iv. 49, pp. 385, 453; 50, p. 475, 1879 and 1880; O. J. Lodge, *Phil. Mag.* (5), i. 373, 1876.

Then, by Article 315, we have from the conditions for the surface through $A$,

$$I = \frac{k_2 - k_1}{k_2 + k_1} E, \qquad E' = \frac{2k_2}{k_2 + k_1} E. \tag{11}$$

For the surface through $B$ we find

$$I_1' = \frac{k_3 - k_2}{k_3 + k_2} E', \qquad E'' = \frac{2k_3}{k_2 + k_3} E'. \tag{12}$$

Similarly for the surface through $A$ again,

$$J_1' = \frac{k_1 - k_2}{k_1 + k_2} I_1', \qquad I_1 = \frac{2k_1}{k_1 + k_2} I_1', \tag{13}$$

and for the surface through $B$,

$$I_2' = \frac{k_3 - k_2}{k_3 + k_2} J_1', \qquad J_1 = \frac{2k_3}{k_3 + k_2} J_1'. \tag{14}$$

If we make $\quad \rho = \dfrac{k_1 - k_2}{k_1 + k_2}$ and $\rho' = \dfrac{k_3 - k_2}{k_3 + k_2}$,

we find for the potential in the first medium,

$$V = \frac{E}{PS} - \rho\frac{E}{PI} + (1-\rho^2)\rho'\frac{E}{PI_1} + \rho'(1-\rho^2)\rho\rho'\frac{E}{PI_2} + \&c.$$

$$+\rho'(1-\rho^2)(\rho\rho')^{n-1}\frac{E}{PI_n} + \ldots \tag{15}$$

For the potential in the third medium we find

$$V = (1+\rho')(1-\rho)E\left\{\frac{1}{PS} + \frac{\rho\rho'}{PJ_1} + \&c. + \frac{(\rho\rho')^n}{PJ_n} + \ldots\right\}^*. \tag{16}$$

* {These expressions may be reduced to definite integrals by the relation

$$\frac{1}{\sqrt{a^2+b^2}} = \int_0^\infty J_0(bt)e^{-at}dt$$

where $J_0$ denotes Bessel's function of zero order.

Hence if we take $S$ as the origin of coordinates, and the normal to the plate as the axis of $x$,

$$\frac{1}{PS} = \int_0^\infty J_0(yt)e^{-xt}dt,$$

$$\frac{1}{PJ_1} = \int_0^\infty J_0(yt)e^{-(x+2c)t}dt,$$

where $c$ is the thickness of the plate,

$$\frac{1}{PJ_2} = \int_0^\infty J_0(yt)e^{-(x+4c)t}dt,$$

and so on. Substituting these values in (16), we see that $V$ equals

$$E(1+\rho')(1-\rho)\int_0^\infty \frac{J_0(yt)e^{-xt}}{1-\rho\rho' e^{-2ct}}dt.$$

The values of this when $y=0$, $x = 2nc$ when $n$ is an integer can easily be found.}

If the first medium is the same as the third, then $k_1 = k_3$ and $\rho = \rho'$, and the potential on the other side of the plate will be

$$V = (1-\rho^2)\, E\left\{\frac{1}{PS} + \frac{\rho^2}{PJ_1} + \&c. + \frac{\rho^{2n}}{PJ_n} + \dots\right\}. \qquad (17)$$

If the plate is a very much better conductor than the rest of the medium, $\rho$ is very nearly equal to 1. If the plate is a nearly perfect insulator, $\rho$ is nearly equal to $-1$, and if the plate differs little in conducting power from the rest of the medium, $\rho$ is a small quantity positive or negative.

The theory of this case was first stated by Green in his 'Theory of Magnetic Induction' (*Essay*, p. 65). His result, however, is correct only when $\rho$ is nearly equal to 1 *. The quantity $g$ which he uses is connected with $\rho$ by the equations

$$g = \frac{2\rho}{3-\rho} = \frac{k_1 - k_2}{k_1 + 2k_2}, \qquad \rho = \frac{3g}{2+g} = \frac{k_1 - k_2}{k_1 + k_2}.$$

If we put $\rho = \dfrac{2\pi\kappa}{1+2\pi\kappa}$, we shall have a solution of the problem of the magnetic induction excited by a magnetic pole in an infinite plate whose coefficient of magnetization is $\kappa$.

## *On Stratified Conductors.*

319.] Let a conductor be composed of alternate strata of thicknesses $c$ and $c'$ of two substances whose coefficients of conductivity are different. Required the coefficients of resistance and conductivity of the compound conductor.

Let the planes of the strata be normal to $z$. Let every symbol relating to the strata of the second kind be accented, and let every symbol relating to the compound conductor be marked with a bar thus, $\overline{X}$. Then

$$\begin{aligned} \overline{X} &= X = X', & (c+c')\,\bar{u} &= cu + c'u', \\ \overline{Y} &= Y = Y', & (c+c')\,\bar{v} &= cv + c'v'; \\ (c+c')\,\overline{Z} &= cZ + c'Z', & \bar{w} &= w = w'. \end{aligned}$$

We must first determine $u$, $u'$, $v$, $v'$, $Z$ and $Z'$ in terms of $\overline{X}$, $\overline{Y}$ and $\bar{w}$ from the equations of resistance, Art. 297, or those

* See Sir W. Thomson's 'Note on Induced Magnetism in a Plate,' *Camb. and Dub. Math. Journ.*, Nov. 1845, *or Reprint*, art. ix. § 156.

of conductivity, Art. 298. If we put $D$ for the determinant of the coefficients of resistance, we find

$$
\begin{aligned}
ur_3 D &= R_2\overline{X} - Q_3\overline{Y} + \bar{w}q_2 D,\\
vr_3 D &= R_1\overline{Y} - P_3\overline{X} + \bar{w}p_1 D,\\
Zr_3 &= -p_2\overline{X} - q_1\overline{Y} + \bar{w}.
\end{aligned}
$$

Similar equations with the symbols accented give the values of $u'$, $v'$ and $Z'$. Having found $\bar{u}$, $\bar{v}$ and $\bar{w}$ in terms of $\overline{X}$, $\overline{Y}$ and $\overline{Z}$, we may write down the equations of conductivity of the stratified conductor. If we make $h = \dfrac{c}{r_3}$ and $h' = \dfrac{c'}{r_3'}$, we find

$$
\bar{p}_1 = \frac{hp_1 + h'p_1'}{h+h'}, \qquad \bar{q}_1 = \frac{hq_1 + h'q_1'}{h+h'},
$$

$$
\bar{p}_2 = \frac{hp_2 + h'p_2'}{h+h'}, \qquad \bar{q}_2 = \frac{hq_2 + h'q_2'}{h+h'},
$$

$$
\bar{p}_3 = \frac{cp_3 + c'p_3'}{c+c'} - \frac{hh'(q_1 - q_1')(q_2 - q_2')}{(h+h')(c+c')},
$$

$$
\bar{q}_3 = \frac{cq_3 + c'q_3'}{c+c'} - \frac{hh'(p_1 - p_1')(p_2 - p_2')}{(h+h')(c+c')},
$$

$$
\bar{r}_1 = \frac{cr_1 + c'r_1'}{c+c'} - \frac{hh'(p_2 - p_2')(q_2 - q_2')}{(h+h')(c+c')},
$$

$$
\bar{r}_2 = \frac{cr_2 + c'r_2'}{c+c'} - \frac{hh'(p_1 - p_1')(q_1 - q_1')}{(h+h')(c+c')},
$$

$$
\bar{r}_3 = \frac{c+c'}{h+h'}.
$$

320.] If neither of the two substances of which the strata are formed has the rotatory property of Art. 303, the value of any $P$ or $p$ will be equal to that of its corresponding $Q$ or $q$. From this it follows that in the stratified conductor also

$$
\bar{p}_1 = \bar{q}_1, \qquad \bar{p}_2 = \bar{q}_2, \qquad \bar{p}_3 = \bar{q}_3,
$$

or there is no rotatory property developed by stratification, unless it exists in one or both of the separate materials.

321.] If we now suppose that there is no rotatory property, and also that the axes of $x$, $y$ and $z$ are the principal axes, then the $p$ and $q$ coefficients vanish, and

$$
\bar{r}_1 = \frac{cr_1 + c'r_1'}{c+c'}, \qquad \bar{r}_2 = \frac{cr_2 + c'r_2'}{c+c'}, \qquad \bar{r}_3 = \frac{c+c'}{\dfrac{c}{r_3} + \dfrac{c'}{r_3'}}.
$$

If we begin with both substances isotropic, but of different conductivities $r$ and $r'$, then, since $\bar{r}_1 - \bar{r}_3 = \dfrac{cc'}{c+c'}\dfrac{(r-r')^2}{(cr'+c'r)}$, the result of stratification will be to make the resistance greatest in the direction of a normal to the strata, and the resistances in all directions in the plane of the strata will be equal.

322.] Take an isotropic substance of conductivity $r$, cut it into exceedingly thin slices of thickness $a$, and place them alternately with slices of a substance whose conductivity is $s$, and thickness $k_1 a$.

Let these slices be normal to $x$. Then cut this compound conductor into very much thicker slices, of thickness $b$, normal to $y$, and alternate these with slices whose conductivity is $s$ and thickness $k_2 b$.

Lastly, cut the new conductor into still thicker slices, of thickness $c$, normal to $z$, and alternate them with slices whose conductivity is $s$ and thickness $k_3 c$.

The result of the three operations will be to cut the substance whose conductivity is $r$ into rectangular parallelepipeds whose dimensions are $a$, $b$ and $c$, where $b$ is exceedingly small compared with $c$, and $a$ is exceedingly small compared with $b$, and to embed these parallelepipeds in the substance whose conductivity is $s$, so that they are separated from each other $k_1 a$ in the direction of $x$, $k_2 b$ in that of $y$, and $k_3 c$ in that of $z$. The conductivities of the conductor so formed in the directions of $x$, $y$, and $z$ are to be found by three applications in order of the results of Art. 321. We thereby obtain

$$r_1 = \frac{\{1+k_1(1+k_2)(1+k_3)\}r + (k_2+k_3+k_2k_3)s}{(1+k_2)(1+k_3)(k_1 r+s)}s,$$

$$r_2 = \frac{(1+k_2+k_2k_3)r + (k_1+k_3+k_1k_2+k_1k_3+k_1k_2k_3)s}{(1+k_3)\{k_2 r+(1+k_1+k_1k_2)s\}}s,$$

$$r_3 = \frac{(1+k_3)(r+(k_1+k_2+k_1k_2)s)}{k_3 r+(1+k_1+k_2+k_2k_3+k_3k_1+k_1k_2+k_1k_2k_3)s}s.$$

The accuracy of this investigation depends upon the three dimensions of the parallelepipeds being of different orders of magnitude, so that we may neglect the conditions to be fulfilled at their edges and angles. If we make $k_1$, $k_2$ and $k_3$ each unity, then

$$r_1 = \frac{5r+3s}{4r+4s}s, \quad r_2 = \frac{3r+5s}{2r+6s}s, \quad r_3 = \frac{2r+6s}{r+7s}s.$$

If $r = 0$, that is, if the medium of which the parallelepipeds are made is a perfect insulator, then

$$r_1 = \tfrac{3}{4}s, \qquad r_2 = \tfrac{5}{8}s, \qquad r_3 = \tfrac{6}{7}s.$$

If $r = \infty$, that is, if the parallelepipeds are perfect conductors,

$$r_1 = \tfrac{5}{4}s, \qquad r_2 = \tfrac{3}{2}s, \qquad r_3 = 2s.$$

In every case, provided $k_1 = k_2 = k_3$, it may be shewn that $r_1$, $r_2$ and $r_3$ are in ascending order of magnitude, so that the greatest conductivity is in the direction of the longest dimensions of the parallelepipeds, and the greatest resistance in the direction of their shortest dimensions.

323.] In a rectangular parallelepiped of a conducting solid, let there be a conducting channel made from one angle to the opposite, the channel being a wire covered with insulating material, and let the lateral dimensions of the channel be so small that the conductivity of the solid is not affected except on account of the current conveyed along the wire.

Let the dimensions of the parallelepiped in the directions of the coordinate axes be $a$, $b$, $c$, and let the conductivity of the channel, extending from the origin to the point $(abc)$, be $abcK$.

The electromotive force acting between the extremities of the channel is

$$aX + bY + cZ,$$

and if $C'$ be the current along the channel

$$C' = Kabc\,(aX + bY + cZ).$$

The current across the face $bc$ of the parallelepiped is $bcu$, and this is made up of that due to the conductivity of the solid and of that due to the conductivity of the channel, or

$$bcu = bc\,(r_1 X + p_3 Y + q_2 Z) + Kabc\,(aX + bY + cZ),$$

or

$$u = (r_1 + Ka^2)\,X + (p_3 + Kab)\,Y + (q_2 + Kca)\,Z.$$

In the same way we may find the values of $v$ and $w$. The coefficients of conductivity as altered by the effect of the channel will be

$$\begin{array}{lll} r_1 + Ka^2, & r_2 + Kb^2, & r_3 + Kc^2, \\ p_1 + Kbc, & p_2 + Kca, & p_3 + Kab, \\ q_1 + Kbc, & q_2 + Kca, & q_3 + Kab. \end{array}$$

In these expressions, the additions to the values of $p_1$, &c., due to the effect of the channel, are equal to the additions to the values of $q_1$, &c. Hence the values of $p_1$ and $q_1$ cannot be rendered unequal by the introduction of linear channels into every element of volume of the solid, and therefore the rotatory property of Art. 303, if it does not exist previously in a solid, cannot be introduced by such means.

324.] *To construct a framework of linear conductors which shall have any given coefficients of conductivity forming a symmetrical system.*

Let the space be divided into equal small cubes, of which let the figure represent one. Let the coordinates of the points $O$, $L$, $M$, $N$, and their potentials be as follows:—

| | $x$ | $y$ | $z$ | *Potential* |
|---|---|---|---|---|
| $O$ | 0 | 0 | 0 | $X+Y+Z$ |
| $L$ | 0 | 1 | 1 | $X$ |
| $M$ | 1 | 0 | 1 | $Y$ |
| $N$ | 1 | 1 | 0 | $Z$. |

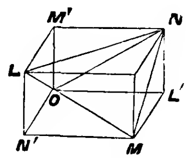


Fig. 25.

Let these four points be connected by six conductors,

$$OL, \quad OM, \quad ON, \quad MN, \quad NL, \quad LM,$$

of which the conductivities are respectively

$$A, \quad B, \quad C, \quad P, \quad Q, \quad R.$$

The electromotive forces along these conductors will be

$$Y+Z, \quad Z+X, \quad X+Y, \quad Y-Z, \quad Z-X, \quad X-Y,$$

and the currents

$$A(Y+Z),\ B(Z+X),\ C(X+Y),\ P(Y-Z),\ Q(Z-X),\ R(X-Y).$$

Of these currents, those which convey electricity in the positive direction of $x$ are those along $LM$, $LN$, $OM$ and $ON$, and the quantity conveyed is

$$u = (B+C+Q+R)X + (C-R)Y \qquad + (B-Q)Z.$$

Similarly

$$v = (C-R)X \qquad + (C+A+R+P)Y + (A-P)Z;$$
$$w = (B-Q)X \qquad + (A-P)Y \qquad + (A+B+P+Q)Z;$$

whence we find by comparison with the equations of conduction, Art. 298,

$$4A = r_2+r_3-r_1+2p_1, \qquad 4P = r_2+r_3-r_1-2p_1,$$
$$4B = r_3+r_1-r_2+2p_2, \qquad 4Q = r_3+r_1-r_2-2p_2,$$
$$4C = r_1+r_2-r_3+2p_3, \qquad 4R = r_1+r_2-r_3-2p_3.$$

# CHAPTER X.

## CONDUCTION IN DIELECTRICS.

325.] We have seen that when electromotive force acts on a dielectric medium it produces in it a state which we have called electric polarization, and which we have described as consisting of electric displacement within the medium in a direction which, in isotropic media, coincides with that of the electromotive force, combined with a superficial charge on every element of volume into which we may suppose the dielectric divided, which is negative on the side towards which the force acts, and positive on the side from which it acts.

When electromotive force acts on a conducting medium it also produces what is called an electric current.

Now dielectric media, with very few, if any, exceptions, are also more or less imperfect conductors, and many media which are not good insulators exhibit phenomena of dielectric induction. Hence we are led to study the state of a medium in which induction and conduction are going on at the same time.

For simplicity we shall suppose the medium isotropic at every point, but not necessarily homogeneous at different points. In this case, the equation of Poisson becomes, by Art. 83,

$$\frac{d}{dx}\left(K\frac{dV}{dx}\right)+\frac{d}{dy}\left(K\frac{dV}{dy}\right)+\frac{d}{dz}\left(K\frac{dV}{dz}\right)+4\pi\rho=0, \tag{1}$$

where $K$ is the 'specific inductive capacity.'

The 'equation of continuity' of electric currents becomes

$$\frac{d}{dx}\left(\frac{1}{r}\frac{dV}{dx}\right)+\frac{d}{dy}\left(\frac{1}{r}\frac{dV}{dy}\right)+\frac{d}{dz}\left(\frac{1}{r}\frac{dV}{dz}\right)-\frac{d\rho}{dt}=0, \tag{2}$$

where $r$ is the specific resistance referred to unit of volume.

When $K$ or $r$ is discontinuous, these equations must be transformed into those appropriate to surfaces of discontinuity.

In a strictly homogeneous medium $r$ and $K$ are both constant, so that we find

$$\frac{d^2V}{dx^2}+\frac{d^2V}{dy^2}+\frac{d^2V}{dz^2}=-4\pi\frac{\rho}{K}=r\frac{d\rho}{dt}, \tag{3}$$

whence
$$\rho=Ce^{-\frac{4\pi}{Kr}t}; \tag{4}$$

or, if we put
$$T=\frac{Kr}{4\pi}, \qquad \rho=Ce^{-\frac{t}{T}}. \tag{5}$$

This result shews that under the action of any external electric forces on a homogeneous medium, the interior of which is originally charged in any manner with electricity, the internal charges will die away at a rate which does not depend on the external forces, so that at length there will be no charge of electricity within the medium, after which no external forces can either produce or maintain a charge in any internal portion of the medium, provided the relation between electromotive force, electric polarization and conduction remains the same. When disruptive discharge occurs these relations cease to be true, and internal charge may be produced.

*On Conduction through a Condenser.*

326.] Let $C$ be the capacity of a condenser, $R$ its resistance, and $E$ the electromotive force which acts on it, that is, the difference of potentials of the surfaces of the metallic electrodes.

Then the quantity of electricity on the side from which the electromotive force acts will be $CE$, and the current through the substance of the condenser in the direction of the electromotive force will be $\frac{E}{R}$.

If the electrification is supposed to be produced by an electromotive force $E$ acting in a circuit of which the condenser forms part, and if $\frac{dQ}{dt}$ represents the current in that circuit, then

$$\frac{dQ}{dt}=\frac{E}{R}+C\frac{dE}{dt}. \tag{6}$$

Let a battery of electromotive force $E_0$ whose resistance together with that of the wire connecting the electrodes is $r_1$ be introduced into this circuit, then

$$\frac{dQ}{dt}=\frac{E_0-E}{r_1}=\frac{E}{R}+C\frac{dE}{dt}. \tag{7}$$

Hence, at any time $t_1$,

$$E(=E_1) = E_0 \frac{R}{R+r_1}\left(1-e^{-\frac{t_1}{T_1}}\right) \text{ where } T_1 = \frac{CRr_1}{R+r_1}. \tag{8}$$

Next, let the circuit $r_1$ be broken for a time $t_2$, putting $r_1$ infinite, we get from (7),

$$E(=E_2) = E_1 e^{-\frac{t_2}{T_2}} \text{ where } T_2 = CR. \tag{9}$$

Finally, let the surfaces of the condenser be connected by means of a wire whose resistance is $r_3$ for a time $t_3$, then putting $E_0 = 0$, $r_1 = r_3$ in (7), we get

$$E(=E_3) = E_2 e^{-\frac{t_3}{T_3}} \text{ where } T_3 = \frac{CRr_3}{R+r_3}. \tag{10}$$

If $Q_3$ is the total discharge through this wire in the time $t_3$,

$$Q_3 = E_0 \frac{CR^2}{(R+r_1)(R+r_3)}\left(1-e^{-\frac{t_1}{T_1}}\right)e^{-\frac{t_2}{T_2}}\left(1-e^{-\frac{t_3}{T_3}}\right). \tag{11}$$

In this way we may find the discharge through a wire which is made to connect the surfaces of a condenser after being charged for a time $t_1$, and then insulated for a time $t_2$. If the time of charging is sufficient, as it generally is, to develop the whole charge, and if the time of discharge is sufficient for a complete discharge, the discharge is

$$Q_3 = E_0 \frac{CR^2}{(R+r_1)(R+r_3)} e^{-\frac{t_2}{CR}}. \tag{12}$$

327.] In a condenser of this kind, first charged in any way, next discharged through a wire of small resistance, and then insulated, no new electrification will appear. In most actual condensers, however, we find that after discharge and insulation a new charge is gradually developed, of the same kind as the original charge, but inferior in intensity. This is called the residual charge. To account for it we must admit that the constitution of the dielectric medium is different from that which we have just described. We shall find, however, that a medium formed of a conglomeration of small pieces of different simple media would possess this property.

## *Theory of a Composite Dielectric.*

328.] We shall suppose, for the sake of simplicity, that the dielectric consists of a number of plane strata of different materials and of area unity, and that the electric forces act in the direction of the normal to the strata.

Let $a_1$, $a_2$, &c. be the thicknesses of the different strata.

$X_1$, $X_2$, &c. the resultant electrical forces within the strata.

$p_1$, $p_2$, &c. the currents due to conduction through the strata.

$f_1$, $f_2$, &c. the electric displacements.

$u_1$, $u_2$, &c. the total currents, due partly to conduction and partly to variation of displacement.

$r_1$, $r_2$, &c. the specific resistances referred to unit of volume.

$K_1$, $K_2$, &c. the specific inductive capacities.

$k_1$, $k_2$, &c. the reciprocals of the specific inductive capacities.

$E$ the electromotive force due to a voltaic battery, placed in the part of the circuit leading from the last stratum towards the first, which we shall suppose good conductors.

$Q$ the total quantity of electricity which has passed through this part of the circuit up to the time $t$.

$R_0$ the resistance of the battery with its connecting wires.

$\sigma_{12}$ the surface-density of electricity on the surface which separates the first and second strata.

Then in the first stratum we have, by Ohm's Law,

$$X_1 = r_1 p_1. \tag{1}$$

By the theory of electrical displacement,

$$X_1 = 4\pi k_1 f_1. \tag{2}$$

By the definition of the total current,

$$u_1 = p_1 + \frac{df_1}{dt}, \tag{3}$$

with similar equations for the other strata, in each of which the quantities have the suffix belonging to that stratum.

To determine the surface-density on any stratum, we have an equation of the form $\sigma_{12} = f_2 - f_1$, (4)

and to determine its variation we have

$$\frac{d\sigma_{12}}{dt} = p_1 - p_2. \tag{5}$$

By differentiating (4) with respect to $t$, and equating the result to (5), we obtain

$$p_1 + \frac{df_1}{dt} = p_2 + \frac{df_2}{dt} = u, \text{ say,} \tag{6}$$

or, by taking account of (3),

$$u_1 = u_2 = \&c. = u. \tag{7}$$

That is, the total current $u$ is the same in all the strata, and is equal to the current through the wire and battery.

We have also, in virtue of equations (1) and (2),

$$u = \frac{1}{r_1} X_1 + \frac{1}{4\pi k_1} \frac{dX_1}{dt}, \tag{8}$$

from which we may find $X_1$ by the inverse operation on $u$,

$$X_1 = \left(\frac{1}{r_1} + \frac{1}{4\pi k_1} \frac{d}{dt}\right)^{-1} u. \tag{9}$$

The total electromotive force $E$ is

$$E = a_1 X_1 + a_2 X_2 + \&c., \tag{10}$$

or $$E = \left\{a_1 \left(\frac{1}{r_1} + \frac{1}{4\pi k_1} \frac{d}{dt}\right)^{-1} + a_2 \left(\frac{1}{r_2} + \frac{1}{4\pi k_2} \frac{d}{dt}\right)^{-1} + \&c.\right\} u, \tag{11}$$

an equation between $E$, the external electromotive force, and $u$, the external current.

If the ratio of $r$ to $k$ is the same in all the strata, the equation reduces itself to

$$E + \frac{r}{4\pi k} \frac{dE}{dt} = (a_1 r_1 + a_2 r_2 + \&c.)\, u, \tag{12}$$

which is the case we have already examined in Art. 326, and in which, as we found, no phenomenon of residual charge can take place.

If there are $n$ substances having different ratios of $r$ to $k$, the general equation (11), when cleared of inverse operations, will be a linear differential equation, of the $n$th order with respect to $E$ and of the $(n-1)$th order with respect to $u$, $t$ being the independent variable.

From the form of the equation it is evident that the order of the different strata is indifferent, so that if there are several strata of the same substance we may suppose them united into one without altering the phenomena.

329.] Let us now suppose that at first $f_1$, $f_2$, &c. are all zero, and that an electromotive force $E_0$ is suddenly made to act, and let us find its instantaneous effect.

Integrating (8) with respect to $t$, we find

$$Q = \int u\, dt = \frac{1}{r_1} \int X_1\, dt + \frac{1}{4\pi k_1} X_1 + \text{const.} \tag{13}$$

Now, since $X_1$ is always in this case finite, $\int X_1 dt$ must be insensible when $t$ is insensible, and therefore, since $X_1$ is originally zero, the instantaneous effect will be

$$X_1 = 4\pi k_1 Q. \tag{14}$$

Hence, by equation (10),

$$E_0 = 4\pi (k_1 a_1 + k_2 a_2 + \&c.)\, Q, \tag{15}$$

and if $C$ be the electric capacity of the system as measured in this instantaneous way,

$$C = \frac{Q}{E_0} = \frac{1}{4\pi(k_1 a_1 + k_2 a_2 + \&c.)} \tag{16}$$

This is the same result that we should have obtained if we had neglected the conductivity of the strata.

Let us next suppose that the electromotive force $E_0$ is continued uniform for an indefinitely long time, or till a uniform current of conduction equal to $p$ is established through the system.

We have then $X_1 = r_1 p$, etc., and therefore by (10),

$$E_0 = (r_1 a_1 + r_2 a_2 + \&c.)\, p. \tag{17}$$

If $R$ be the total resistance of the system,

$$R = \frac{E_0}{p} = r_1 a_1 + r_2 a_2 + \&c. \tag{18}$$

In this state we have by (2),

$$f_1 = \frac{r_1}{4\pi k_1} p,$$

so that

$$\sigma_{12} = \left(\frac{r_2}{4\pi k_2} - \frac{r_1}{4\pi k_1}\right) p. \tag{19}$$

If we now suddenly connect the extreme strata by means of a conductor of small resistance, $E$ will be suddenly changed from its original value $E_0$ to zero, and a quantity $Q$ of electricity will pass through the conductor.

To determine $Q$ we observe that if $X_1'$ be the new value of $X_1$, then by (13),

$$X_1' = X_1 + 4\pi k_1 Q. \tag{20}$$

Hence, by (10), putting $E = 0$,

$$0 = a_1 X_1 + \&c. + 4\pi (a_1 k_1 + a_2 k_2 + \&c.)\, Q, \tag{21}$$

or

$$0 = E_0 + \frac{1}{C} Q. \tag{22}$$

Hence $Q = -CE_0$ where $C$ is the capacity, as given by equation (16). The instantaneous discharge is therefore equal to the instantaneous charge.

Let us next suppose the connexion broken immediately after this discharge. We shall then have $u = 0$, so that by equation (8),

$$X_1 = X_1' e^{-\frac{4\pi k_1}{r_1}t}, \tag{23}$$

where $X_1'$ is the initial value after the discharge.

Hence, at any time $t$, we have by (23) and (20)

$$X_1 = E_0 \left\{\frac{r_1}{R} - 4\pi k_1 C\right\} e^{-\frac{4\pi k_1}{r_1}t}.$$

The value of $E$ at any time is therefore

$$= E_0 \left\{\left(\frac{a_1 r_1}{R} - 4\pi a_1 k_1 C\right) e^{-\frac{4\pi k_1}{r_1}t} + \left(\frac{a_2 r_2}{R} - 4\pi a_2 k_2 C\right) e^{-\frac{4\pi k_2}{r_2}t} + \&c.\right\}, \tag{24}$$

and the instantaneous discharge after any time $t$ is $EC$. This is called the residual discharge.

If the ratio of $r$ to $k$ is the same for all the strata, the value of $E$ will be reduced to zero. If, however, this ratio is not the same, let the terms be arranged according to the values of this ratio in descending order of magnitude.

The sum of all the coefficients is evidently zero, so that when $t = 0$, $E = 0$. The coefficients are also in descending order of magnitude, and so are the exponential terms when $t$ is positive. Hence, when $t$ is positive, $E$ will be positive *, so that the residual discharge is always of the same sign as the primary discharge.

When $t$ is indefinitely great all the terms disappear unless any of the strata are perfect insulators, in which case $r_1$ is infinite for that stratum, and $R$ is infinite for the whole system, and the final value of $E$ is not zero but

$$E = E_0 (1 - 4\pi a_1 k_1 C). \tag{25}$$

Hence, when some, but not all, of the strata are perfect insulators, a residual discharge may be permanently preserved in the system.

330.] We shall next determine the total discharge through a wire of resistance $R_0$ kept permanently in connexion with the extreme strata of the system, supposing the system first charged by means of a long-continued application of the electromotive force $E$.

* {This is perhaps more easily seen if we write (24) as

$$E = \frac{E_0 4\pi C}{R} \Sigma \left[\frac{a_p a_q}{r_p r_q}\left\{\frac{k_q}{r_q} - \frac{k_p}{r_p}\right\}\left\{e^{-\frac{4\pi k_p}{r_p}t} - e^{-\frac{4\pi k_q}{r_q}t}\right\}\right].\}$$

At any instant we have

$$E = a_1 r_1 p_1 + a_2 r_2 p_2 + \&c. + R_0 u = 0, \tag{26}$$

and also, by (3),
$$u = p_1 + \frac{df_1}{dt}. \tag{27}$$

Hence
$$(R+R_0)u = a_1 r_1 \frac{df_1}{dt} + a_2 r_2 \frac{df_2}{dt} + \&c. \tag{28}$$

Integrating with respect to $t$ in order to find $Q$, we get

$$(R+R_0)\,Q = a_1 r_1 (f_1' - f_1) + a_2 r_2 (f_2' - f_2) + \&c., \tag{29}$$

where $f_1$ is the initial, and $f_1'$ the final value of $f_1$.

In this case $f_1' = 0$, and by (2) and (20) $f_1 = E_0\left(\frac{r_1}{4\pi k_1 R} - C\right)$.

Hence
$$(R+R_0)Q = -\frac{E}{4\pi R}\left(\frac{a_1 r_1^2}{k_1} + \frac{a_2 r_2^2}{k_2} + \&c.\right) + E_0 CR, \tag{30}$$

$$= -\frac{CE_0}{R}\Sigma\Sigma\left[a_1 a_2 k_1 k_2 \left(\frac{r_1}{k_1} - \frac{r_2}{k_2}\right)\right], \tag{31}$$

where the summation is extended to all quantities of this form belonging to every pair of strata.

It appears from this that $Q$ is always negative, that is to say, in the opposite direction to that of the current employed in charging the system.

This investigation shews that a dielectric composed of strata of different kinds may exhibit the phenomena known as electric absorption and residual discharge, although none of the substances of which it is made exhibit these phenomena when alone. An investigation of the cases in which the materials are arranged otherwise than in strata would lead to similar results, though the calculations would be more complicated, so that we may conclude that the phenomena of electric absorption may be expected in the case of substances composed of parts of different kinds, even though these individual parts should be microscopically small *.

It by no means follows that every substance which exhibits this phenomenon is so composed, for it may indicate a new kind of electric polarization of which a homogeneous substance may

* {Rowland and Nichols have shewn that crystals of Iceland Spar which are very homogeneous shew no Electric Absorption, *Phil. Mag.* xi. p. 414, 1881. Muraoka found that while paraffin and xylol shewed no residual charge when separate, a layer of xylol on a layer of paraffin did. *Wied. Ann.* 40, 331, 1890.}

be capable, and this in some cases may perhaps resemble electro-chemical polarization much more than dielectric polarization.

The object of the investigation is merely to point out the true mathematical character of the so-called electric absorption, and to shew how fundamentally it differs from the phenomena of heat which seem at first sight analogous.

331.] If we take a thick plate of any substance and heat it on one side, so as to produce a flow of heat through it, and if we then suddenly cool the heated side to the same temperature as the other, and leave the plate to itself, the heated side of the plate will again become hotter than the other by conduction from within.

Now an electrical phenomenon exactly analogous to this can be produced, and actually occurs in telegraph cables, but its mathematical laws, though exactly agreeing with those of heat, differ entirely from those of the stratified condenser.

In the case of heat there is true absorption of the heat into the substance with the result of making it hot. To produce a truly analogous phenomenon in electricity is impossible, but we may imitate it in the following way in the form of a lecture-room experiment.

Let $A_1$, $A_2$, &c. be the inner conducting surfaces of a series of condensers, of which $B_0$, $B_1$, $B_2$, &c. are the outer surfaces.

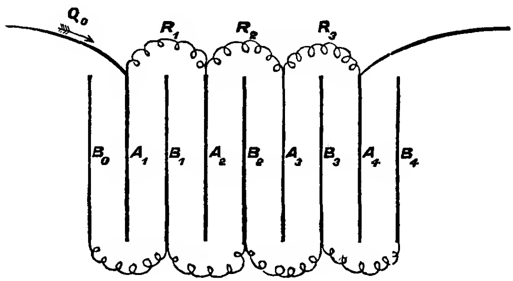


Fig. 26.

Let $A_1$, $A_2$, &c. be connected in series by connexions of resistances $R$, and let a current be passed along this series from left to right.

Let us first suppose the plates $B_0$, $B_1$, $B_2$, each insulated and free from charge. Then the total quantity of electricity on each of the plates $B$ must remain zero, and since the electricity on the plates $A$ is in each case equal and opposite to that of the opposed surface they will not be electrified, and no alteration of the current will be observed.

But let the plates $B$ be all connected together, or let each be connected with the earth. Then, since the potential of $A_1$ is positive, while that of the plates $B$ is zero, $A_1$ will be positively electrified and $B_1$ negatively.

If $P_1$, $P_2$, &c. are the potentials of the plates $A_1$, $A_2$, &c., and $C$ the capacity of each, and if we suppose that a quantity of electricity equal to $Q_0$ passes through the wire on the left, $Q_1$ through the connexion $R_1$, and so on, then the quantity which exists on the plate $A_1$ is $Q_0 - Q_1$, and we have

$$Q_0 - Q_1 = CP_1.$$

Similarly $$Q_1 - Q_2 = CP_2,$$

and so on.

But by Ohm's Law we have

$$P_1 - P_2 = R_1 \frac{dQ_1}{dt},$$

$$P_2 - P_3 = R_2 \frac{dQ_2}{dt}.$$

We have supposed the values of $C$ the same for each plate, if we suppose those of $R$ the same for each wire, we shall have a series of equations of the form

$$Q_0 - 2Q_1 + Q_2 = RC\frac{dQ_1}{dt},$$

$$Q_1 - 2Q_2 + Q_3 = RC\frac{dQ_2}{dt}.$$

If there are $n$ quantities of electricity to be determined, and if either the total electromotive force, or some other equivalent condition be given, the differential equation for determining any one of them will be linear and of the $n$th order.

By an apparatus arranged in this way, Mr. Varley succeeded in imitating the electrical action of a cable 12,000 miles long.

When an electromotive force is made to act along the wire on the left hand, the electricity which flows into the system is at first principally occupied in charging the different condensers beginning with $A_1$, and only a very small fraction of the current appears at the right hand till a considerable time has elapsed. If galvanometers be placed in circuit at $R_1$, $R_2$, &c. they will be affected by the current one after another, the interval between the times of equal indications being greater as we proceed to the right.

332.] In the case of a telegraph cable the conducting wire is separated from conductors outside by a cylindrical sheath of gutta-percha, or other insulating material. Each portion of the cable thus becomes a condenser, the outer surface of which is always at potential zero. Hence, in a given portion of the cable, the quantity of free electricity at the surface of the conducting wire is equal to the product of the potential into the capacity of the portion of the cable considered as a condenser.

If $a_1$, $a_2$ are the outer and inner radii of the insulating sheath, and if $K$ is its specific dielectric capacity, the capacity of unit of length of the cable is, by Art. 126,

$$c = \frac{K}{2\log\frac{a_1}{a_2}}. \tag{1}$$

Let $v$ be the potential at any point of the wire, which we may consider as the same at every part of the same section.

Let $Q$ be the total quantity of electricity which has passed through that section since the beginning of the current. Then the quantity which at the time $t$ exists between sections at $x$ and at $x+\delta x$, is

$$Q-\left(Q+\frac{dQ}{dx}\delta x\right), \quad \text{or} \quad -\frac{dQ}{dx}\delta x,$$

and this is, by what we have said, equal to $cv\delta x$.

Hence
$$cv = -\frac{dQ}{dx}. \tag{2}$$

Again, the electromotive force at any section is $-\frac{dv}{dx}$, and by Ohm's Law,

$$-\frac{dv}{dx} = k\frac{dQ}{dt}, \tag{3}$$

where $k$ is the resistance of unit of length of the conductor, and $\frac{dQ}{dt}$ is the strength of the current. Eliminating $Q$ between (2) and (3), we find

$$ck\frac{dv}{dt} = \frac{d^2v}{dx^2}. \tag{4}$$

This is the partial differential equation which must be solved in order to obtain the potential at any instant at any point of the cable. It is identical with that which Fourier gives to determine the temperature at any point of a stratum through which heat is flowing in a direction normal to the stratum. In the case of heat $c$ represents the capacity of unit of volume, or what Fourier denotes by $CD$, and $k$ represents the reciprocal of the conductivity.

If the sheath is not a perfect insulator, and if $k_1$ is the resistance of unit of length of the sheath to conduction through it in a radial direction, then if $\rho_1$ is the specific resistance of the insulating material, it is easy to shew that

$$k_1 = \frac{1}{2\pi}\rho_1 \log_e \frac{a_1}{a_2}. \tag{5}$$

The equation (2) will no longer be true, for the electricity is expended not only in charging the wire to the extent represented by $cv$, but in escaping at a rate represented by $v/k_1$. Hence the rate of expenditure of electricity will be

$$-\frac{d^2 Q}{dx\,dt} = c\frac{dv}{dt} + \frac{1}{k_1}v, \tag{6}$$

whence, by comparison with (3), we get

$$ck\frac{dv}{dt} = \frac{d^2 v}{dx^2} - \frac{k}{k_1}v, \tag{7}$$

and this is the equation of conduction of heat in a rod or ring as given by Fourier *.

333.] If we had supposed that a body when raised to a high potential becomes electrified throughout its substance as if electricity were compressed into it, we should have arrived at equations of this very form. It is remarkable that Ohm himself, misled by the analogy between electricity and heat, entertained an opinion of this kind, and was thus, by means of an erroneous opinion, led to employ the equations of Fourier to express the true laws of conduction of electricity through a long wire, long before the real reason of the appropriateness of these equations had been suspected.

### *Mechanical Illustration of the Properties of a Dielectric.*

334.] Five tubes of equal sectional area $A$, $B$, $C$, $D$ and $P$ are arranged in circuit as in the figure. $A$, $B$, $C$ and $D$ are vertical and equal, and $P$ is horizontal.

* *Théorie de la Chaleur*, Art. 105.

The lower halves of $A$, $B$, $C$, $D$ are filled with mercury, their upper halves and the horizontal tube $P$ are filled with water.

A tube with a stopcock $Q$ connects the lower part of $A$ and $B$ with that of $C$ and $D$, and a piston $P$ is made to slide in the horizontal tube.

Let us begin by supposing that the level of the mercury in the four tubes is the same, and that it is indicated by $A_0$, $B_0$, $C_0$, $D_0$, that the piston is at $P_0$, and that the stopcock $Q$ is shut.

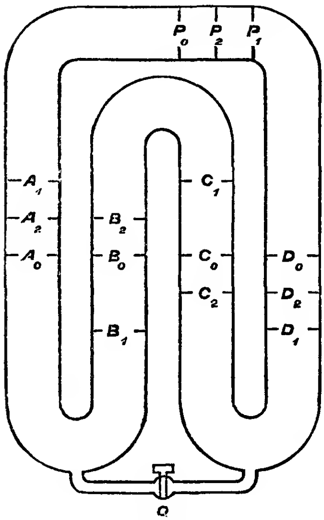

Fig. 27.

Now let the piston be moved from $P_0$ to $P_1$, a distance $a$. Then since the sections of all the tubes are equal, the level of the mercury in $A$ and $C$ will rise a distance $a$, or to $A_1$ and $C_1$, and the mercury in $B$ and $D$ will sink an equal distance $a$, or to $B_1$ and $D_1$.

The difference of pressure on the two sides of the piston will be represented by $4a$.

This arrangement may serve to represent the state of a dielectric acted on by an electromotive force $4a$.

The excess of water in the tube $D$ may be taken to represent a positive charge of electricity on one side of the dielectric, and the excess of mercury in the tube $A$ may represent the negative charge on the other side. The excess of pressure in the tube $P$ on the side of the piston next $D$ will then represent the excess of potential on the positive side of the dielectric.

If the piston is free to move it will move back to $P_0$ and be in equilibrium there. This represents the complete discharge of the dielectric.

During the discharge there is a reversed motion of the liquids throughout the whole tube, and this represents that change of electric displacement which we have supposed to take place in a dielectric.

I have supposed every part of the system of tubes filled with incompressible liquids, in order to represent the property of all electric displacement that there is no real accumulation of electricity at any place.

Let us now consider the effect of opening the stopcock $Q$ while the piston $P$ is at $P_1$.

The levels of $A_1$ and $D_1$ will remain unchanged, but those of $B$ and $C$ will become the same, and will coincide with $B_0$ and $C_0$.

The opening of the stopcock $Q$ corresponds to the existence of a part of the dielectric which has a slight conducting power, but which does not extend through the whole dielectric so as to form an open channel.

The charges on the opposite sides of the dielectric remain insulated, but their difference of potential diminishes.

In fact, the difference of pressure on the two sides of the piston sinks from $4a$ to $2a$ during the passage of the fluid through $Q$.

If we now shut the stopcock $Q$ and allow the piston $P$ to move freely, it will come to equilibrium at a point $P_2$, and the discharge will be apparently only half of the charge.

The level of the mercury in $A$ and $B$ will be $\frac{1}{2}a$ above its original level, and the level in the tubes $C$ and $D$ will be $\frac{1}{2}a$ below its original level. This is indicated by the levels $A_2$, $B_2$, $C_2$, $D_2$.

If the piston is now fixed and the stopcock opened, mercury will flow from $B$ to $C$ till the level in the two tubes is again at $B_0$ and $C_0$. There will then be a difference of pressure $= a$ on the two sides of the piston $P$. If the stopcock is then closed and the piston $P$ left free to move, it will again come to equilibrium at a point $P_3$, half way between $P_2$ and $P_0$. This corresponds to the residual charge which is observed when a charged dielectric is first discharged and then left to itself. It gradually recovers part of its charge, and if this is again discharged a third charge is formed, the successive charges diminishing in quantity. In the case of the illustrative experiment each charge is half of the preceding, and the discharges, which are $\frac{1}{2}$, $\frac{1}{4}$, &c. of the original charge, form a series whose sum is equal to the original charge.

If, instead of opening and closing the stopcock, we had allowed it to remain nearly, but not quite, closed during the whole experiment, we should have had a case resembling that of the electrification of a dielectric which is a perfect insulator and yet exhibits the phenomenon called 'electric absorption.'

To represent the case in which there is true conduction through the dielectric we must either make the piston leaky, or we must establish a communication between the top of the tube $A$ and the top of the tube $D$.

In this way we may construct a mechanical illustration of the properties of a dielectric of any kind, in which the two electricities are represented by two real fluids, and the electric potential is represented by fluid pressure. Charge and discharge are represented by the motion of the piston $P$, and electromotive force by the resultant force on the piston.

# CHAPTER XI.

## THE MEASUREMENT OF ELECTRIC RESISTANCE.

335.] In the present state of electrical science, the determination of the electric resistance of a conductor may be considered as the cardinal operation in electricity, in the same sense that the determination of weight is the cardinal operation in chemistry.

The reason of this is that the determination in absolute measure of other electrical magnitudes, such as quantities of electricity, electromotive forces, currents, &c., requires in each case a complicated series of operations, involving generally observations of time, measurements of distances, and determinations of moments of inertia, and these operations, or at least some of them, must be repeated for every new determination, because it is impossible to preserve a unit of electricity, or of electromotive force, or of current, in an unchangeable state, so as to be available for direct comparison.

But when the electric resistance of a properly shaped conductor of a properly chosen material has been once determined, it is found that it always remains the same for the same temperature, so that the conductor may be used as a standard of resistance, with which that of other conductors can be compared, and the comparison of two resistances is an operation which admits of extreme accuracy.

When the unit of electrical resistance has been fixed on, material copies of this unit, in the form of 'Resistance Coils,' are prepared for the use of electricians, so that in every part of the world electrical resistances may be expressed in terms of the same unit. These unit resistance coils are at present the only examples of material electric standards which can be preserved, copied, and used for the purpose of measurement *. Measures of electrical capacity, which are also of great

* {The Clark's cell as a standard of Electromotive Force may now claim to be an exception to this statement.}

importance, are still defective, on account of the disturbing influence of electric absorption.

336.] The unit of resistance may be an entirely arbitrary one, as in the case of Jacobi's Etalon, which was a certain copper wire of 22·4932 grammes weight, 7·61975 metres length, and 0·667 millimetres diameter. Copies of this have been made by Leyser of Leipsig, and are to be found in different places.

According to another method the unit may be defined as the resistance of a portion of a definite substance of definite dimensions. Thus, Siemen's unit is defined as the resistance of a column of mercury of one metre in length, and one square millimetre in section, at the temperature of 0°C.

337.] Finally, the unit may be defined with reference to the electrostatic or the electromagnetic system of units. In practice the electromagnetic system is used in all telegraphic operations, and therefore the only systematic units actually in use are those of this system.

In the electromagnetic system, as we shall shew at the proper place, a resistance is a quantity of the dimensions of a velocity, and may therefore be expressed as a velocity. See Art. 628.

338.] The first actual measurements on this system were made by Weber, who employed as his unit one millimetre per second. Sir W. Thomson afterwards used one foot per second as a unit, but a large number of electricians have now agreed to use the unit of the British Association, which professes to represent a resistance which, expressed as a velocity, is ten millions of metres per second. The magnitude of this unit is more convenient than that of Weber's unit, which is too small. It is sometimes referred to as the B.A. unit, but in order to connect it with the name of the discoverer of the laws of resistance, it is called the Ohm.

339.] To recollect its value in absolute measure it is useful to know that ten millions of metres is professedly the distance from the pole to the equator, measured along the meridian of Paris. A body, therefore, which in one second travels along a meridian from the pole to the equator would have a velocity which, on the electromagnetic system, is professedly represented by an Ohm.

I say professedly, because, if more accurate researches should prove that the Ohm, as constructed from the British Association's material standards, is not really represented by this velocity, electricians would not alter their standards, but would apply a correction *. In the same way the metre is professedly one ten-millionth of a certain quadrantal arc, but though this is found not to be exactly true, the length of the metre has not been altered, but the dimensions of the earth are expressed by a less simple number.

According to the system of the British Association, the absolute value of the unit is *originally chosen* so as to represent as nearly as possible a quantity derived from the electromagnetic absolute system.

340.] When a material unit representing this abstract quantity has been made, other standards are constructed by copying this unit, a process capable of extreme accuracy—of much greater accuracy than, for instance, the copying of foot-rules from a standard foot.

These copies, made of the most permanent materials, are distributed over all parts of the world, so that it is not likely that any difficulty will be found in obtaining copies of them if the original standards should be lost.

But such units as that of Siemens can without very great labour be reconstructed with considerable accuracy, so that as the relation of the Ohm to Siemens unit is known, the Ohm can be reproduced even without having a standard to copy, though the labour is much greater and the accuracy much less than by the method of copying.

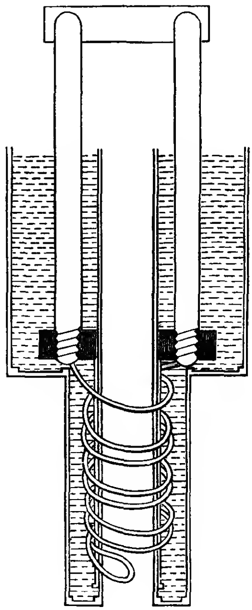

Fig. 28.

Finally, the Ohm may be reproduced by the electromagnetic method by which it was originally determined. This method, which is considerably more laborious than the determination of a foot from

* {Lord Rayleigh's and Mrs. Sidgwick's experiments have shewn that the British Association Unit is only ·9867 earth quadrants a second, it is thus smaller than was intended by nearly 1·3 per cent. The Congress of Electricians at Paris in 1884 adopted a new unit of resistance, the 'Legal Ohm,' which is defined as the resistance at 0°C. of a column of mercury 106 centimetres long and 1 square millimetre in cross section.}

the seconds pendulum, is probably inferior in accuracy to that last mentioned. On the other hand, the determination of the electromagnetic unit in terms of the Ohm with an amount of accuracy corresponding to the progress of electrical science, is a most important physical research and well worthy of being repeated.

The actual resistance coils constructed to represent the Ohm were made of an alloy of two parts of silver and one of platinum in the form of wires from ·5 millimetres to ·8 millimetres diameter, and from one to two metres in length. These wires were soldered to stout copper electrodes. The wire itself was covered with two layers of silk, imbedded in solid paraffin, and enclosed in a thin brass case, so that it can be easily brought to a temperature at which its resistance is accurately one Ohm. This temperature is marked on the insulating support of the coil. (See Fig. 28.)

### *On the Forms of Resistance Coils.*

341.] A Resistance Coil is a conductor capable of being easily placed in the voltaic circuit, so as to introduce into the circuit a known resistance.

The electrodes or ends of the coil must be such that no appreciable error may arise from the mode of making the connexions. For resistances of considerable magnitude it is sufficient that the electrodes should be made of stout copper wires or rods well amalgamated with mercury at the ends, and that the ends should be made to press on flat amalgamated copper surfaces placed in mercury cups.

For very great resistances it is sufficient that the electrodes should be thick pieces of brass, and that the connexions should be made by inserting a wedge of brass or copper into the interval between them. This method is found very convenient.

The resistance coil itself consists of a wire well covered with silk, the ends of which are soldered permanently to the electrodes.

The coil must be so arranged that its temperature may be easily observed. For this purpose the wire is coiled on a tube and covered with another tube, so that it may be placed in a vessel of water, and that the water may have access to the inside and the outside of the coil.

To avoid the electromagnetic effects of the current in the coil the wire is first doubled back on itself and then coiled on the tube, so that at every part of the coil there are equal and opposite currents in the adjacent parts of the wire.

When it is desired to keep two coils at the same temperature the wires are sometimes placed side by side and coiled up together. This method is especially useful when it is more important to secure equality of resistance than to know the absolute value of the resistance, as in the case of the equal arms of Wheatstone's Bridge (Art. 347).

When measurements of resistance were first attempted, a resistance coil, consisting of an uncovered wire coiled in a spiral groove round a cylinder of insulating material, was much used. It was called a Rheostat. The accuracy with which it was found possible to compare resistances was soon found to be inconsistent with the use of any instrument in which the contacts are not more perfect than can be obtained in the rheostat. The rheostat, however, is still used for adjusting the resistance where accurate measurement is not required.

Resistance coils are generally made of those metals whose resistance is greatest and which vary least with temperature. German silver fulfils these conditions very well, but some specimens are found to change their properties during the lapse of years. Hence, for standard coils, several pure metals, and also an alloy of platinum and silver, have been employed, and the relative resistance of these during several years has been found constant up to the limits of modern accuracy.

342.] For very great resistances, such as several millions of Ohms, the wire must be either very long or very thin, and the construction of the coil is expensive and difficult. Hence tellurium and selenium have been proposed as materials for constructing standards of great resistance. A very ingenious and easy method of construction has been lately proposed by Phillips *. On a piece of ebonite or ground glass a fine pencil-line is drawn. The ends of this filament of plumbago are connected to metallic electrodes, and the whole is then covered with insulating varnish. If it should be found that the resistance of such a pencil-line remains constant, this will be the best method of obtaining a resistance of several millions of Ohms.

* *Phil. Mag.*, July, 1870.

343.] There are various arrangements by which resistance coils may be easily introduced into a circuit.

For instance, a series of coils of which the resistances are 1, 2, 4, 8, 16, &c., arranged according to the powers of 2, may be placed in a box in series.

The electrodes consist of stout brass plates, so arranged on the outside of the box that by inserting a brass plug or wedge between two of them as a shunt, the resistance of the corresponding coil may be put out of the circuit. This arrangement was introduced by Siemens.

Each interval between the electrodes is marked with the resistance of the corresponding coil, so that if we wish to make

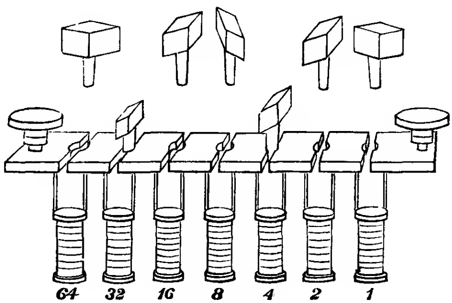


Fig. 29.

the resistance in the box equal to 107 we express 107 in the binary scale as $64+32+8+2+1$ or 1101011. We then take the plugs out of the holes corresponding to 64, 32, 8, 2 and 1, and leave the plugs in 16 and 4.

This method, founded on the binary scale, is that in which the smallest number of separate coils is needed, and it is also that which can be most readily tested. For if we have another coil equal to 1 we can test the quality of 1 and $1'$, then that of $1+1'$ and 2, then that of $1+1'+2$ and 4, and so on.

The only disadvantage of the arrangement is that it requires a familiarity with the binary scale of notation, which is not generally possessed by those accustomed to express every number in the decimal scale.

344.] A box of resistance coils may be arranged in a different way for the purpose of measuring conductivities instead of resistances.

The coils are placed so that one end of each is connected with a long thick piece of metal which forms one electrode of the box, and the other end is connected with a stout piece of brass plate as in the former case.

The other electrode of the box is a long brass plate, such that by inserting brass plugs between it and the electrodes of the coils it may be connected to the first electrode through any given set of coils. The conductivity of the box is then the sum of the conductivities of the coils.

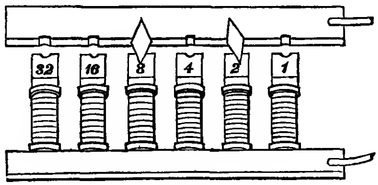

Fig. 30.

In the figure, in which the resistances of the coils are 1, 2, 4, &c., and the plugs are inserted at 2 and 8, the conductivity of the box is $\frac{1}{2}+\frac{1}{8}=\frac{5}{8}$, and the resistance of the box is therefore $\frac{8}{5}$ or 1·6.

This method of combining resistance coils for the measurement of fractional resistances was introduced by Sir W. Thomson under the name of the method of multiple arcs. See Art. 276.

*On the Comparison of Resistances.*

345.] If $E$ is the electromotive force of a battery, and $R$ the resistance of the battery and its connexions, including the galvanometer used in measuring the current, and if the strength of the current is $I$ when the battery connexions are closed, and $I_1$, $I_2$ when additional resistances $r_1$, $r_2$ are introduced into the circuit, then, by Ohm's Law,

$$E = IR = I_1(R+r_1) = I_2(R+r_2).$$

Eliminating $E$, the electromotive force of the battery, and $R$ the resistance of the battery and its connexions, we get Ohm's formula

$$\frac{r_1}{r_2} = \frac{(I-I_1)I_2}{(I-I_2)I_1}.$$

This method requires a measurement of the ratios of $I$, $I_1$ and $I_2$, and this implies a galvanometer graduated for absolute measurements.

If the resistances $r_1$ and $r_2$ are equal, then $I_1$ and $I_2$ are equal, and we can test the equality of currents by a galvanometer which is not capable of determining their ratios.

But this is rather to be taken as an example of a faulty method than as a practical method of determining resistance. The electromotive force $E$ cánnot be maintained rigorously constant, and the internal resistance of the battery is also exceedingly variable, so that any methods in which these are assumed to be even for a short time constant are not to be depended on.

346.] The comparison of resistances can be made with extreme accuracy by either of two methods, in which the result is independent of variations of $R$ and $E$.

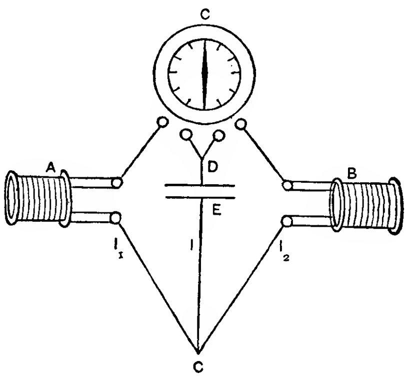


Fig. 31.

The first of these methods depends on the use of the differential galvanometer, an instrument in which there are two coils, the currents in which are independent of each other, so that when the currents are made to flow in opposite directions they act in opposite directions on the needle, and when the ratio of these currents is that of $m$ to $n$ they have no resultant effect on the galvanometer needle.

Let $I_1$, $I_2$ be the currents through the two coils of the galvanometer, then the deflexion of the needle may be written

$$\delta = mI_1 - nI_2.$$

Now let the battery current $I$ be divided between the coils of the galvanometer, and let resistances $A$ and $B$ be introduced into the first and second coils respectively. Let the remainder of the resistances of the coils and their connexions be $\alpha$ and $\beta$ respectively, and let the resistance of the battery and its connexions between $C$ and $D$ be $r$, and its electromotive force $E$.

Then we find, by Ohm's Law, for the difference of potentials between $C$ and $D$,

$$I_1(A+a) = I_2(B+\beta) = E - Ir,$$

and since

$$I_1 + I_2 = I,$$

$$I_1 = E\frac{B+\beta}{D}, \qquad I_2 = E\frac{A+a}{D}, \qquad I = E\frac{A+a+B+\beta}{D},$$

where

$$D = (A+a)(B+\beta) + r(A+a+B+\beta).$$

The deflexion of the galvanometer needle is therefore

$$\delta = \frac{E}{D}\{m(B+\beta) - n(A+a)\},$$

and if there is no observable deflexion, then we know that the quantity enclosed in brackets cannot differ from zero by more than a certain small quantity, depending on the power of the battery, the suitableness of the arrangement, the delicacy of the galvanometer, and the accuracy of the observer.

Suppose that $B$ has been adjusted so that there is no apparent deflexion.

Now let another conductor $A'$ be substituted for $A$, and let $A'$ be adjusted till there is no apparent deflexion. Then evidently to a first approximation $A' = A$.

To ascertain the degree of accuracy of this estimate, let the altered quantities in the second observation be accented, then

$$m(B+\beta) - n(A+a) = \frac{D}{E}\delta,$$

$$m(B+\beta) - n(A'+a) = \frac{D'}{E'}\delta'.$$

Hence

$$n(A'-A) = \frac{D}{E}\delta - \frac{D'}{E'}\delta'.$$

If $\delta$ and $\delta'$, instead of being both apparently zero, had been only observed to be equal, then, unless we also could assert that $E = E'$, the right-hand side of the equation might not be zero. In fact, the method would be a mere modification of that already described.

The merit of the method consists in the fact that the thing observed is the absence of any deflexion, or in other words, the method is a Null method, one in which the non-existence of a force is asserted from an observation in which the force, if it had been different from zero by more than a certain small amount, would have produced an observable effect.

Null methods are of great value where they can be employed, but they can only be employed where we can cause two equal and opposite quantities of the same kind to enter into the experiment together.

In the case before us both $\delta$ and $\delta'$ are quantities too small to be observed, and therefore any change in the value of $E$ will not affect the accuracy of the result.

The actual degree of accuracy of this method might be ascertained by making a number of observations in each of which $A'$ is separately adjusted, and comparing the result of each observation with the mean of the whole series.

But by putting $A'$ out of adjustment by a known quantity, as, for instance, by inserting at $A$ or at $B$ an additional resistance equal to a hundredth part of $A$ or of $B$, and then observing the resulting deviation of the galvanometer needle we can estimate the number of degrees corresponding to an error of one per cent. To find the actual degree of precision we must estimate the smallest deflexion which could not escape observation, and compare it with the deflexion due to an error of one per cent.

*If the comparison is to be made between $A$ and $B$, and if the positions of $A$ and $B$ are exchanged, then the second equation becomes

$$m(A+\beta)-n(B+\alpha)=\frac{D'}{E}\delta',$$

whence

$$(m+n)(B-A)=\frac{D}{E}\delta-\frac{D'}{E'}\delta'.$$

If $m$ and $n$, $A$ and $B$, $\alpha$ and $\beta$, $E$ and $E'$ are approximately equal, then

$$B-A=\frac{1}{2nE}(A+\alpha)(A+\alpha+2r)(\delta-\delta').$$

Here $\delta-\delta'$ may be taken to be the smallest observable deflexion of the galvanometer.

If the galvanometer wire be made longer and thinner, retaining the same total mass, then $n$ will vary as the length of the wire and $\alpha$ as the square of the length. Hence there will be a minimum value of $\dfrac{(A+\alpha)(A+\alpha+2r)}{n}$ when

$$\alpha=\tfrac{1}{3}(A+r)\left\{2\sqrt{1-\frac{3}{4}\frac{r^2}{(A+r)^2}}-1\right\}.$$

* This investigation is taken from Weber's treatise on Galvanometry. *Göttingen Transactions*, x. p. 65.

If we suppose $r$, the battery resistance, negligible compared with $A$, this gives $$a = \tfrac{1}{3}A;$$

or, *the resistance of each coil of the galvanometer should be one-third of the resistance to be measured.*

We then find $$B-A = \frac{8}{9}\frac{A^2}{nE}(\delta-\delta').$$

If we allow the current to flow through one only of the coils of the galvanometer, and if the deflexion thereby produced is $\Delta$ (supposing the deflexion strictly proportional to the deflecting force), then

$$\Delta = \frac{nE}{A+a+r} = \frac{3}{4}\frac{nE}{A} \text{ if } r = 0 \text{ and } a = \frac{1}{3}A.$$

Hence $$\frac{B-A}{A} = \frac{2}{3}\frac{\delta-\delta'}{\Delta}.$$

In the differential galvanometer two currents are made to produce equal and opposite effects on the suspended needle. The force with which either current acts on the needle depends not only on the strength of the current, but on the position of the windings of the wire with respect to the needle. Hence, unless the coil is very carefully wound, the ratio of $m$ to $n$ may change when the position of the needle is changed, and therefore it is necessary to determine this ratio by proper methods during each course of experiments if any alteration of the position of the needle is suspected.

The other null method, in which Wheatstone's Bridge is used, requires only an ordinary galvanometer, and the observed zero deflexion of the needle is due, not to the opposing action of two currents, but to the non-existence of a current in the wire. Hence we have not merely a null deflexion, but a null current as the phenomenon observed, and no errors can arise from want of regularity or change of any kind in the coils of the galvanometer. The galvanometer is only required to be sensitive enough to detect the existence and direction of a current, without in any way determining its value or comparing its value with that of another current.

347.] Wheatstone's Bridge consists essentially of six conductors connecting four points. An electromotive force $E$ is made to act between two of the points by means of a voltaic battery introduced between $B$ and $C$. The current between the other two points $O$ and $A$ is measured by a galvanometer.

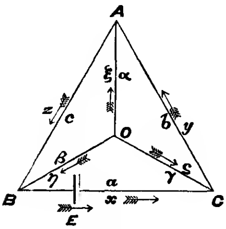

Fig. 32.

Under certain circumstances this current becomes zero. The conductors $BC$ and $OA$ are then said to be *conjugate* to each other, which implies a certain relation between the resistances of the other four conductors, and this relation is made use of in measuring resistances.

If the current in $OA$ is zero, the potential at $O$ must be equal to that at $A$. Now when we know the potentials at $B$ and $C$ we can determine those at $O$ and $A$ by the rule given in Art. 275, provided there is no current in $OA$,

$$O = \frac{B\gamma + C\beta}{\beta + \gamma}, \quad A = \frac{Bb + Cc}{b + c},$$

whence the condition is $\qquad b\beta = c\gamma,$

where $b$, $c$, $\beta$, $\gamma$ are the resistances in $CA$, $AB$, $BO$, and $OC$ respectively.

To determine the degree of accuracy attainable by this method we must ascertain the strength of the current in $OA$ when this condition is not fulfilled exactly.

Let $A$, $B$, $C$ and $O$ be the four points. Let the currents along $BC$, $CA$ and $AB$ be $x$, $y$ and $z$, and the resistances of these conductors $a$, $b$ and $c$. Let the currents along $OA$, $OB$ and $OC$ be $\xi$, $\eta$, $\zeta$, and the resistances $\alpha$, $\beta$ and $\gamma$. Let an electromotive force $E$ act along $BC$. Required the current $\xi$ along $OA$.

Let the potentials at the points $A$, $B$, $C$ and $O$ be denoted by the symbols $A$, $B$, $C$ and $O$. The equations of conduction are

$$\begin{aligned} ax &= B - C + E, & \alpha\xi &= O - A, \\ by &= C - A, & \beta\eta &= O - B, \\ cz &= A - B, & \gamma\zeta &= O - C; \end{aligned}$$

with the equations of continuity

$$\begin{aligned} \xi + y - z &= 0, \\ \eta + z - x &= 0, \\ \zeta + x - y &= 0. \end{aligned}$$

By considering the system as made up of three circuits $OBC$, $OCA$ and $OAB$, in which the currents are $x$, $y$, $z$ respectively, and applying Kirchhoff's rule to each cycle, we eliminate the values of the potentials $O$, $A$, $B$, $C$, and the currents $\xi$, $\eta$, $\zeta$, and obtain the following equations for $x$, $y$ and $z$,

$$\begin{aligned}
(a+\beta+\gamma)x-\gamma y \quad & -\beta z && = E, \\
-\gamma x \quad + (b+\gamma+a)y & -az && = 0, \\
-\beta x \quad -ay \quad & +(c+a+\beta)z && = 0.
\end{aligned}$$

Hence, if we put

$$D = \begin{vmatrix} a+\beta+\gamma & -\gamma & -\beta \\ -\gamma & b+\gamma+a & -a \\ -\beta & -a & c+a+\beta \end{vmatrix},$$

we find $$\xi = \frac{E}{D}(b\beta - c\gamma),$$

and $$x = \frac{E}{D}\{(b+\gamma)(c+\beta)+a(b+c+\beta+\gamma)\}.$$

348.] The value of $D$ may be expressed in the symmetrical form,

$$D = abc + bc(\beta+\gamma) + ca(\gamma+a) \\ + ab(a+\beta) + (a+b+c)(\beta\gamma+\gamma a+a\beta)^*,$$

or, since we suppose the battery in the conductor $a$ and the galvanometer in $a$, we may put $B$ the battery resistance for $a$ and $G$ the galvanometer resistance for $a$. We then find

$$D = BG(b+c+\beta+\gamma) + B(b+\gamma)(c+\beta) \\ + G(b+c)(\beta+\gamma) + bc(\beta+\gamma) + \beta\gamma(b+c).$$

If the electromotive force $E$ were made to act along $OA$, the resistance of $OA$ being still $a$, and if the galvanometer were placed in $BC$, the resistance of $BC$ being still $a$, then the value of $D$ would remain the same, and the current in $BC$ due to the electromotive force $E$ acting along $OA$ would be equal to the current in $OA$ due to the electromotive force $E$ acting in $BC$.

But if we simply disconnect the battery and the galvanometer, and without altering their respective resistances connect the battery to $O$ and $A$ and the galvanometer to $B$ and $C$, then in the value of $D$ we must exchange the values of $B$ and $G$. If $D'$ be the value of $D$ after this exchange, we find

$$\begin{aligned} D - D' &= (G-B)\{(b+c)(\beta+\gamma)-(b+\gamma)(\beta+c)\}, \\ &= (B-G)\{(b-\beta)(c-\gamma)\}. \end{aligned}$$

* {$D$ is the sum of the products of the resistances taken 3 at a time, leaving out the product of any three that meet in a point.}

Let us suppose that the resistance of the galvanometer is greater than that of the battery.

Let us also suppose that in its original position the galvanometer connects the junction of the two conductors of least resistance $\beta$, $\gamma$ with the junction of the two conductors of greatest resistance $b$, $c$, or, in other words, we shall suppose that if the quantities $b$, $c$, $\gamma$, $\beta$ are arranged in order of magnitude, $b$ and $c$ stand together, and $\gamma$ and $\beta$ stand together. Hence the quantities $b-\beta$ and $c-\gamma$ are of the same sign, so that their product is positive, and therefore $D-D'$ is of the same sign as $B-G$.

If therefore the galvanometer is made to connect the junction of the two greatest resistances with that of the two least, and if the galvanometer resistance is greater than that of the battery, then the value of $D$ will be less, and the value of the deflexion of the galvanometer greater, than if the connexions are exchanged.

The rule therefore for obtaining the greatest galvanometer deflexion in a given system is as follows:

Of the two resistances, that of the battery and that of the galvanometer, connect the greater resistance so as to join the two greatest to the two least of the four other resistances.

349.] We shall suppose that we have to determine the ratio of the resistances of the conductors $AB$ and $AC$, and that this is to be done by finding a point $O$ on the conductor $BOC$, such that when the points $A$ and $O$ are connected by a wire, in the course of which a galvanometer is inserted, no sensible deflexion of the galvanometer needle occurs when the battery is made to act between $B$ and $C$.

The conductor $BOC$ may be supposed to be a wire of uniform resistance divided into equal parts, so that the ratio of the resistances of $BO$ and $OC$ may be read off at once.

Instead of the whole conductor being a uniform wire, we may make the part near $O$ of such a wire, and the parts on each side may be coils of any form, the resistances of which are accurately known.

We shall now use a different notation instead of the symmetrical notation with which we commenced.

Let the whole resistance of $BAC$ be $R$.

Let $c = mR$ and $b = (1-m)R$.

Let the whole resistance of $BOC$ be $S$.

Let $\beta = nS$ and $\gamma = (1-n)S$.

The value of $n$ is read off directly, and that of $m$ is deduced from it when there is no sensible deviation of the galvanometer.

Let the resistance of the battery and its connexions be $B$, and that of the galvanometer and its connexions $G$.

We find as before

$$D = G\{BR+BS+RS\} + m(1-m)R^2(B+S) + n(1-n)S^2(B+R) \\ +(m+n-2mn)\,BRS,$$

and if $\xi$ is the current in the galvanometer wire

$$\xi = \frac{ERS}{D}(n-m).$$

In order to obtain the most accurate results we must make the deviation of the needle as great as possible compared with the value of $(n-m)$. This may be done by properly choosing the dimensions of the galvanometer and the standard resistance wire.

It will be shewn, when we come to Galvanometry, Art. 716, that when the form of a galvanometer wire is changed while its mass remains constant, the deviation of the needle for unit current is proportional to the length, but the resistance increases as the square of the length. Hence the maximum deflexion is shewn to occur when the resistance of the galvanometer wire is equal to the constant resistance of the rest of the circuit.

In the present case, if $\delta$ is the deviation,

$$\delta = C\sqrt{G}\,\xi,$$

where $C$ is some constant, and $G$ is the galvanometer resistance which varies as the square of the length of the wire. Hence we find that in the value of $D$, when $\delta$ is a maximum, the part involving $G$ must be made equal to the rest of the expression.

If we also put $m = n$, as is the case if we have made a correct observation, we find the best value of $G$ to be

$$G = n(1-n)(R+S).$$

This result is easily obtained by considering the resistance from $A$ to $O$ through the system, remembering that $BC$, being conjugate to $AO$, has no effect on this resistance.

In the same way we should find that if the total area of the acting surfaces of the battery is given, since in this case $E$ is proportional to $\sqrt{B}$, the most advantageous arrangement of the battery is when

$$B = \frac{RS}{R+S}.$$

Finally, we shall determine the value of $S$ such that a given change in the value of $n$ may produce the greatest galvanometer deflexion. By differentiating the expression for $\xi$ with respect to $S$ we find it is a maximum when

$$S^2 = \frac{BR}{B+R}\left(R + \frac{G}{n(1-n)}\right).$$

If we have a great many determinations of resistance to make in which the actual resistance has nearly the same value, then it may be worth while to prepare a galvanometer and a battery for this purpose. In this case we find that the best arrangement is

$$S = R, \qquad B = \tfrac{1}{2}R, \qquad G = 2n(1-n)R,$$

and if $n = \frac{1}{2}$, $G = \frac{1}{2}R$.

*On the Use of Wheatstone's Bridge.*

350.] We have already explained the general theory of Wheatstone's Bridge, we shall now consider some of its applications.

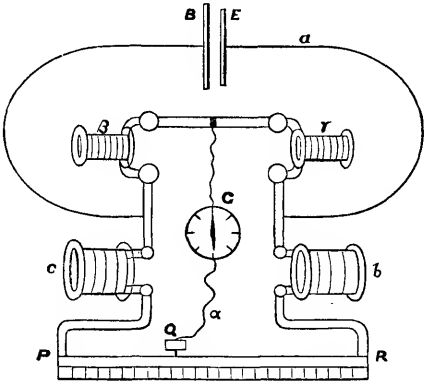


Fig. 33.

The comparison which can be effected with the greatest exactness is that of two equal resistances.

Let us suppose that $\beta$ is a standard resistance coil, and that we wish to adjust $\gamma$ to be equal in resistance to $\beta$.

Two other coils, $b$ and $c$, are prepared which are equal or nearly equal to each other, and the four coils are placed with their electrodes in mercury cups so that the current of the battery is divided between two branches, one consisting of $\beta$ and $\gamma$ and the other of $b$ and $c$. The coils $b$ and $c$ are connected by a wire $PR$, as uniform in its resistance as possible, and furnished with a scale of equal parts.

The galvanometer wire connects the junction of $\beta$ and $\gamma$ with a point $Q$ of the wire $PR$, and the point of contact $Q$ is made to vary till on closing first the battery circuit and then the galvanometer circuit, no deflexion of the galvanometer needle is observed.

The coils $\beta$ and $\gamma$ are then made to change places, and a new position is found for $Q$. If this new position is the same as the old one, then we know that the exchange of $\beta$ and $\gamma$ has produced no change in the proportions of the resistances, and therefore $\gamma$ is rightly adjusted. If $Q$ has to be moved, the direction and amount of the change will indicate the nature and amount of the alteration of the length of the wire of $\gamma$, which will make its resistance equal to that of $\beta$.

If the resistances of the coils $b$ and $c$, each including part of the wire $PR$ up to its zero reading, are equal to that of $b$ and $c$ divisions of the wire respectively, then, if $x$ is the scale reading of $Q$ in the first case, and $y$ that in the second,

$$\frac{c+x}{b-x} = \frac{\beta}{\gamma}, \qquad \frac{c+y}{b-y} = \frac{\gamma}{\beta},$$

whence
$$\frac{\gamma^2}{\beta^2} = 1 + \frac{(b+c)(y-x)}{(c+x)(b-y)}.$$

Since $b-y$ is nearly equal to $c+x$, and both are great with respect to $x$ or $y$, we may write this

$$\frac{\gamma^2}{\beta^2} = 1 + 4\frac{y-x}{b+c},$$

and
$$\gamma = \beta\left(1 + 2\frac{y-x}{b+c}\right).$$

When $\gamma$ is adjusted as well as we can, we substitute for $b$ and $c$ other coils of (say) ten times greater resistance.

The remaining difference between $\beta$ and $\gamma$ will now produce a ten times greater difference in the position of $Q$ than with the original coils $b$ and $c$, and in this way we can continually increase the accuracy of the comparison.

The adjustment by means of the wire with sliding contact piece is more quickly made than by means of a resistance box, and it is capable of continuous variation.

The battery must never be introduced instead of the galvanometer into the wire with a sliding contact, for the passage of a powerful current at the point of contact would injure the surface of the wire. Hence this arrangement is adapted for the case in which the resistance of the galvanometer is greater than that of the battery.

When $\gamma$ the resistance to be measured, $a$ the resistance of the battery, and $\alpha$ the resistance of the galvanometer, are given, the best values of the other resistances have been shewn by Mr. Oliver Heaviside (*Phil. Mag.*, Feb. 1873) to be

$$c = \sqrt{a\alpha},$$

$$b = \sqrt{a\gamma\frac{\alpha+\gamma}{a+\gamma}},$$

$$\beta = \sqrt{\alpha\gamma\frac{a+\gamma}{\alpha+\gamma}}.$$

*On the Measurement of Small Resistances.*

351.] When a short and thick conductor is introduced into a circuit its resistance is so small compared with the resistance occasioned by unavoidable faults in the connexions, such as want of contact or imperfect soldering, that no correct value of the resistance can be deduced from experiments made in the way described above.

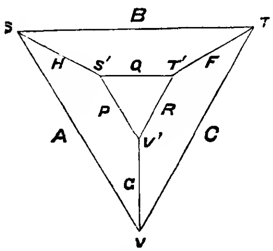


Fig. 34.

The object of such experiments is generally to determine the specific resistance of the substance, and it is resorted to in cases when the substance cannot be obtained in the form of a long thin wire, or when the resistance to transverse as well as to longitudinal conduction has to be measured.

Sir W. Thomson* has described a method applicable to such cases, which we may take as an example of a system of nine conductors.

* *Proc. R. S.*, June 6, 1861.

The most important part of the method consists in measuring the resistance, not of the whole length of the conductor, but of the part between two marks on the conductor at some little distance from its ends.

The resistance which we wish to measure is that experienced by a current whose intensity is uniform in any section of the conductor, and which flows in a direction parallel to its axis. Now close to the extremities, when the current is introduced by means of electrodes, either soldered, amalgamated, or simply pressed to the ends of the conductor, there is generally a want of uniformity in the distribution of the current in the conductor. At a short distance from the extremities the current becomes

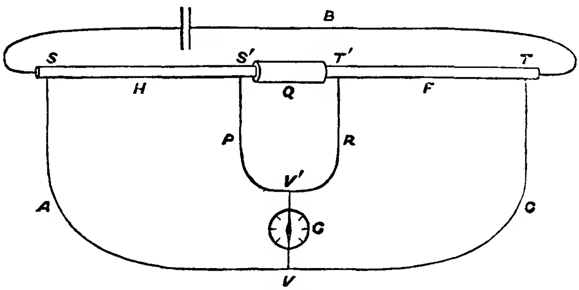


Fig. 35.

sensibly uniform. The student may examine for himself the investigation and the diagrams of Art. 193, where a current is introduced into a strip of metal with parallel sides through one of the sides, but soon becomes itself parallel to the sides.

The resistances of the conductors between certain marks $S$, $S'$ and $T$, $T'$ are to be compared.

The conductors are placed in series, and with connexions as perfectly conducting as possible, in a battery circuit of small resistance. A wire $SVT$ is made to touch the conductors at $S$ and $T$, and $S'V'T'$ is another wire touching them at $S'$ and $T'$.

The galvanometer wire connects the points $V$ and $V'$ of these wires.

The wires $SVT$ and $S'V'T'$ are of resistance so great that the resistance due to imperfect connexion at $S$, $T$, $S'$ or $T'$ may be neglected in comparison with the resistance of the wire, and

$V, V'$ are taken so that the resistances in the branches of either wire leading to the two conductors are nearly in the ratio of the resistances of the two conductors.

Call $H$ and $F$ the resistances of the conductors $SS'$ and $T'T$.
„ $A$ and $C$ those of the branches $SV$ and $VT$.
„ $P$ and $R$ those of the branches $S'V'$ and $V'T'$.
„ $Q$ that of the connecting piece $S'T'$.
„ $B$ that of the battery and its connexions.
„ $G$ that of the galvanometer and its connexions.

The symmetry of the system may be understood from the skeleton diagram. Fig. 34.

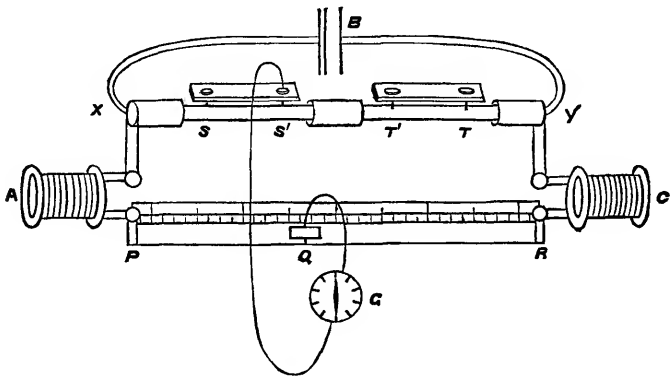


Fig. 36.

The condition that $B$ the battery and $G$ the galvanometer may be conjugate conductors is, in this case *,

$$\frac{F}{C}-\frac{H}{A}+\left(\frac{R}{C}-\frac{P}{A}\right)\frac{Q}{P+Q+R}=0.$$

Now the resistance of the connector $Q$ is as small as we can make it. If it were zero this equation would be reduced to

$$\frac{F}{C}=\frac{H}{A},$$

and the ratio of the resistances of the conductors to be compared would be that of $C$ to $A$, as in Wheatstone's Bridge in the ordinary form.

In the present case the value of $Q$ is small compared with $P$ or with $R$, so that if we select the points $V, V'$ so that the

* {This may easily be deduced by the rule given in the Appendix to Chap. vi.}

ratio of $R$ to $C$ is nearly equal to that of $P$ to $A$, the last term of the equation will vanish, and we shall have

$$F:H::C:A.$$

The success of this method depends in some degree on the perfection of the contact between the wires and the tested conductors at $S$, $S'$, $T'$ and $T$. In the following method, employed by Messrs. Matthiessen and Hockin *, this condition is dispensed with.

352.] The conductors to be tested are arranged in the manner already described, with the connexions as well made as possible, and it is required to compare the resistance between the marks $SS'$ on the first conductor with the resistance between the marks $T'T$ on the second.

Two conducting points or sharp edges are fixed in a piece of insulating material so that the distance between them can be accurately measured. This apparatus is laid on the conductor to be tested, and the points of contact with the conductor are then at a known distance $SS'$. Each of these contact pieces is connected with a mercury cup, into which one electrode of the galvanometer may be plunged.

The rest of the apparatus is arranged, as in Wheatstone's Bridge, with resistance coils or boxes $A$ and $C$, and a wire $PR$ with a sliding contact piece $Q$, to which the other electrode o the galvanometer is connected.

Now let the galvanometer be connected to $S$ and $Q$, and let $A_1$ and $C_1$ be so arranged, and the position of $Q$, (viz. $Q_1$,) so determined, that there is no current in the galvanometer wire.

Then we know that $$\frac{XS}{SY} = \frac{A_1 + PQ_1}{C_1 + Q_1R},$$

where $XS$, $PQ_1$, &c. stand for the resistances in these conductors.

From this we get

$$\frac{XS}{XY} = \frac{A_1 + PQ_1}{A_1 + C_1 + PR}.$$

Now let the electrode of the galvanometer be connected to $S'$, and let resistance be transferred from $C$ to $A$ (by carrying resistance coils from one side to the other) till electric equilibrium of the galvanometer wire can be obtained by placing $Q$ at some

* *Laboratory.* Matthiessen and Hockin on Alloys.

point of the wire, say $Q_2$. Let the values of $C$ and $A$ be now $C_2$ and $A_2$, and let

$$A_2+C_2+PR = A_1+C_1+PR = R.$$

Then we have, as before

$$\frac{XS'}{XY} = \frac{A_2+PQ_2}{R}.$$

Whence
$$\frac{SS'}{XY} = \frac{A_2-A_1+Q_1Q_2}{R}.$$

In the same way, placing the apparatus on the second conductor at $TT'$ and again transferring resistance, we get, when the electrode is in $T'$,

$$\frac{XT'}{XY} = \frac{A_3+PQ_3}{R},$$

and when it is in $T$,

$$\frac{XT}{XY} = \frac{A_4+PQ_4}{R}.$$

Whence
$$\frac{T'T}{XY} = \frac{A_4-A_3+Q_3Q_4}{R}.$$

We can now deduce the ratio of the resistances $SS'$ and $T'T$, for

$$\frac{SS'}{T'T} = \frac{A_2-A_1+Q_1Q_2}{A_4-A_3+Q_3Q_4}.$$

When great accuracy is not required we may dispense with the resistance coils $A$ and $C$, and we then find

$$\frac{SS'}{T'T} = \frac{Q_1Q_2}{Q_3Q_4}.$$

The readings of the position of $Q$ on a wire of a metre in length cannot be depended on to less than a tenth of a millimetre, and the resistance of the wire may vary considerably in different parts owing to inequality of temperature, friction, &c. Hence, when great accuracy is required, coils of considerable resistance are introduced at $A$ and $C$, and the ratios of the resistances of these coils can be determined more accurately than the ratio of the resistances of the parts into which the wire is divided at $Q$.

It will be observed that in this method the accuracy of the determination depends in no degree on the perfection of the contacts at $S$, $S'$ or $T$, $T'$.

This method may be called the differential method of using

Wheatstone's Bridge, since it depends on the comparison of observations separately made.

An essential condition of accuracy in this method is that the resistance of the connexions should continue the same during the course of the four observations required to complete the determination. Hence the series of observations ought always to be repeated in order to detect any change in the resistances *.

*On the Comparison of Great Resistances.*

353.] When the resistances to be measured are very great, the comparison of the potentials at different points of the system may be made by means of a delicate electrometer, such as the Quadrant Electrometer described in Art. 219.

If the conductors whose resistances are to be measured are placed in series, and the same current passed through them by means of a battery of great electromotive force, the difference of the potentials at the extremities of each conductor will be proportional to the resistance of that conductor. Hence, by connecting the electrodes of the electrometer with the extremities, first of one conductor and then of the other, the ratio of their resistances may be determined.

This is the most direct method of determining resistances. It involves the use of an electrometer whose readings may be depended on, and we must also have some guarantee that the current remains constant during the experiment.

Four conductors of great resistance may also be arranged as in Wheatstone's Bridge, and the Bridge itself may consist of the electrodes of an electrometer instead of those of a galvanometer. The advantage of this method is that no permanent current is required to produce the deviation of the electrometer, whereas the galvanometer cannot be deflected unless a current passes through the wire.

354.] When the resistance of a conductor is so great that the current which can be sent through it by any available electromotive force is too small to be directly measured by a galvanometer, a condenser may be used in order to accumulate the electricity for a certain time, and then, by discharging the condenser through a galvanometer, the quantity accumulated

* {For another method of comparing small resistances, see Lord Rayleigh, *Proceedings of the Cambridge Philosophical Society*, vol. v. p. 50.}

may be estimated. This is Messrs. Bright and Clark's method of testing the joints of submarine cables.

355.] But the simplest method of measuring the resistance of such a conductor is to charge a condenser of great capacity and to connect its two surfaces with the electrodes of an electrometer and also with the extremities of the conductor. If $E$ is the difference of potentials as shewn by the electrometer, $S$ the capacity of the condenser, and $Q$ the charge on either surface, $R$ the resistance of the conductor and $x$ the current in it, then, by the theory of condensers,

$$Q = SE.$$

By Ohm's Law, $$E = Rx,$$

and by the definition of a current,

$$x = -\frac{dQ}{dt}.$$

Hence $$-Q = RS\frac{dQ}{dt},$$

and $$Q = Q_0 e^{-\frac{t}{RS}},$$

where $Q_0$ is the charge at first when $t = 0$.

Similarly $$E = E_0 e^{-\frac{t}{RS}}$$

where $E_0$ is the original reading of the electrometer, and $E$ the same after a time $t$. From this we find

$$R = \frac{t}{S\{\log_e E_0 - \log_e E\}},$$

which gives $R$ in absolute measure. In this expression a knowledge of the value of the unit of the electrometer scale is not required.

If $S$, the capacity of the condenser, is given in electrostatic measure as a certain number of metres, then $R$ is also given in electrostatic measure as the reciprocal of a velocity.

If $S$ is given in electromagnetic measure its dimensions are $\frac{T^2}{L}$, and $R$ is a velocity.

Since the condenser itself is not a perfect insulator it is necessary to make two experiments. In the first we determine the resistance of the condenser itself, $R_0$, and in the second, that of the condenser when the conductor is made to connect its surfaces. Let this be $R'$. Then the resistance, $R$, of the conductor is given by the equation

$$\frac{1}{R} = \frac{1}{R'} - \frac{1}{R_0}.$$

This method has been employed by MM. Siemens.

*Thomson's* Method for the Determination of the Resistance of a Galvanometer.*

356.] An arrangement similar to Wheatstone's Bridge has been employed with advantage by Sir W. Thomson in determining the resistance of the galvanometer when in actual

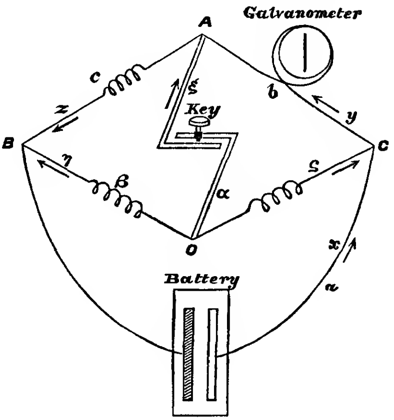


Fig. 37.

use. It was suggested to Sir W. Thomson by Mance's Method. See Art. 357.

Let the battery be placed, as before, between $B$ and $C$ in the figure of Article 347, but let the galvanometer be placed in $CA$ instead of in $OA$. If $b\beta - c\gamma$ is zero, then the conductor $OA$ is conjugate to $BC$, and, as there is no current produced in $OA$ by the battery in $BC$, the strength of the current in any other conductor is independent of the resistance in $OA$. Hence, if the galvanometer is placed in $CA$ its deflexion will remain the same whether the resistance of $OA$ is small or great. We therefore observe whether the deflexion of the galvanometer remains the same when $O$ and $A$ are joined by a conductor

* *Proc. R. S.*, Jan. 19, 1871.

of small resistance, as when this connexion is broken, and if, by properly adjusting the resistances of the conductors, we obtain this result, we know that the resistance of the galvanometer is

$$b = \frac{c\gamma}{\beta},$$

where $c$, $\gamma$, and $\beta$ are resistance coils of known resistance.

It will be observed that though this is not a null method, in the sense of there being no current in the galvanometer, it is so in the sense of the fact observed being the negative one, that the deflexion of the galvanometer is not changed when a certain contact is made. An observation of this kind is of greater value than an observation of the equality of two different deflexions of the same galvanometer, for in the latter case there is time for alteration in the strength of the battery or the sensitiveness of the galvanometer, whereas when the deflexion remains constant, in spite of certain changes which we can repeat at pleasure, we are sure that the current is quite independent of these changes.

The determination of the resistance of the coil of a galvanometer can easily be effected in the ordinary way of using Wheatstone's Bridge by placing another galvanometer in $OA$. By the method now described the galvanometer itself is employed to measure its own resistance.

*Mance's* Method of Determining the Resistance of a Battery.*

357.] The measurement of the resistance of a battery when in action is of a much higher order of difficulty, since the resistance of the battery is found to change considerably for some time after the strength of the current through it is changed. In many of the methods commonly used to measure the resistance of a battery such alterations of the strength of the current through it occur in the course of the operations, and therefore the results are rendered doubtful.

In Mance's method, which is free from this objection, the battery is placed in $BC$ and the galvanometer in $CA$. The connexion between $O$ and $B$ is then alternately made and broken.

Now the deflexion of the galvanometer needle will remain unaltered, however the resistance in $OB$ be changed, provided that $OB$ and $AC$ are conjugate. This may be regarded as a particular

* *Proc. R. S.*, Jan. 19, 1871.

case of the result proved in Art. 347, or may be seen directly on the elimination of $z$ and $\beta$ from the equations of that article, viz. we then have

$$(a\alpha - c\gamma)\,x + (c\gamma + c\alpha + cb + b\alpha)\,y = E\alpha.$$

If $y$ is independent of $x$, and therefore of $\beta$, we must have $a\alpha = c\gamma$. The resistance of the battery is thus obtained in terms of $c$, $\gamma$, $\alpha$.

When the condition $a\alpha = c\gamma$ is fulfilled, the current $y$ through the galvanometer is given by

$$y = \frac{E\alpha}{cb + \alpha(a+b+c)}, = \frac{E\gamma}{ab + \gamma(a+b+c)}.$$

To test the sensibility of the method let us suppose that the condition $c\gamma = a\alpha$ is nearly, but not accurately, fulfilled,

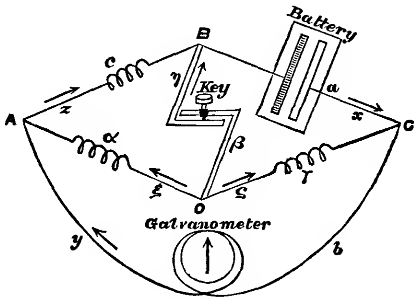


Fig. 38.

and that $y_0$ is the current through the galvanometer when $O$ and $B$ are connected by a conductor of no sensible resistance, and $y_1$ the current when $O$ and $B$ are completely disconnected.

To find these values we must make $\beta$ equal to 0 and to $\infty$ in the general formula for $y$, and compare the results.

The general value for $y$ is

$$\frac{c\gamma + \beta\gamma + \gamma\alpha + \alpha\beta}{D}E,$$

where $D$ denotes the same expression as in Art. 348. Putting $\beta = 0$, we get

$$y_0 = \frac{\gamma E}{ab + \gamma(a+b+c) + \dfrac{c(a\alpha - c\gamma)}{\alpha + c}}$$

$$= y + \frac{c(c\gamma - a\alpha)}{\gamma(c+\alpha)}\frac{y^2}{E} \text{ approximately,}$$

putting $\beta = \infty$, we get

$$y_1 = \frac{E}{a+b+c+\dfrac{ab}{\gamma}-\dfrac{(\alpha a - c\gamma)b}{(\gamma+a)\gamma}},$$

$$= y - \frac{b(c\gamma - \alpha a)}{\gamma(\gamma+a)}\frac{y^2}{E}.$$

From these values we find

$$\frac{y_0 - y_1}{y} = \frac{\alpha}{\gamma}\frac{c\gamma - \alpha a}{(c+a)(a+\gamma)}.$$

The resistance, $c$, of the conductor $AB$ should be equal to $a$, that of the battery; $\alpha$ and $\gamma$ should be equal and as small as possible; and $b$ should be equal to $a+\gamma$.

Since a galvanometer is most sensitive when its deflexion is small, we should bring the needle nearly to zero by means of fixed magnets before making contact between $O$ and $B$.

In this method of measuring the resistance of the battery, the current in the galvanometer is not in any way interfered with during the operation, so that we may ascertain the resistance of the battery for any given strength of current in the galvanometer so as to determine how the strength of the current affects the resistance *.

If $y$ is the current in the galvanometer, the actual current through the battery is $x_0$ with the key down and $x_1$ with the key up, where

$$x_0 = y\left(1 + \frac{b}{\gamma} + \frac{ac}{\gamma(a+c)}\right), \qquad x_1 = y\left(1 + \frac{b}{a+\gamma}\right),$$

the resistance of the battery is

$$a = \frac{c\gamma}{\alpha},$$

and the electromotive force of the battery is

$$E = y\left(b + c + \frac{c}{\alpha}(b+\gamma)\right).$$

The method of Art. 356 for finding the resistance of the galvanometer differs from this only in making and breaking contact

* [In the *Philosophical Magazine* for 1877, vol. i. pp. 515–525, Mr. Oliver Lodge has pointed out as a defect in Mance's method that as the electromotive force of the battery depends upon the current passing through the battery, the deflexion of the galvanometer needle cannot be the same in the two cases when the key is down or up, if the equation $\alpha a = c\gamma$ is true. Mr. Lodge describes a modification of Mance's method which he has employed with success.]

between $O$ and $A$ instead of between $O$ and $B$, and by exchanging $a$ and $\beta$, $a$ and $b$, we obtain for this case

$$\frac{y_0-y_1}{y} = \frac{\beta}{\gamma}\frac{c\gamma-b\beta}{(c+\beta)(\beta+\gamma)}.$$

*On the Comparison of Electromotive Forces.*

358.] The following method of comparing the electromotive forces of voltaic and thermoelectric arrangements, when no current passes through them, requires only a set of resistance coils and a constant battery.

Let the electromotive force $E$ of the battery be greater than that of either of the electromotors to be compared, then, if a

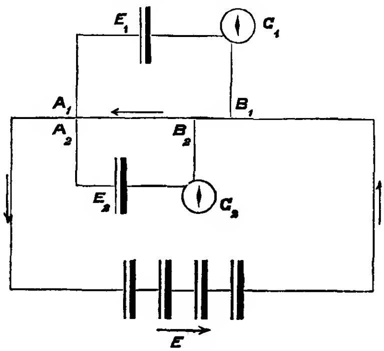


Fig. 39.

sufficient resistance, $R_1$, be interposed between the points $A_1$, $B_1$ of the primary circuit $EB_1A_1E$, the electromotive force from $B_1$ to $A_1$ may be made equal to that of the electromotor $E_1$. If the electrodes of this electromotor are now connected with the points $A_1$, $B_1$ no current will flow through the electromotor. By placing a galvanometer $G_1$ in the circuit of the electromotor $E_1$, and adjusting the resistance between $A_1$ and $B_1$ till the galvanometer $G_1$ indicates no current, we obtain the equation

$$E_1 = R_1 C,$$

where $R_1$ is the resistance between $A_1$ and $B_1$, and $C$ is the strength of the current in the primary circuit.

In the same way, by taking a second electromotor $E_2$ and placing its electrodes at $A_2$ and $B_2$, so that no current is indicated by the galvanometer $G_2$,

$$E_2 = R_2 C,$$

where $R_2$ is the resistance between $A_2$ and $B_2$. If the observations of the galvanometers $G_1$ and $G_2$ are simultaneous, the value of $C$, the current in the primary circuit, is the same in both equations, and we find

$$E_1 : E_2 :: R_1 : R_2 .$$

In this way the electromotive forces of two electromotors may be compared. The absolute electromotive force of an electromotor may be measured either electrostatically by means of the electrometer, or electromagnetically by means of an absolute galvanometer.

This method, in which, at the time of the comparison, there is no current through either of the electromotors, is a modification of Poggendorff's method, and is due to Mr. Latimer Clark, who has deduced the following values of electromotive forces:

| | | | | Concentrated solution of | | Volts. |
|---|---|---|---|---|---|---|
| *Daniell* | I. | Amalgamated Zinc | $H_2SO_4 + 4$ aq. | $Cu\,SO_4$ | Copper | = 1·079 |
| | II. | ,, | $H_2SO_4 + 12$ aq. | $Cu\,SO_4$ | Copper | = 0·978 |
| | III. | ,, | $H_2SO_4 + 12$ aq. | $Cu\,(NO_3)_2$ | Copper | = 1·00 |
| *Bunsen* | I. | ,, | ,, ,, | $H\,NO_3$ | Carbon | = 1·964 |
| | II. | ,, | ,, ,, | sp. g. 1·38 | Carbon | = 1·888 |
| *Grove* | | ,, | $H_2SO_4 + 4$ aq. | $H\,NO_3$ | Platinum | = 1·956 |

*A Volt is an electromotive force equal to* 100,000,000 *units of the centimetre-gramme-second system.*

# CHAPTER XII.

## ON THE ELECTRIC RESISTANCE OF SUBSTANCES.

359.] THERE are three classes in which we may place different substances in relation to the passage of electricity through them.

The first class contains all the metals and their alloys, some sulphurets, and other compounds containing metals, to which we must add carbon in the form of gas-coke, and selenium in the crystalline form.

In all these substances conduction takes place without any decomposition, or alteration of the chemical nature of the substance, either in its interior or where the current enters and leaves the body. In all of them the resistance increases as the temperature rises *.

The second class consists of substances which are called electrolytes, because the current is associated with a decomposition of the substance into two components which appear at the electrodes. As a rule a substance is an electrolyte only when in the liquid form, though certain colloid substances, such as glass at 100° C, which are apparently solid, are electrolytes †. It would appear from the experiments of Sir B. C. Brodie that certain gases are capable of electrolysis by a powerful electromotive force.

In all substances which conduct by electrolysis the resistance diminishes as the temperature rises.

The third class consists of substances the resistance of which is so great that it is only by the most refined methods that the passage of electricity through them can be detected. These are called Dielectrics. To this class belong a considerable number of solid bodies, many of which are electrolytes when melted, some liquids, such as turpentine, naphtha, melted paraffin, &c.,

* {Carbon is an exception to this statement; and Feussner has lately found that the resistance of an alloy of manganese and copper diminishes as the temperature increases.}

† {W. Kohlrausch has shown that the haloid salts of silver conduct electrolytically when solid, Wied. *Ann.* 17. p. 642, 1882.}

and all gases and vapours. Carbon in the form of diamond, and selenium in the amorphous form, belong to this class.

The resistance of this class of bodies is enormous compared with that of the metals. It diminishes as the temperature rises. It is difficult, on account of the great resistance of these substances, to determine whether the feeble current which we can force through them is or is not associated with electrolysis.

*On the Electric Resistance of Metals.*

360.] There is no part of electrical research in which more numerous or more accurate experiments have been made than in the determination of the resistance of metals. It is of the utmost importance in the electric telegraph that the metal of which the wires are made should have the smallest attainable resistance. Measurements of resistance must therefore be made before selecting the materials. When any fault occurs in the line, its position is at once ascertained by measurements of resistance, and these measurements, in which so many persons are now employed, require the use of resistance coils, made of metal the electrical properties of which have been carefully tested.

The electrical properties of metals and their alloys have been studied with great care by MM. Matthiessen, Vogt, and Hockin, and by MM. Siemens, who have done so much to introduce exact electrical measurements into practical work.

It appears from the researches of Dr. Matthiessen, that the effect of temperature on the resistance is nearly the same for a considerable number of the *pure* metals, the resistance at 100°C being to that at 0°C in the ratio of 1·414 to 1, or 100 to 70·7. For pure iron the ratio is 1·6197, and for pure thallium 1·458.

The resistance of metals has been observed by Dr. C. W. Siemens * through a much wider range of temperature, extending from the freezing-point to 350°C, and in certain cases to 1000°C. He finds that the resistance increases as the temperature rises, but that the rate of increase diminishes as the temperature rises. The formula, which he finds to agree very closely both with the resistances observed at low temperatures by Dr. Matthiessen and with his own observations through a range of 1000°C, is

$$r = \alpha T^{\frac{1}{2}} + \beta T + \gamma,$$

* *Proc. R. S.*, April 27, 1871.

where $T$ is the absolute temperature reckoned from $-273°C$, and $\alpha, \beta, \gamma$ are constants. Thus, for

$$\text{Platinum}\ldots\ldots r = 0{\cdot}039369\, T^{\frac{1}{2}} + 0{\cdot}00216407\, T - 0{\cdot}2413\,*,$$

$$\text{Copper}\ldots\ldots\ldots r = 0{\cdot}026577\, T^{\frac{1}{2}} + 0{\cdot}0031443\, T - 0{\cdot}22751,$$

$$\text{Iron}\ldots\ldots\ldots\ldots r = 0{\cdot}072545\, T^{\frac{1}{2}} + 0{\cdot}0038133\, T - 1{\cdot}23971.$$

From data of this kind the temperature of a furnace may be determined by means of an observation of the resistance of a platinum wire placed in the furnace.

Dr. Matthiessen found that when two metals are combined to form an alloy, the resistance of the alloy is in most cases greater than that calculated from the resistance of the component metals and their proportions. In the case of alloys of gold and silver, the resistance of the alloy is greater than that of either pure gold or pure silver, and, within certain limiting proportions of the constituents, it varies very little with a slight alteration of the proportions. For this reason Dr. Matthiessen recommended an alloy of two parts by weight of gold and one of silver as a material for reproducing the unit of resistance.

The effect of change of temperature on electric resistance is generally less in alloys than in pure metals.

Hence ordinary resistance coils are made of German silver, on account of its great resistance and its small variation with temperature.

An alloy of silver and platinum is also used for standard coils.

361.] The electric resistance of some metals changes when the metal is annealed; and until a wire has been tested by being repeatedly raised to a high temperature without permanently altering its resistance, it cannot be relied on as a measure of resistance. Some wires alter in resistance in course of time without having been exposed to changes of temperature. Hence it is important to ascertain the specific resistance of mercury, a metal which being fluid has always the same molecular structure, and which can be easily purified by distillation and treatment

* {Mr. Callendar's recent researches in the Cavendish Laboratory on the Resistance of Platinum have shown that these expressions do not accord with the facts at high temperatures. Siemens' formula for platinum requires the temperature coefficient of the resistance to become constant at high temperatures and equal to ·0021; while the experiments seem to indicate a much slower rate of increase if not a decrease at very high temperatures. H. L. Callendar, 'On the Practical Measurement of Temperature,' *Phil Trans.* 178 A. pp. 161–230.}

with nitric acid. Great care has been bestowed in determining the resistance of this metal by W. and C. F. Siemens, who introduced it as a standard. Their researches have been supplemented by those of Matthiessen and Hockin.

The specific resistance of mercury was deduced from the observed resistance of a tube of length $l$ containing a mass $w$ of mercury, in the following manner.

No glass tube is of exactly equal bore throughout, but if a small quantity of mercury is introduced into the tube and occupies a length $\lambda$ of the tube, the middle point of which is distant $x$ from one end of the tube, then the area $s$ of the section near this point will be $s = \frac{C}{\lambda}$, where $C$ is some constant.

The mass of mercury which fills the whole tube is

$$w = \rho \int s\,dx = \rho C \Sigma \left(\frac{1}{\lambda}\right) \frac{l}{n},$$

where $n$ is the number of points, at equal distances along the tube, where $\lambda$ has been measured, and $\rho$ is the mass of unit of volume.

The resistance of the whole tube is

$$R = \int \frac{r}{s} dx = \frac{r}{C} \Sigma(\lambda) \frac{l}{n},$$

where $r$ is the specific resistance per unit of volume.

Hence
$$wR = r\rho \Sigma(\lambda) \Sigma \left(\frac{1}{\lambda}\right) \frac{l^2}{n^2},$$

and
$$r = \frac{wR}{\rho l^2} \frac{n^2}{\Sigma(\lambda) \Sigma \left(\frac{1}{\lambda}\right)}$$

gives the specific resistance of unit of volume.

To find the resistance of unit of length and unit of mass we must multiply this by the density.

It appears from the experiments of Matthiessen and Hockin that the resistance of a uniform column of mercury of one metre in length, and weighing one gramme at 0°C, is 13·071 B.A. units, whence it follows that if the specific gravity of mercury is 13·595, the resistance of a column of one metre in length and one square millimetre in section is 0·96146 B.A. units.

362.] In the following table $R$ is the resistance in B.A. units of a column one metre long and one gramme weight at 0°C, and $r$ is the resistance in centimetres per second of a cube of one centimetre, according to the experiments of Matthiessen* assuming the B.A. unit to be ·98677 Earth quadrants.

| | Specific gravity. | | *R*. | *r*. | Percentage increment of resistance for 1°C at 20°C. |
|---|---|---|---|---|---|
| Silver . . . . . . | 10·50 | hard drawn | 0·1689 | 1588 | 0·377 |
| Copper . . . . . | 8·95 | hard drawn | 0·1469 | 1620 | 0·388 |
| Gold . . . . . . | 19·27 | hard drawn | 0·4150 | 2125 | 0·365 |
| Lead . . . . . . | 11·391 | pressed | 2·257 | 19584 | 0·387 |
| Mercury†. . . . | 13·595 | liquid | 13·071 | 94874 | 0·072 |
| Gold 2, Silver 1 . | 15·218 | hard or annealed | 1·668 | 18326 | 0·065 |
| Selenium at 100°C | | crystalline form | | $6 \times 10^{13}$ | 1·00 |

*On the Electric Resistance of Electrolytes.*

363.] The measurement of the electric resistance of electrolytes is rendered difficult on account of the polarization of the electrodes, which causes the observed difference of potentials of the metallic electrodes to be greater than the electromotive force which actually produces the current.

This difficulty can be overcome in various ways. In certain cases we can get rid of polarization by using electrodes of proper material, as, for instance, zinc electrodes in a solution of sulphate of zinc. By making the surface of the electrodes very large compared with the section of the part of the electrolyte whose resistance is to be measured, and by using only currents of short duration in opposite directions alternately, we can make the measurements before any considerable intensity of polarization has been excited by the passage of the current.

Finally, by making two different experiments, in one of which the path of the current through the electrolyte is much longer than in the other, and so adjusting the electromotive force that the actual current, and the time during which it flows, are nearly the same in each case, we can eliminate the effect of polarization altogether.

* *Phil. Mag.*, May, 1865.

† {More recent experiments have given a different value for the specific resistance of mercury. The following are recent determinations of the resistance in B.A. units of a column of mercury one metre long and one square millimetre in cross section at 0°C:—

| | |
|---|---|
| Lord Rayleigh and Mrs. Sidgwick, *Phil. Trans.* Part I. 1883 . . | ·95412, |
| Glazebrook and Fitzpatrick, *Phil. Trans.* A. 1888 . . . . . | ·95352, |
| Hutchinson and Wilkes, *Phil. Mag.* (5). 28. 17. 1889 . . . . | ·95341.} |

364.] In the experiments of Dr. Paalzow * the electrodes were in the form of large disks placed in separate flat vessels filled with the electrolyte, and the connexion was made by means of a long siphon filled with the electrolyte and dipping into both vessels. Two such siphons of different lengths were used.

The observed resistances of the electrolyte in these siphons being $R_1$ and $R_2$, the siphons were next filled with mercury, and their resistances when filled with mercury were found to be $R_1'$ and $R_2'$.

The ratio of the resistance of the electrolyte to that of a mass of mercury at 0°C of the same form was then found from the formula

$$\rho = \frac{R_1 - R_2}{R_1' - R_2'}.$$

To deduce from the values of $\rho$ the resistance of a centimetre in length having a section of a square centimetre, we must multiply them by the value of $r$ for mercury at 0°C. See Art. 361.

The results given by Paalzow are as follow :—

*Mixtures of Sulphuric Acid and Water.*

| | Temp. | Resistance compared with mercury. |
|---|---|---|
| $H_2SO_4$ | ......15°C | 96950 |
| $H_2SO_4 + 14\,H_2O$ | ......19°C | 14157 |
| $H_2SO_4 + 13\,H_2O$ | ......22°C | 13310 |
| $H_2SO_4 + 499\,H_2O$ | ......22°C | 184773 |

*Sulphate of Zinc and Water.*

| | | |
|---|---|---|
| $ZnSO_4 + 33\,H_2O$ | ......23°C | 194400 |
| $ZnSO_4 + 24\,H_2O$ | ......23°C | 191000 |
| $ZnSO_4 + 107\,H_2O$ | ......23°C | 354000 |

*Sulphate of Copper and Water.*

| | | |
|---|---|---|
| $CuSO_4 + 45\,H_2O$ | ......22°C | 202410 |
| $CuSO_4 + 105\,H_2O$ | ......22°C | 339341 |

*Sulphate of Magnesium and Water.*

| | | |
|---|---|---|
| $MgSO_4 + 34\,H_2O$ | ......22°C | 199180 |
| $MgSO_4 + 107\,H_2O$ | ......22°C | 324600 |

* *Berlin Monatsbericht*, July, 1868.

*Hydrochloric Acid and Water.*

| | Temp. | Resistance compared with mercury. |
|---|---|---|
| $HCl + 15 H_2O$ | 23°C | 13626 |
| $HCl + 500 H_2O$ | 23°C | 86679 |

365.] MM. F. Kohlrausch and W. A. Nippoldt* have determined the resistance of mixtures of sulphuric acid and water. They used alternating magneto-electric currents, the electromotive force of which varied from $\frac{1}{2}$ to $\frac{1}{74}$ of that of a Grove's cell, and by means of a thermoelectric copper-iron pair they reduced the electromotive force to $\frac{1}{429000}$ of that of a Grove's cell. They found that Ohm's law was applicable to this electrolyte throughout the range of these electromotive forces.

The resistance is a minimum in a mixture containing about one-third of sulphuric acid.

The resistance of electrolytes diminishes as the temperature increases. The percentage increment of conductivity for a rise of 1°C is given in the following table:—

*Resistance of Mixtures of Sulphuric Acid and Water at* 22°C *in terms of Mercury at* 0°C. MM. Kohlrausch and Nippoldt.

| Specific gravity at 18°5. | Percentage of $H_2SO_4$. | Resistance at 22°C (Hg = 1). | Percentage increment of conductivity for 1°C. |
|---|---|---|---|
| 0·9985 | 0·0 | 746300 | 0·47 |
| 1·00 | 0·2 | 465100 | 0·47 |
| 1·0504 | 8·3 | 34530 | 0·653 |
| 1·0989 | 14·2 | 18946 | 0·646 |
| 1·1431 | 20·2 | 14990 | 0·799 |
| 1·2045 | 28·0 | 13133 | 1·317 |
| 1·2631 | 35·2 | 13132 | 1·259 |
| 1·3163 | 41·5 | 14286 | 1·410 |
| 1·3597 | 46·0 | 15762 | 1·674 |
| 1·3994 | 50·4 | 17726 | 1·582 |
| 1·4482 | 55·2 | 20796 | 1·417 |
| 1·5026 | 60·3 | 25574 | 1·794 |

*On the Electrical Resistance of Dielectrics.*

366.] A great number of determinations of the resistance of gutta-percha, and other materials used as insulating media,

* Pogg., *Ann.* cxxxviii. pp. 280, 370, 1869.

in the manufacture of telegraphic cables, have been made in order to ascertain the value of these materials as insulators.

The tests are generally applied to the material after it has been used to cover the conducting wire, the wire being used as one electrode, and the water of a tank, in which the cable is plunged, as the other. Thus the current is made to pass through a cylindrical coating of the insulator of great area and small thickness.

It is found that when the electromotive force begins to act, the current, as indicated by the galvanometer, is by no means constant. The first effect is of course a transient current of considerable intensity, the total quantity of electricity being that required to charge the surfaces of the insulator with the superficial distribution of electricity corresponding to the electromotive force. This first current therefore is a measure not of the conductivity, but of the capacity of the insulating layer.

But even after this current has been allowed to subside the residual current is not constant, and does not indicate the true conductivity of the substance. It is found that the current continues to decrease for at least half an hour, so that a determination of the resistance deduced from the current will give a greater value if a certain time is allowed to elapse than if taken immediately after applying the battery.

Thus, with Hooper's insulating material the apparent resistance at the end of ten minutes was four times, and at the end of nineteen hours twenty-three times that observed at the end of one minute. When the direction of the electromotive force is reversed, the resistance falls as low or lower than at first and then gradually rises.

These phenomena seem to be due to a condition of the gutta-percha, which, for want of a better name, we may call polarization, and which we may compare on the one hand with that of a series of Leyden jars charged by cascade, and, on the other, with Ritter's secondary pile, Art. 271.

If a number of Leyden jars of great capacity are connected in series by means of conductors of great resistance (such as wet cotton threads in the experiments of M. Gaugain), then an electromotive force acting on the series will produce a current, as indicated by a galvanometer, which will gradually diminish till the jars are fully charged.

The apparent resistance of such a series will increase, and if the dielectric of the jars is a perfect insulator it will increase without limit. If the electromotive force be removed and connexion made between the ends of the series, a reverse current will be observed, the total quantity of which, in the case of perfect insulation, will be the same as that of the direct current. Similar effects are observed in the case of the secondary pile, with the difference that the final insulation is not so good, and that the capacity per unit of surface is immensely greater.

In the case of the cable covered with gutta-percha, &c., it is found that after applying the battery for half an hour, and then connecting the wire with the external electrode, a reverse current takes place, which goes on for some time, and gradually reduces the system to its original state.

These phenomena are of the same kind with those indicated by the 'residual discharge' of the Leyden jar, except that the amount of the polarization is much greater in gutta-percha, &c. than in glass.

This state of polarization seems to be a directed property of the material, which requires for its production not only electromotive force, but the passage, by displacement or otherwise, of a considerable quantity of electricity, and this passage requires a considerable time. When the polarized state has been set up, there is an internal electromotive force acting in the substance in the reverse direction, which will continue till it has either produced a reversed current equal in total quantity to the first, or till the state of polarization has quietly subsided by means of true conduction through the substance.

The whole theory of what has been called residual discharge, absorption of electricity, electrification, or polarization, deserves a careful investigation, and will probably lead to important discoveries relating to the internal structure of bodies.

367.] The resistance of the greater number of dielectrics diminishes as the temperature rises.

Thus the resistance of gutta-percha is about twenty times as great at 0°C as at 24°C. Messrs. Bright and Clark have found that the following formula gives results agreeing with their experiments. If $r$ is the resistance of gutta-percha at temperature $T$ centigrade, then the resistance at temperature $T+t$ will be $$R = r \times C^t,$$

where $C$ varies between 0·8878 and 0·9 for different specimens of gutta-percha.

Mr. Hockin has verified the curious fact that it is not until some hours after the gutta-percha has taken its final temperature that the resistance reaches its corresponding value.

The effect of temperature on the resistance of india-rubber is not so great as on that of gutta-percha.

The resistance of gutta-percha increases considerably on the application of pressure.

The resistance, in Ohms, of a cubic metre of various specimens of gutta-percha used in different cables is as follows *.

| Name of Cable. | |
|---|---|
| Red Sea ........................ | $\cdot 267 \times 10^{12}$ to $\cdot 362 \times 10^{12}$ |
| Malta-Alexandria ............ | $1\cdot 23 \times 10^{12}$ |
| Persian Gulf .................. | $1\cdot 80 \times 10^{12}$ |
| Second Atlantic ............... | $3\cdot 42 \times 10^{12}$ |
| Hooper's Persian Gulf Core | $74\cdot 7 \times 10^{12}$ |
| Gutta-percha at 24°C......... | $3\cdot 53 \times 10^{12}$ |

368.] The following table, calculated from the experiments of M. Buff, described in Art. 271, shews the resistance of a cubic metre of glass in Ohms at different temperatures.

| Temperature. | Resistance. |
|---|---|
| 200°C | 227000 |
| 250° | 13900 |
| 300° | 1480 |
| 350° | 1035 |
| 400° | 735 |

369.] Mr. C. F. Varley† has recently investigated the conditions of the current through rarefied gases, and finds that the electromotive force $E$ is equal to a constant $E_0$ together with a part depending on the current according to Ohm's Law, thus

$$E = E_0 + RC.$$

For instance, the electromotive force required to cause the current to begin in a certain tube was that of 323 Daniell's cells, but an electromotive force of 304 cells was just sufficient to maintain the current. The intensity of the current, as measured by the galvanometer, was proportional to the number

* Jenkin's *Cantor Lectures*.

† *Proc. R. S.*, Jan. 12, 1871.

of cells above 304. Thus for 305 cells the deflexion was 2, for 306 it was 4, for 307 it was 6, and so on up to 380, or $304 + 76$ for which the deflexion was 150, or $76 \times 1{\cdot}97$.

From these experiments it appears that there is a kind of polarization of the electrodes, the electromotive force of which is equal to that of 304 Daniell's cells, and that up to this electromotive force the battery is occupied in establishing this state of polarization. When the maximum polarization is established, the excess of electromotive force above that of 304 cells is devoted to maintaining the current according to Ohm's Law.

The law of the current in a rarefied gas is therefore very similar to the law of the current through an electrolyte in which we have to take account of the polarization of the electrodes.

In connexion with this subject we should study Thomson's results, that the electromotive force required to produce a spark in air was found to be proportional not to the distance, but to the distance together with a constant quantity. The electromotive force corresponding to this constant quantity may be regarded as the intensity of polarization of the electrodes.

370.] MM. Wiedemann and Rühlmann have recently* investigated the passage of electricity through gases. The electric current was produced by Holtz's machine, and the discharge took place between spherical electrodes within a metallic vessel containing rarefied gas. The discharge was in general discontinuous, and the interval of time between successive discharges was measured by means of a mirror revolving along with the axis of Holtz's machine. The images of the series of discharges were observed by means of a heliometer with a divided object-glass, which was adjusted till one image of each discharge coincided with the other image of the next discharge. By this method very consistent results were obtained. It was found that the quantity of electricity in each discharge is independent of the strength of the current and of the material of the electrodes, and that it depends on the nature and density of the gas, and on the distance and form of the electrodes.

* *Berichte der Königl. Sächs. Gesellschaft*, Leipzig, Oct. 20, 1871.

These researches confirm the statement of Faraday* that the electric tension (see Art. 48) required to cause a disruptive discharge to begin at the electrified surface of a conductor is a little less when the electrification is negative than when it is positive, but that when a discharge does take place, much more electricity passes at each discharge when it begins at a positive surface. They also tend to support the hypothesis stated in Art. 57, that the stratum of gas condensed on the surface of the electrode plays an important part in the phenomenon, and they indicate that this condensation is greatest at the positive electrode.

* *Exp. Res.*, 1501.

END OF VOL. I.

# PART III.

## MAGNETISM.

## CHAPTER I.

### ELEMENTARY THEORY OF MAGNETISM.

371.] CERTAIN bodies, as, for instance, the iron ore called load-stone, the earth itself, and pieces of steel which have been subjected to certain treatment, are found to possess the following properties, and are called Magnets.

If, near any part of the earth's surface except the Magnetic Poles, a magnet be suspended so as to turn freely about a vertical axis, it will in general tend to set itself in a certain azimuth, and if disturbed from this position it will oscillate about it. An unmagnetized body has no such tendency, but is in equilibrium in all azimuths alike.

372.] It is found that the force which acts on the body tends to cause a certain line in the body, called the Axis of the Magnet, to become parallel to a certain line in space, called the Direction of the Magnetic Force.

Let us suppose the magnet suspended so as to be free to turn in all directions about a fixed point. To eliminate the action of its weight we may suppose this point to be its centre of gravity. Let it come to a position of equilibrium. Mark two points on the magnet, and note their positions in space. Then let the magnet be placed in a new position of equilibrium, and note the positions in space of the two marked points on the magnet.

Since the axis of the magnet coincides with the direction of magnetic force in both positions, we have to find that line in the magnet which occupies the same position in space before and after the motion. It appears, from the theory of the motion of bodies of invariable form, that such a line always exists, and that a motion equivalent to the actual motion might have taken place by simple rotation round this line.

To find the line, join the first and last positions of each of the marked points, and draw planes bisecting these lines at right angles. The intersection of these planes will be the line required, which indicates the direction of the axis of the magnet and the direction of the magnetic force in space.

The method just described is not convenient for the practical determination of these directions. We shall return to this subject when we treat of Magnetic Measurements.

The direction of the magnetic force is found to be different at different parts of the earth's surface. If the end of the axis of the magnet which points in a northerly direction be marked, it has been found that the direction in which it sets itself in general deviates from the true meridian to a considerable extent, and that the marked end points on the whole downwards in the northern hemisphere and upwards in the southern.

The azimuth of the direction of the magnetic force, measured from the true north in a westerly direction, is called the Variation, or the Magnetic Declination. The angle between the direction of the magnetic force and the horizontal plane is called the Magnetic Dip. These two angles determine the direction of the magnetic force, and, when the magnetic intensity is also known, the magnetic force is completely determined. The determination of the values of these three elements at different parts of the earth's surface, the discussion of the manner in which they vary according to the place and time of observation, and the investigation of the causes of the magnetic force and its variations, constitute the science of Terrestrial Magnetism.

373.] Let us now suppose that the axes of several magnets have been determined, and the end of each which points north marked. Then, if one of these magnets be freely suspended and another brought near it, it is found that two marked ends repel each other, that a marked and an unmarked end attract each other, and that two unmarked ends repel each other.

If the magnets are in the form of long rods or wires, uniformly and longitudinally magnetized, (see below, Art. 384,) it is found that the greatest manifestation of force occurs when the end of one magnet is held near the end of the other, and that the phenomena can be accounted for by supposing that like ends of the magnets repel each other, that unlike ends attract each other, and that the intermediate parts of the magnets have no sensible mutual action.

The ends of a long thin magnet are commonly called its Poles. In the case of an indefinitely thin magnet, uniformly magnetized throughout its length, the extremities act as centres of force, and the rest of the magnet appears devoid of magnetic action. In all actual magnets the magnetization deviates from uniformity, so that no single points can be taken as the poles. Coulomb, however, by using long thin rods magnetized with care, succeeded in establishing the law of force between two like magnetic poles * {the medium between them being air}.

*The repulsion between two like magnetic poles is in the straight line joining them, and is numerically equal to the product of the strengths of the poles divided by the square of the distance between them.*

374.] This law, of course, assumes that the strength of each pole is measured in terms of a certain unit, the magnitude of which may be deduced from the terms of the law.

The unit-pole is a pole which points north, and is such that, when placed at unit distance in air from another unit-pole, it repels it with unit of force, the unit of force being defined as in Art. 6. A pole which points south is reckoned negative.

If $m_1$ and $m_2$ are the strengths of two magnetic poles, $l$ the distance between them, and $f$ the force of repulsion, all expressed numerically, then

$$f = \frac{m_1 m_2}{l^2}.$$

But if $[m]$, $[L]$ and $[F]$ be the concrete units of magnetic pole, length and force, then

$$f[F] = \left[\frac{m}{L}\right]^2 \frac{m_1 m_2}{l^2},$$

whence it follows that

$$[m^2] = [L^2 F] = \left[L^2 \frac{ML}{T^2}\right],$$

$$\text{or} \qquad [m] = [L^{\frac{3}{2}} T^{-1} M^{\frac{1}{2}}].$$

* Coulomb, *Mém. de l'Acad.* 1785, p. 603, and in Biot's *Traité de Physique*, tome iii.

The dimensions of the unit-pole are therefore $\frac{3}{2}$ as regards length, $(-1)$ as regards time, and $\frac{1}{2}$ as regards mass. These dimensions are the same as those of the electrostatic unit of electricity, which is specified in exactly the same way in Arts. 41, 42.

375.] The accuracy of this law may be considered to have been established by the experiments of Coulomb with the Torsion Balance, and confirmed by the experiments of Gauss and Weber, and of all observers in magnetic observatories, who are every day making measurements of magnetic quantities, and who obtain results which would be inconsistent with each other if the law of force had been erroneously assumed. It derives additional support from its consistency with the laws of electromagnetic phenomena.

376.] The quantity which we have hitherto called the strength of a pole may also be called a quantity of 'Magnetism,' provided we attribute no properties to 'Magnetism' except those observed in the poles of magnets.

Since the expression of the law of force between given quantities of 'Magnetism' has exactly the same mathematical form as the law of force between quantities of 'Electricity' of equal numerical value, much of the mathematical treatment of magnetism must be similar to that of electricity. There are, however, other properties of magnets which must be borne in mind, and which may throw some light on the electrical properties of bodies.

*Relation between the Poles of a Magnet.*

377.] The quantity of magnetism at one pole of a magnet is always equal and opposite to that at the other, or more generally thus:—

*In every Magnet the total quantity of Magnetism* (reckoned algebraically) *is zero.*

Hence in a field of force which is uniform and parallel throughout the space occupied by the magnet, the force acting on the marked end of the magnet is exactly equal, opposite and parallel to that on the unmarked end, so that the resultant of the forces is a statical couple, tending to place the axis of the magnet in a determinate direction, but not to move the magnet as a whole in any direction.

This may be easily proved by putting the magnet into a small vessel and floating it in water. The vessel will turn in a certain direction, so as to bring the axis of the magnet as near as possible to the direction of the earth's magnetic force, but there will be no motion of the vessel as a whole in any direction; so that there can be no excess of the force towards the north over that towards the south, or the reverse. It may also be shewn from the fact that magnetizing a piece of steel does not alter its weight. It does alter the apparent position of its centre of gravity, causing it in these latitudes to shift along the axis towards the north. The centre of inertia, as determined by the phenomena of rotation, remains unaltered.

378.] If the middle of a long thin magnet be examined, it is found to possess no magnetic properties, but if the magnet be broken at that point, each of the pieces is found to have a magnetic pole at the place of fracture, and this new pole is exactly equal and opposite to the other pole belonging to that piece. It is impossible, either by magnetization, or by breaking magnets, or by any other means, to procure a magnet whose poles are unequal.

If we break the long thin magnet into a number of short pieces we shall obtain a series of short magnets, each of which has poles of nearly the same strength as those of the original long magnet. This multiplication of poles is not necessarily a creation of energy, for we must remember that after breaking the magnet we have to do work to separate the parts, in consequence of their attraction for one another.

379.] Let us now put all the pieces of the magnet together as at first. At each point of junction there will be two poles exactly equal and of opposite kinds, placed in contact, so that their united action on any other pole will be null. The magnet, thus rebuilt, has therefore the same properties as at first, namely two poles, one at each end, equal and opposite to each other, and the part between these poles exhibits no magnetic action.

Since, in this case, we know the long magnet to be made up of little short magnets, and since the phenomena are the same as in the case of the unbroken magnet, we may regard the magnet, even before being broken, as made up of small particles, each of which has two equal and opposite poles. If we suppose all magnets to be made up of such particles, it is evident that since the algebraical quantity of magnetism in each particle is zero, the quantity in the whole magnet will also be zero, or in other words, its poles will be of equal strength but of opposite kind.

*Theory of 'Magnetic Matter.'*

380.] Since the form of the law of magnetic action is identical with that of electric action, the same reasons which can be given for attributing electric phenomena to the action of one 'fluid' or two 'fluids' can also be used in favour of the existence of a magnetic matter, or of two kinds of magnetic matter, fluid or otherwise. In fact, a theory of magnetic matter, if used in a purely mathematical sense, cannot fail to explain the phenomena, provided new laws are freely introduced to account for the actual facts.

One of these new laws must be that the magnetic fluids cannot pass from one molecule or particle of the magnet to another, but that the process of magnetization consists in separating to a certain extent the two fluids within each particle, and causing the one fluid to be more concentrated at one end, and the other fluid to be more concentrated at the other end of the particle. This is the theory of Poisson.

A particle of a magnetizable body is, on this theory, analogous to a small insulated conductor without charge, which on the two-fluid theory contains indefinitely large but exactly equal quantities of the two electricities. When an electromotive force acts on the conductor, it separates the electricities, causing them to become manifest at opposite sides of the conductor. In a similar manner, according to this theory, the magnetizing force causes the two kinds of magnetism, which were originally in a neutralized state, to be separated, and to appear at opposite sides of the magnetized particle.

In certain substances, such as soft iron and those magnetic substances which cannot be permanently magnetized, this magnetic condition, like the electrification of the conductor, disappears when the inducing force is removed *. In other substances, such as hard steel, the magnetic condition is produced with difficulty, and, when produced, remains after the removal of the inducing force.

This is expressed by saying that in the latter case there is a Coercive Force, tending to prevent alteration in the magnetization, which must be overcome before the power of a magnet can be either increased or diminished. In the case of the

* {See foot-note to page 48.}

electrified body this would correspond to a kind of electric resistance, which, unlike the resistance observed in metals, would be equivalent to complete insulation for electromotive forces below a certain value.

This theory of magnetism, like the corresponding theory of electricity, is evidently too large for the facts, and requires to be restricted by artificial conditions. For it not only gives no reason why one body may not differ from another on account of having more of both fluids, but it enables us to say what would be the properties of a body containing an excess of one magnetic fluid. It is true that a reason is given why such a body cannot exist, but this reason is only introduced as an after-thought to explain this particular fact. It does not grow out of the theory.

381.] We must therefore seek for a mode of expression which shall not be capable of expressing too much, and which shall leave room for the introduction of new ideas as these are developed from new facts. This, I think, we shall obtain if we begin by saying that the particles of a magnet are Polarized.

*Meaning of the term 'Polarization.'*

When a particle of a body possesses properties related to a certain line or direction in the body, and when the body, retaining these properties, is turned so that this direction is reversed, then if as regards other bodies these properties of the particle are reversed, the particle, in reference to these properties, is said to be polarized, and the properties are said to constitute a particular kind of polarization.

Thus we may say that the rotation of a body about an axis constitutes a kind of polarization, because if, while the rotation continues, the direction of the axis is turned end for end, the body will be rotating in the opposite direction as regards space.

A conducting particle through which there is a current of electricity may be said to be polarized, because if it were turned round, and if the current continued to flow in the same direction as regards the particle, its direction in space would be reversed.

In short, if any mathematical or physical quantity is of the nature of a vector, as defined in Art. 11, then any body or particle to which this directed quantity or vector belongs may be said to be Polarized *, because it has opposite properties in the two opposite directions or poles of the directed quantity.

The poles of the earth, for example, have reference to its rotation, and have accordingly different names.

### *Meaning of the term 'Magnetic Polarization.'*

382.] In speaking of the state of the particles of a magnet as magnetic polarization, we imply that each of the smallest parts into which a magnet may be divided has certain properties related to a definite direction through the particle, called its Axis of Magnetization, and that the properties related to one end of this axis are opposite to the properties related to the other end.

The properties which we attribute to the particle are of the same kind as those which we observe in the complete magnet, and in assuming that the particles possess these properties, we only assert what we can prove by breaking the magnet up into small pieces, for each of these is found to be a magnet.

### *Properties of a Magnetized Particle.*

383.] Let the element $dx\,dy\,dz$ be a particle of a magnet, and let us assume that its magnetic properties are those of a magnet the strength of whose positive pole is $m$, and whose length is $ds$. Then if $P$ is any point in space distant $r$ from the positive pole and $r'$ from the negative pole, the magnetic potential at $P$ will be $\dfrac{m}{r}$ due to the positive pole, and $-\dfrac{m}{r'}$ due to the negative pole, or

$$V = \frac{m}{rr'}(r'-r). \tag{1}$$

If $ds$, the distance between the poles, is very small, we may put

$$r'-r = ds\cos\epsilon, \tag{2}$$

* The word Polarization has been used in a sense not consistent with this in Optics, where a ray of light is said to be polarized when it has properties relating to its sides, which are identical on opposite sides of the ray. This kind of polarization refers to another kind of Directed Quantity, which may be called a Dipolar Quantity, in opposition to the former kind, which may be called a Unipolar.

When a dipolar quantity is turned end for end it remains the same as before. Tensions and Pressures in solid bodies, Extensions, Compressions, and Distortions and most of the optical, electrical, and magnetic properties of crystallized bodies are dipolar quantities.

The property produced by magnetism in transparent bodies of twisting the plane of polarization of the incident light, is, like magnetism itself, a unipolar property. The rotatory property referred to in Art. 303 is also unipolar.

where $\epsilon$ is the angle between the vector drawn from the magnet to $P$ and the axis of the magnet *, or in the limit

$$V = \frac{m\,d\,s}{r^2}\cos\epsilon. \tag{3}$$

### *Magnetic Moment.*

384.] The product of the length of a uniformly and longitudinally magnetized bar magnet into the strength of its positive pole is called its Magnetic Moment.

### *Intensity of Magnetization.*

The intensity of magnetization of a magnetic particle is the ratio of its magnetic moment to its volume. We shall denote it by $I$.

The magnetization at any point of a magnet may be defined by its intensity and its direction. Its direction may be defined by its direction-cosines $\lambda$, $\mu$, $\nu$.

### *Components of Magnetization.*

The magnetization at a point of a magnet (being a vector or directed quantity) may be expressed in terms of its three components referred to the axes of coordinates. Calling these $A$, $B$, $C$,

$$A = I\lambda, \qquad B = I\mu, \qquad C = I\nu, \tag{4}$$

and the numerical value of $I$ is given by the equation

$$I^2 = A^2 + B^2 + C^2. \tag{5}$$

385.] If the portion of the magnet which we consider is the differential element of volume $dx\,dy\,dz$, and if $I$ denotes the intensity of magnetization of this element, its magnetic moment is $I\,dx\,dy\,dz$. Substituting this for $mds$ in equation (3), and remembering that

$$r\cos\epsilon = \lambda(\xi - x) + \mu(\eta - y) + \nu(\zeta - z), \tag{6}$$

where $\xi$, $\eta$, $\zeta$ are the coordinates of the extremity of the vector $r$ drawn from the point $(x, y, z)$, we find for the potential at the point $(\xi, \eta, \zeta)$ due to the magnetized element at $(x, y, z)$,

$$\{A(\xi - x) + B(\eta - y) + C(\zeta - z)\}\frac{1}{r^3}\,dx\,dy\,dz. \tag{7}$$

To obtain the potential at the point $(\xi, \eta, \zeta)$ due to a magnet of finite dimensions, we must find the integral of this expression for

* {The positive direction of the axis is from the negative to the positive pole.}

every element of volume included within the space occupied by the magnet, or

$$V=\iiint\{A(\xi-x)+B(\eta-y)+C(\zeta-z)\}\frac{1}{r^3}dx\,dy\,dz. \qquad (8)$$

Integrated by parts, this becomes

$$V=\iint A\frac{1}{r}dy\,dz+\iint B\frac{1}{r}dz\,dx+\iint C\frac{1}{r}dx\,dy$$
$$-\iiint\frac{1}{r}\left(\frac{dA}{dx}+\frac{dB}{dy}+\frac{dC}{dz}\right)dx\,dy\,dz,$$

where the double integration in the first three terms refers to the surface of the magnet, and the triple integration in the fourth to the space within it.

If $l, m, n$ denote the direction-cosines of the normal drawn outwards from the element of surface $dS$, we may write, as in Art. 21, for the sum of the first three terms

$$\iint(lA+mB+nC)\frac{1}{r}dS,$$

where the integration is to be extended over the whole surface of the magnet.

If we now introduce two new symbols $\sigma$ and $\rho$, defined by the equations

$$\sigma=lA+mB+nC,$$
$$\rho=-\left(\frac{dA}{dx}+\frac{dB}{dy}+\frac{dC}{dz}\right),$$

the expression for the potential may be written

$$V=\iint\frac{\sigma}{r}dS+\iiint\frac{\rho}{r}dx\,dy\,dz.$$

386.] This expression is identical with that for the electric potential due to a body on the surface of which there is an electrification whose surface-density is $\sigma$, while throughout its substance there is a bodily electrification whose volume-density is $\rho$. Hence, if we assume $\sigma$ and $\rho$ to be the surface- and volume-densities of the distribution of an imaginary substance, which we have called 'magnetic matter,' the potential due to this imaginary distribution will be identical with that due to the actual magnetization of every element of the magnet.

The surface-density $\sigma$ is the resolved part of the intensity of magnetization $I$ in the direction of the normal to the surface drawn outwards, and the volume-density $\rho$ is the 'convergence' (see Art. 25) of the magnetization at a given point in the magnet.

This method of representing the action of a magnet as due to a distribution of 'magnetic matter' is very convenient, but we must always remember that it is only an artificial method of representing the action of a system of polarized particles.

*On the Action of one Magnetic Molecule on another.*

387.] If, as in the chapter on Spherical Harmonics, Art. 129 *b*, we make

$$\frac{d}{dh} = l\frac{d}{dx} + m\frac{d}{dy} + n\frac{d}{dz}, \tag{1}$$

where $l$, $m$, $n$ are the direction-cosines of the axis $h$, then the potential due to a magnetic molecule at the origin, whose axis is parallel to $h_1$, and whose magnetic moment is $m_1$, is

$$V_1 = -\frac{d}{dh_1}\frac{m_1}{r} = \frac{m_1}{r^2}\lambda_1, \tag{2}$$

where $\lambda_1$ is the cosine of the angle between $h_1$ and $r$.

Again, if a second magnetic molecule whose moment is $m_2$, and whose axis is parallel to $h_2$, is placed at the extremity of the radius vector $r$, the potential energy due to the action of the one magnet on the other is

$$W = m_2\frac{dV_1}{dh_2} = -m_1 m_2\frac{d^2}{dh_1 dh_2}\left(\frac{1}{r}\right), \tag{3}$$

$$= \frac{m_1 m_2}{r^3}(\mu_{12} - 3\,\lambda_1\lambda_2), \tag{4}$$

where $\mu_{12}$ is the cosine of the angle which the axes make with each other, and $\lambda_1$, $\lambda_2$ are the cosines of the angles which they make with $r$.

Let us next determine the moment of the couple with which the first magnet tends to turn the second round its centre.

Let us suppose the second magnet turned through an angle $d\phi$ in a plane perpendicular to a third axis $h_3$, then the work done against the magnetic forces will be $\frac{dW}{d\phi}d\phi$, and the moment of the forces on the magnet in this plane will be

$$-\frac{dW}{d\phi} = -\frac{m_1 m_2}{r^3}\left(\frac{d\mu_{12}}{d\phi} - 3\,\lambda_1\frac{d\lambda_2}{d\phi}\right). \tag{5}$$

The actual moment acting on the second magnet may therefore be considered as the resultant of two couples, of which the first acts in a plane parallel to the axes of both magnets, and tends to *increase* the angle between them with a couple whose moment is

$$\frac{m_1 m_2}{r^3} \sin (h_1 h_2), \tag{6}$$

while the second couple acts in the plane passing through $r$ and the axis of the second magnet, and tends to *diminish* the angle between these directions with a couple whose moment is

$$\frac{3\, m_1 m_2}{r^3} \cos (r h_1) \sin (r h_2), \tag{7}$$

where $(rh_1)$, $(rh_2)$, $(h_1 h_2)$ denote the angles between the lines $r$, $h_1$, $h_2$*.

To determine the force acting on the second magnet in a direction parallel to a line $h_3$, we have to calculate

$$-\frac{dW}{dh_3} = m_1 m_2 \frac{d^3}{dh_1 dh_2 dh_3}\left(\frac{1}{r}\right), \tag{8}$$

$$= -m_1 m_2 \frac{\underline{|3}\, Y_3}{r^4}, \text{ by Art. } 129\,c,$$

$$= 3\frac{m_1 m_2}{r^4}\{\lambda_1 \mu_{23} + \lambda_2 \mu_{31} + \lambda_3 \mu_{12} - 5\,\lambda_1 \lambda_2 \lambda_3\}, \text{ by Art. 133}, \tag{9}$$

$$= 3\lambda_3 \frac{m_1 m_2}{r^4}(\mu_{12} - 5\lambda_1 \lambda_2) + 3\mu_{13}\frac{m_1 m_2}{r^4}\lambda_2 + 3\,\mu_{23}\frac{m_1 m_2}{r^4}\lambda_1. \tag{10}$$

If we suppose the actual force compounded of three forces, $R$, $H_1$ and $H_2$, in the directions of $r$, $h_1$ and $h_2$ respectively, then the force in the direction of $h_3$ is

$$\lambda_3 R + \mu_{13} H_1 + \mu_{23} H_2. \tag{11}$$

* {If $\theta_1$, $\theta_2$ are the angles which the axes of the magnets make with $r$, $\psi$ the angle between the planes containing $r$ and the axes of the first and second magnet respectively, then

$$\mu_{12} - 3\,\lambda_1 \lambda_2 = -2\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2 \cos\psi.$$

Thus the couple acting on the second magnet is equivalent to a couple whose axis is $r$ and whose moment $-dW/d\psi$ tending to increase $\psi$ is

$$\frac{m_1 m_2}{r^3} \sin\theta_1 \sin\theta_2 \sin\psi,$$

together with a couple in the plane of $r$ and the axis of the second magnet whose moment $-dW/d\theta_2$ tending to increase $\theta_2$ is

$$-\frac{m_1 m_2}{r^3}\{2\cos\theta_1 \sin\theta_2 + \sin\theta_1 \cos\theta_2 \cos\psi\}.$$

These couples are equivalent to those given by (6) and (7).}

Since the direction of $h_3$ is arbitrary, we must have

$$\left.\begin{aligned} R &= \frac{3\,m_1 m_2}{r^4}(\mu_{12} - 5\,\lambda_1\lambda_2),\\ H_1 = \frac{3\,m_1 m_2}{r^4}\lambda_2, &\qquad H_2 = \frac{3\,m_1 m_2}{r^4}\lambda_1.\end{aligned}\right\} \tag{12}$$

The force $R$ is a repulsion, tending to increase $r$; $H_1$ and $H_2$ act on the second magnet in the directions of the axes of the first and second magnets respectively.

This analysis of the forces acting between two small magnets was first given in terms of the Quaternion Analysis by Professor Tait in the *Quarterly Math. Journ.* for Jan. 1860. See also his work on *Quaternions*, Arts. 442–443, 2nd Edition.

## *Particular Positions.*

388.] (1) If $\lambda_1$ and $\lambda_2$ are each equal to 1, that is, if the axes of the magnets are in one straight line and in the same direction, $\mu_{12} = 1$, and the force between the magnets is a repulsion

$$R + H_1 + H_2 = -\frac{6\,m_1 m_2}{r^4}. \tag{13}$$

The negative sign indicates that the force is an attraction.

(2) If $\lambda_1$ and $\lambda_2$ are zero, and $\mu_{12}$ unity, the axes of the magnets are parallel to each other and perpendicular to $r$, and the force is a repulsion

$$\frac{3\,m_1 m_2}{r^4}. \tag{14}$$

In neither of these cases is there any couple.

(3) If $\qquad \lambda_1 = 1$ and $\lambda_2 = 0$, then $\mu_{12} = 0$. (15)

The force on the second magnet will be $\frac{3\,m_1 m_2}{r^4}$ in the direction of its axis, and the couple will be $\frac{2\,m_1 m_2}{r^3}$, tending to turn it

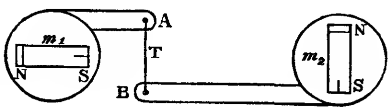


Fig. 1.

parallel to the first magnet. This is equivalent to a single force $\frac{3\,m_1 m_2}{r^4}$ acting parallel to the direction of the axis of the second magnet, and cutting $r$ at a point two-thirds of its length from $m_2$*.

Thus in the figure (1) two magnets are made to float on water, $m_2$ being in the direction of the axis of $m_1$, but having its own axis at right angles to that of $m_1$. If two points, $A$, $B$, rigidly connected with $m_1$ and $m_2$ respectively, are connected by means of a string $T$, the system will be in equilibrium, provided $T$ cuts the line $m_1 m_2$ at right angles at a point one-third of the distance from $m_1$ to $m_2$.

(4) If we allow the second magnet to turn freely about its centre till it comes to a position of stable equilibrium, $W$ will then be a minimum as regards $h_2$, and therefore the resolved part of the force due to $m_2$, taken in the direction of $h_1$, will be a maximum. Hence, if we wish to produce the greatest possible magnetic force at a given point in a given direction by means of magnets, the positions of whose centres are given, then, in order to determine the proper directions of the axes of these magnets to produce this effect, we have only to place a magnet in the given direction at the given point, and to observe the direction of stable equilibrium of the axis of a second magnet when its centre is placed at each of the other given points. The magnets must then be placed with their axes in the directions indicated by that of the second magnet.

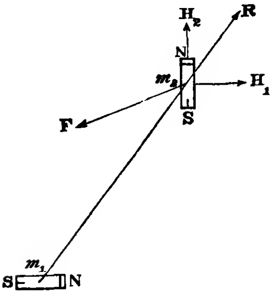

Fig. 2.

Of course, in performing this experiment we must take account of terrestrial magnetism, if it exists.

Let the second magnet be in a position of stable equilibrium as regards its direction, then since the couple acting on it vanishes, the axis of the second magnet must be in the same plane with that of the first. Hence

$$(h_1 h_2) = (h_1 r) + (r h_2), \tag{16}$$

* {In case (3) the first magnet is said to be 'end on' to the second, and the second 'broadside on' to the first, we can easily prove by formulae (6) and (7) that if the first magnet were 'broadside on' to the second the couple on the second would be $m_1 m_2/r^3$. Thus the couple when the deflecting magnet is 'end on' is twice as great as when it is 'broadside on.' Gauss has proved that if the law of force were inversely as the $p$'th power of the distance between the poles the couple when the deflecting magnet is 'end on' would be $p$ times as great as when it is 'broadside on.' By comparing the couples in these positions we can verify the law of the inverse square more accurately than is possible by the torsion balance.}

and the couple being

$$\frac{m_1 m_2}{r^3}(\sin(h_1 h_2) - 3\cos(h_1 r)\sin(r h_2)), \tag{17}$$

we find when this is zero

$$\tan(h_1 r) = 2\tan(r h_2), \tag{18}$$

or

$$\tan H_1 m_2 R = 2\tan R m_2' H_2. \tag{19}$$

When this position has been taken up by the second magnet the value of $W$ becomes

$$m_2 \frac{dV_1}{dh_2},$$

where $h_2$ is in the direction of the line of force due to $m_1$ at $m_2$.

Hence $$W = -m_2\sqrt{\overline{\frac{dV_1}{dx}}\Big|^2 + \overline{\frac{dV_1}{dy}}\Big|^2 + \overline{\frac{dV_1}{dz}}\Big|^2}. \tag{20}$$

Hence the second magnet will tend to move towards places of greater resultant force.

The force on the second magnet may be decomposed into a force $R$, which in this case is always attractive towards the first magnet, and a force $H_1$ parallel to the axis of the first magnet, where

$$R = 3\frac{m_1 m_2}{r^4}\frac{4\lambda_1^2+1}{\sqrt{3\lambda_1^2+1}}, \qquad H_1 = 3\frac{m_1 m_2}{r^4}\frac{\lambda_1}{\sqrt{3\lambda_1^2+1}}. \tag{21}$$

In Fig. XIV, at the end of this volume, the lines of force and equipotential surfaces in two dimensions are drawn. The magnets which produce them are supposed to be two long cylindrical rods the sections of which are represented by the circular blank spaces, and these rods are magnetized transversely in the direction of the arrows.

If we remember that there is a tension along the lines of force, it is easy to see that each magnet will tend to turn in the direction of the motion of the hands of a watch.

That on the right hand will also, as a whole, tend to move towards the top, and that on the left hand towards the bottom of the page.

### *On the Potential Energy of a Magnet placed in a Magnetic Field.*

389.] Let $V$ be the magnetic potential due to any system of magnets acting on the magnet under consideration. We shall call $V$ the potential of the external magnetic force.

If a small magnet whose strength is $m$, and whose length is $ds$, be placed so that its positive pole is at a point where the potential is $V$, and its negative pole at a point where the potential is $V'$, the potential energy of this magnet will be $m\,(V - V')$, or, if $ds$ is measured from the negative pole to the positive,

$$m\frac{dV}{ds}ds. \tag{1}$$

If $I$ is the intensity of the magnetization, and $\lambda$, $\mu$, $\nu$ its direction-cosines, we may write,

$$m\,ds = I\,dx\,dy\,dz,$$

$$\text{and} \quad \frac{dV}{ds} = \lambda\frac{dV}{dx} + \mu\frac{dV}{dy} + \nu\frac{dV}{dz},$$

and, finally, if $A$, $B$, $C$ are the components of magnetization,

$$A = \lambda I, \qquad B = \mu I, \qquad C = \nu I,$$

so that the expression (1) for the potential energy of the element of the magnet becomes

$$\left(A\frac{dV}{dx} + B\frac{dV}{dy} + C\frac{dV}{dz}\right)dx\,dy\,dz. \tag{2}$$

To obtain the potential energy of a magnet of finite size, we must integrate this expression for every element of the magnet. We thus obtain

$$W = \iiint\left(A\frac{dV}{dx} + B\frac{dV}{dy} + C\frac{dV}{dz}\right)dx\,dy\,dz \tag{3}$$

as the value of the potential energy of the magnet with respect to the magnetic field in which it is placed.

The potential energy is here expressed in terms of the components of magnetization and of those of the magnetic force arising from external causes.

By integration by parts we may express it in terms of the distribution of magnetic matter and of magnetic potential, thus,

$$W = \iint(Al + Bm + Cn)\,V\,dS - \iiint V\left(\frac{dA}{dx} + \frac{dB}{dy} + \frac{dC}{dz}\right)dx\,dy\,dz, \tag{4}$$

where $l$, $m$, $n$ are the direction-cosines of the normal at the element of surface $dS$. If we substitute in this equation the expressions for the surface- and volume-density of magnetic matter as given in Art. 385, the expression becomes

$$W = \iint V\sigma\,dS + \iiint V\rho\,dx\,dy\,dz. \tag{5}$$

We may write equation (3) in the form

$$W = -\iiint (A\alpha + B\beta + C\gamma)\, dx\, dy\, dz, \tag{6}$$

where $\alpha$, $\beta$ and $\gamma$ are the components of the external magnetic force.

*On the Magnetic Moment and Axis of a Magnet.*

390.] If throughout the whole space occupied by the magnet the external magnetic force is uniform in direction and magnitude, the components $\alpha$, $\beta$, $\gamma$ will be constant quantities, and if we write

$$\iiint A\, dx\, dy\, dz = lK, \quad \iiint B\, dx\, dy\, dz = mK, \quad \iiint C\, dx\, dy\, dz = nK, \tag{7}$$

the integrations being extended over the whole substance of the magnet, the value of $W$ may be written

$$W = -K\,(l\alpha + m\beta + n\gamma). \tag{8}$$

In this expression $l$, $m$, $n$ are the direction-cosines of the axis of the magnet, and $K$ is the magnetic moment of the magnet. If $\epsilon$ is the angle which the axis of the magnet makes with the direction of the magnetic force $\mathfrak{H}$, the value of $W$ may be written

$$W = -K\mathfrak{H}\cos\epsilon. \tag{9}$$

If the magnet is suspended so as to be free to turn about a vertical axis, as in the case of an ordinary compass needle, let the azimuth of the axis of the magnet be $\phi$, and let it be inclined at an angle $\theta$ to the horizontal plane. Let the force of terrestrial magnetism be in a direction whose azimuth is $\delta$ and dip $\zeta$, then

$$\alpha = \mathfrak{H}\cos\zeta\cos\delta, \quad \beta = \mathfrak{H}\cos\zeta\sin\delta, \quad \gamma = \mathfrak{H}\sin\zeta; \tag{10}$$

$$l = \cos\theta\cos\phi, \quad m = \cos\theta\sin\phi, \quad n = \sin\theta; \tag{11}$$

whence $$W = -K\mathfrak{H}\{\cos\zeta\cos\theta\cos(\phi-\delta) + \sin\zeta\sin\theta\}. \tag{12}$$

The moment of the force tending to increase $\phi$ by turning the magnet round a vertical axis is

$$-\frac{dW}{d\phi} = -K\mathfrak{H}\cos\zeta\cos\theta\sin(\phi-\delta). \tag{13}$$

*On the Expansion of the Potential of a Magnet in Solid Harmonics.*

391.] Let $V$ be the potential due to a unit pole placed at the point $(\xi, \eta, \zeta)$. The value of $V$ at the point $x$, $y$, $z$ is

$$V = \{(\xi-x)^2 + (\eta-y)^2 + (\zeta-z)^2\}^{-\frac{1}{2}}. \tag{1}$$

This expression may be expanded in terms of spherical harmonics, with their centre at the origin. We have then

$$V = V_0 + V_1 + V_2 + \&c., \tag{2}$$

where $V_0 = \dfrac{1}{r}$, $r$ being the distance of $(\xi, \eta, \zeta)$ from the origin, (3)

$$V_1 = \frac{\xi x + \eta y + \zeta z}{r^3}, \tag{4}$$

$$V_2 = \frac{3\,(\xi x + \eta y + \zeta z)^2 - (x^2 + y^2 + z^2)\,(\xi^2 + \eta^2 + \zeta^2)}{2\,r^5}, \tag{5}$$

&c.

To determine the value of the potential energy when the magnet is placed in the field of force expressed by this potential, we have to integrate the expression for $W$ in equation (3) of Art. 389 with respect to $x$, $y$ and $z$, considering $\xi$, $\eta$, $\zeta$ and $r$ as constants.

If we consider only the terms introduced by $V_0$, $V_1$ and $V_2$ the result will depend on the following volume-integrals,

$$lK = \iiint A\,dx\,dy\,dz,\quad mK = \iiint B\,dx\,dy\,dz,\quad nK = \iiint C\,dx\,dy\,dz; \tag{6}$$

$$L = \iiint Ax\,dx\,dy\,dz,\quad M = \iiint By\,dx\,dy\,dz,\quad N = \iiint Cz\,dx\,dy\,dz; \tag{7}$$

$$P = \iiint (Bz + Cy)\,dx\,dy\,dz,\qquad Q = \iiint (Cx + Az)\,dx\,dy\,dz,$$

$$R = \iiint (Ay + Bx)\,dx\,dy\,dz. \tag{8}$$

We thus find for the value of the potential energy of the magnet placed in presence of the unit pole at the point $(\xi, \eta, \zeta)$,

$$W = K\frac{l\xi + m\eta + n\zeta}{r^3}$$

$$+ \frac{\xi^2(2L - M - N) + \eta^2(2M - N - L) + \zeta^2(2N - L - M) + 3(P\eta\zeta + Q\zeta\xi + R\xi\eta)}{r^5} \tag{9}$$

$$+ \&c.$$

This expression may also be regarded as the potential energy of the unit pole in presence of the magnet, or more simply as the potential at the point $\xi$, $\eta$, $\zeta$ due to the magnet.

*On the Centre of a Magnet and its Primary and Secondary Axes.*

392.] This expression may be simplified by altering the directions of the coordinates and the position of the origin. In the first place, we shall make the direction of the axis of $x$ parallel to the axis of the magnet. This is equivalent to making

$$l = 1, \quad m = 0, \quad n = 0. \tag{10}$$

If we change the origin of coordinates to the point $(x', y', z')$, the directions of the axes remaining unchanged, the volume-integrals $lK$, $mK$ and $nK$ will remain unchanged, but the others will be altered as follows:

$$L' = L - lKx', \qquad M' = M - mKy', \qquad N' = N - nKz'; \tag{11}$$

$$P' = P - K(mz' + ny'),\; Q' = Q - K(nx' + lz'),\; R' = R - K(ly' + mx'). \tag{12}$$

If we now make the direction of the axis of $x$ parallel to the axis of the magnet, and put

$$x' = \frac{2L - M - N}{2K}, \qquad y' = \frac{R}{K}, \qquad z' = \frac{Q}{K}, \tag{13}$$

then for the new axes $M$ and $N$ have their values unchanged, and the value of $L'$ becomes $\frac{1}{2}(M+N)$. $P$ remains unchanged, and $Q$ and $R$ vanish. We may therefore write the potential thus,

$$K\frac{\xi}{r^3} + \frac{\frac{3}{2}(\eta^2 - \zeta^2)(M-N) + 3P\eta\zeta}{r^5} + \dots. \tag{14}$$

We have thus found a point, fixed with respect to the magnet, such that the second term of the potential assumes the most simple form when this point is taken as origin of coordinates. This point we therefore define as the centre of the magnet, and the axis drawn through it in the direction formerly defined as the direction of the magnetic axis may be defined as the principal axis of the magnet.

We may simplify the result still more by turning the axes of $y$ and $z$ round that of $x$ through half the angle whose tangent is $\frac{P}{M-N}$. This will cause $P$ to become zero, and the final form of the potential may be written

$$K\frac{\xi}{r^3} + \frac{3}{2}\frac{(\eta^2 - \zeta^2)(M-N)}{r^5} + \&c. \tag{15}$$

This is the simplest form of the first two terms of the potential of a magnet. When the axes of $y$ and $z$ are thus placed they may be called the Secondary axes of the magnet.

We may also determine the centre of a magnet by finding the position of the origin of coordinates, for which the surface-integral of the square of the second term of the potential, extended over a sphere of unit radius, is a minimum.

The quantity which is to be made a minimum is, by Art. 141,

$$4(L^2+M^2+N^2-MN-NL-LM)+3(P^2+Q^2+R^2). \qquad (16)$$

The changes in the values of this quantity due to a change of position of the origin may be deduced from equations (11) and (12). Hence the conditions of a minimum are

$$\left.\begin{aligned} 2\,l\,(2\,L-M-N)+3\,n\,Q+3\,m\,R &= 0,\\ 2\,m\,(2\,M-N-L)+3\,l\,R+3\,n\,P &= 0,\\ 2\,n\,(2\,N-L-M)+3\,m\,P+3\,l\,Q &= 0. \end{aligned}\right\} \qquad (17)$$

If we assume $l = 1$, $m = 0$, $n = 0$, these conditions become

$$2\,L-M-N = 0, \quad Q = 0, \quad R = 0, \qquad (18)$$

which are the conditions made use of in the previous investigation.

This investigation may be compared with that by which the potential of a system of gravitating matter is expanded. In the latter case, the most convenient point to assume as the origin is the centre of gravity of the system, and the most convenient axes are the principal axes of inertia through that point.

In the case of the magnet, the point corresponding to the centre of gravity is at an infinite distance in the direction of the axis, and the point which we call the centre of the magnet is a point having different properties from those of the centre of gravity. The quantities $L$, $M$, $N$ correspond to the moments of inertia, and $P$, $Q$, $R$ to the products of inertia of a material body, except that $L$, $M$, and $N$ are not necessarily positive quantities.

When the centre of the magnet is taken as the origin, the spherical harmonic of the second order is of the sectorial form, having its axis coinciding with that of the magnet, and this is true of no other point.

When the magnet is symmetrical on all sides of this axis, as in the case of a figure of revolution, the term involving the harmonic of the second order disappears entirely.

393.] At all parts of the earth's surface, except some parts of the Polar regions, one end of a magnet points towards the north, or at least in a northerly direction, and the other in a southerly direction. In speaking of the ends of a magnet we shall adopt the popular method of calling the end which points to the north the north end of the magnet. When, however, we speak in the language of the theory of magnetic fluids we shall use the words Boreal and Austral. Boreal magnetism is an imaginary kind of matter supposed to be most abundant in the northern parts of the earth, and Austral magnetism is the imaginary magnetic matter which prevails in the southern regions of the earth. The magnetism of the north end of a magnet is Austral, and that of the south end is Boreal. When therefore we speak of the north and south ends of a magnet we do not compare the magnet with the earth as the great magnet, but merely express the position which the magnet endeavours to take up when free to move. When, on the other hand, we wish to compare the distribution of imaginary magnetic fluid in the magnet with that in the earth we shall use the more grandiloquent words Boreal and Austral magnetism.

394.] In speaking of a field of magnetic force we shall use the phrase Magnetic North to indicate the direction in which the north end of a compass needle would point if placed in the field of force.

In speaking of a line of magnetic force we shall always suppose it to be traced from magnetic south to magnetic north, and shall call this direction positive. In the same way the direction of magnetization of a magnet is indicated by a line drawn from the south end of the magnet towards the north end, and the end of the magnet which points north is reckoned the positive end.

We shall consider Austral magnetism, that is, the magnetism of that end of a magnet which points north, as positive. If we denote its numerical value by $m$, then the magnetic potential

$$V = \Sigma \left(\frac{m}{r}\right),$$

and the positive direction of a line of force is that in which $V$ diminishes.

# CHAPTER II.

## MAGNETIC FORCE AND MAGNETIC INDUCTION.

395.] We have already (Art. 385) determined the magnetic potential at a given point due to a magnet, the magnetization of which is given at every point of its substance, and we have shewn that the mathematical result may be expressed either in terms of the actual magnetization of every element of the magnet, or in terms of an imaginary distribution of 'magnetic matter,' partly condensed on the surface of the magnet and partly diffused throughout its substance.

The magnetic potential, as thus defined, is found by the same mathematical process, whether the given point is outside the magnet or within it. The force exerted on a unit magnetic pole placed at any point outside the magnet is deduced from the potential by the same process of differentiation as in the corresponding electrical problem. If the components of this force are $\alpha, \beta, \gamma$,

$$\alpha = -\frac{dV}{dx}, \quad \beta = -\frac{dV}{dy}, \quad \gamma = -\frac{dV}{dz}. \tag{1}$$

To determine by experiment the magnetic force at a point within the magnet we must begin by removing part of the magnetized substance, so as to form a cavity within which we are to place the magnetic pole. The force acting on the pole will depend, in general, on the form of this cavity, and on the inclination of the walls of the cavity to the direction of magnetization. Hence it is necessary, in order to avoid ambiguity in speaking of the magnetic force within a magnet, to specify the form and position of the cavity within which the force is to be measured. It is manifest that when the form and position of the cavity is specified, the point within it at which the magnetic pole is placed must be regarded as no longer within the substance of the magnet, and therefore the ordinary methods of determining the force become at once applicable.

396.] Let us now consider a portion of a magnet in which the direction and intensity of the magnetization are uniform. Within this portion let a cavity be hollowed out in the form of a cylinder, the axis of which is parallel to the direction of magnetization, and let a magnetic pole of unit strength be placed at the middle point of the axis.

Since the generating lines of this cylinder are in the direction of magnetization, there will be no superficial distribution of magnetism on the curved surface, and since the circular ends of the cylinder are perpendicular to the direction of magnetization, there will be a uniform superficial distribution, of which the surface-density is $I$ for the negative end, and $-I$ for the positive end.

Let the length of the axis of the cylinder be $2b$, and its radius $a$. Then the force arising from this superficial distribution on a magnetic pole placed at the middle point of the axis is that due to the attraction of the disk on the positive side, and the repulsion of the disk on the negative side. These two forces are equal and in the same direction, and their sum is

$$R = 4\pi I\left(1 - \frac{b}{\sqrt{a^2+b^2}}\right). \tag{2}$$

From this expression it appears that the force depends, not on the absolute dimensions of the cavity, but on the ratio of the length to the diameter of the cylinder. Hence, however small we make the cavity, the force arising from the surface distribution on its walls will remain, in general, finite.

397.] We have hitherto supposed the magnetization to be uniform and in the same direction throughout the whole of the portion of the magnet from which the cylinder is hollowed out. When the magnetization is not thus restricted, there will in general be a distribution of imaginary magnetic matter through the substance of the magnet. The cutting out of the cylinder will remove part of this distribution, but since in similar solid figures the forces at corresponding points are proportional to the linear dimensions of the figures, the alteration of the force on the magnetic pole due to the volume-density of magnetic matter will diminish indefinitely as the size of the cavity is diminished, while the effect due to the surface-density on the walls of the cavity remains, in general, finite.

If, therefore, we assume the dimensions of the cylinder so small that the magnetization of the part removed may be regarded as everywhere parallel to the axis of the cylinder, and of constant magnitude $I$, the force on a magnetic pole placed at the middle point of the axis of the cylindrical hollow will be compounded of two forces. The first of these is that due to the distribution of magnetic matter on the outer surface of the magnet, and throughout its interior, exclusive of the portion hollowed out. The components of this force are $\alpha$, $\beta$ and $\gamma$, derived from the potential by equations (1). The second is the force $R$, acting along the axis of the cylinder in the direction of magnetization. The value of this force depends on the ratio of the length to the diameter of the cylindric cavity.

398.] *Case I.* Let this ratio be very great, or let the diameter of the cylinder be small compared with its length. Expanding the expression for $R$ in powers of $\frac{a}{b}$, we find

$$R = 4\pi I\left\{\frac{1}{2}\frac{a^2}{b^2} - \frac{3}{8}\frac{a^4}{b^4} + \&c.\right\}, \tag{3}$$

a quantity which vanishes when the ratio of $b$ to $a$ is made infinite. Hence, when the cavity is a very narrow cylinder with its axis parallel to the direction of magnetization, the magnetic force within the cavity is not affected by the surface distribution on the ends of the cylinder, and the components of this force are simply $\alpha$, $\beta$, $\gamma$, where

$$\alpha = -\frac{dV}{dx}, \quad \beta = -\frac{dV}{dy}, \quad \gamma = -\frac{dV}{dz}. \tag{4}$$

We shall define the force within a cavity of this form as the magnetic force within the magnet. Sir William Thomson has called this the Polar definition of magnetic force. When we have occasion to consider this force as a vector we shall denote it by $\mathfrak{H}$.

399.] *Case II.* Let the length of the cylinder be very small compared with its diameter, so that the cylinder becomes a thin disk. Expanding the expression for $R$ in powers of $\frac{b}{a}$, it becomes

$$R = 4\pi I\left\{1 - \frac{b}{a} + \frac{1}{2}\frac{b^3}{a^3} - \&c.\right\}, \tag{5}$$

the ultimate value of which, when the ratio of $a$ to $b$ is made infinite, is $4\pi I$.

Hence, when the cavity is in the form of a thin disk, whose plane is normal to the direction of magnetization, a unit magnetic pole placed at the middle of the axis experiences a force $4\pi I$ in the direction of magnetization, arising from the superficial magnetism on the circular surfaces of the disk *.

Since the components of $I$ are $A$, $B$ and $C$, the components of this force are $4\pi A$, $4\pi B$, and $4\pi C$. This must be compounded with the force whose components are $\alpha$, $\beta$, $\gamma$.

400.] Let the actual force on the unit pole be denoted by the vector $\mathfrak{B}$, and its components by $a$, $b$ and $c$, then

$$\left.\begin{aligned} a &= \alpha + 4\pi A, \\ b &= \beta + 4\pi B, \\ c &= \gamma + 4\pi C. \end{aligned}\right\} \tag{6}$$

We shall define the force within a hollow disk, whose plane sides are normal to the direction of magnetization, as the Magnetic Induction within the magnet. Sir William Thomson has called this the Electromagnetic definition of magnetic force.

The three vectors, the magnetization $\mathfrak{I}$, the magnetic force $\mathfrak{H}$, and the magnetic induction $\mathfrak{B}$, are connected by the vector equation

$$\mathfrak{B} = \mathfrak{H} + 4\pi\mathfrak{I}. \tag{7}$$

## *Line-Integral of Magnetic Force.*

401.] Since the magnetic force, as defined in Art. 398, is that due to the distribution of free magnetism on the surface and through the interior of the magnet, and is not affected by the surface-magnetism of the cavity, it may be derived directly from the general expression for the potential of the magnet, and the

* *On the force within cavities of other forms.*

1. Any narrow crevasse. The force arising from the surface-magnetism is $4\pi I\cos\epsilon$ in the direction of the normal to the plane of the crevasse, where $\epsilon$ is the angle between this normal and the direction of magnetization. When the crevasse is parallel to the direction of magnetization the force is the magnetic force $\mathfrak{H}$; when the crevasse is perpendicular to the direction of magnetization the force is the magnetic induction $\mathfrak{B}$.

2. In an infinitely elongated cylinder, the axis of which makes an angle $\epsilon$ with the direction of magnetization, the force arising from the surface-magnetism is $2\pi I\sin\epsilon$, perpendicular to the axis in the plane containing the axis and the direction of magnetization.

3. In a sphere the force arising from surface magnetism is $\frac{4}{3}\pi I$ in the direction of magnetization.

line-integral of the magnetic force taken along any curve from the point $A$ to the point $B$ is

$$\int_A^B \left(\alpha\frac{dx}{ds}+\beta\frac{dy}{ds}+\gamma\frac{dz}{ds}\right)ds = V_A - V_B, \tag{8}$$

where $V_A$ and $V_B$ denote the potentials at $A$ and $B$ respectively.

*Surface-Integral of Magnetic Induction.*

402.] The magnetic induction through the surface $S$ is defined as the value of the integral

$$Q = \iint \mathfrak{B}\cos\epsilon\, dS, \tag{9}$$

where $\mathfrak{B}$ denotes the magnitude of the magnetic induction at the element of surface $dS$, and $\epsilon$ the angle between the direction of the induction and the normal to the element of surface, and the integration is to be extended over the whole surface, which may be either closed or bounded by a closed curve.

If $a, b, c$ denote the components of the magnetic induction, and $l, m, n$ the direction-cosines of the normal, the surface-integral may be written

$$Q = \iint (la+mb+nc)\,dS. \tag{10}$$

If we substitute for the components of the magnetic induction their values in terms of those of the magnetic force, and the magnetization as given in Art. 400, we find

$$Q = \iint (l\alpha+m\beta+n\gamma)\,dS + 4\pi\iint (lA+mB+nC)\,dS. \tag{11}$$

We shall now suppose that the surface over which the integration extends is a closed one, and we shall investigate the value of the two terms on the right-hand side of this equation.

Since the mathematical form of the relation between magnetic force and free magnetism is the same as that between electric force and free electricity, we may apply the result given in Art. 77 to the first term in the value of $Q$ by substituting $\alpha, \beta, \gamma$, the components of magnetic force, for $X, Y, Z$, the components of electric force in Art. 77, and $M$, the algebraic sum of the free magnetism within the closed surface, for $e$, the algebraic sum of the free electricity.

We thus obtain the equation

$$\iint (l\alpha+m\beta+n\gamma)\,dS = 4\pi M. \tag{12}$$

Since every magnetic particle has two poles, which are equal in numerical magnitude but of opposite signs, the algebraic sum of the magnetism of the particle is zero. Hence, those particles which are entirely within the closed surface $S$ can contribute nothing to the algebraic sum of the magnetism within $S$. The value of $M$ must therefore depend only on those magnetic particles which are cut by the surface $S$.

Consider a small element of the magnet of length $s$ and transverse section $k^2$, magnetized in the direction of its length, so that the strength of its poles is $m$. The moment of this small magnet will be $ms$, and the intensity of its magnetization, being the ratio of the magnetic moment to the volume, will be

$$I = \frac{m}{k^2}. \tag{13}$$

Let this small magnet be cut by the surface $S$, so that the direction of magnetization makes an angle $\epsilon'$ with the normal drawn outwards from the surface, then if $dS$ denotes the area of the section,

$$k^2 = dS \cos\epsilon'. \tag{14}$$

The negative pole $-m$ of this magnet lies within the surface $S$.

Hence, if we denote by $dM$ the part of the free magnetism within $S$ which is contributed by this little magnet,

$$\begin{aligned} dM &= -m = -Ik^2, \\ &= -I\cos\epsilon' dS. \end{aligned} \tag{15}$$

To find $M$, the algebraic sum of the free magnetism within the closed surface $S$, we must integrate this expression over the closed surface, so that

$$M = -\iint I\cos\epsilon' dS,$$

or writing $A$, $B$, $C$ for the components of magnetization, and $l$, $m$, $n$ for the direction-cosines of the normal drawn outwards,

$$M = -\iint (lA + mB + nC)\, dS. \tag{16}$$

This gives us the value of the integral in the second term on the right-hand side of equation (11). The value of $Q$ in that equation may therefore be found from equations (12) and (16),

$$Q = 4\pi M - 4\pi M = 0, \tag{17}$$

or, *the surface-integral of the magnetic induction through any closed surface is zero.*

403.] If we assume as the closed surface that of the differential element of volume $dx\,dy\,dz$, we obtain the equation

$$\frac{da}{dx}+\frac{db}{dy}+\frac{dc}{dz}=0. \tag{18}$$

This is the solenoidal condition, which is always satisfied by the components of the magnetic induction.

Since the distribution of magnetic induction is solenoidal, the induction through any surface bounded by a closed curve depends only on the form and position of the closed curve, and not on that of the surface itself.

404.] Surfaces at every point of which

$$la+mb+nc=0 \tag{19}$$

are called surfaces of no induction, and the intersection of two such surfaces is called a line of induction. The conditions that a curve, $s$, may be a line of induction are

$$\frac{1}{a}\frac{dx}{ds}=\frac{1}{b}\frac{dy}{ds}=\frac{1}{c}\frac{dz}{ds}. \tag{20}$$

A system of lines of induction drawn through every point of a closed curve forms a tubular surface called a Tube of induction.

The induction across any section of such a tube is the same. If the induction is unity the tube is called a Unit tube of induction.

All that Faraday* says about lines of magnetic force and magnetic sphondyloids is mathematically true, if understood of the lines and tubes of magnetic induction.

The magnetic force and the magnetic induction are identical outside the magnet, but within the substance of the magnet they must be carefully distinguished.

In a straight uniformly magnetized bar the magnetic force due to the magnet itself is from the end which points north, which we call the positive pole, towards the south end or negative pole, both within the magnet and in the space without.

The magnetic induction, on the other hand, is from the positive pole to the negative outside the magnet, and from the negative pole to the positive within the magnet, so that the lines and tubes of induction are re-entering or cyclic figures.

* *Exp. Res.*, series xxviii.

The importance of the magnetic induction as a physical quantity will be more clearly seen when we study electromagnetic phenomena. When the magnetic field is explored by a moving wire, as in Faraday's *Exp. Res.* 3076, it is the magnetic induction and not the magnetic force which is directly measured.

### *The Vector-Potential of Magnetic Induction.*

405.] Since, as we have shewn in Art. 403, the magnetic induction through a surface bounded by a closed curve depends on the closed curve, and not on the form of the surface which is bounded by it, it must be possible to determine the induction through a closed curve by a process depending only on the nature of that curve, and not involving the construction of a surface forming a diaphragm of the curve.

This may be done by finding a vector $\mathfrak{A}$ related to $\mathfrak{B}$, the magnetic induction, in such a way that the line-integral of $\mathfrak{A}$, extended round the closed curve, is equal to the surface-integral of $\mathfrak{B}$, extended over a surface bounded by the closed curve.

If, in Art. 24, we write $F$, $G$, $H$ for the components of $\mathfrak{A}$, and $a$, $b$, $c$ for the components of $\mathfrak{B}$, we find for the relation between these components

$$a = \frac{dH}{dy} - \frac{dG}{dz}, \quad b = \frac{dF}{dz} - \frac{dH}{dx}, \quad c = \frac{dG}{dx} - \frac{dF}{dy}. \tag{21}$$

The vector $\mathfrak{A}$, whose components are $F$, $G$, $H$, is called the vector-potential of magnetic induction.

If a magnetic molecule whose moment is $m$ and the direction of whose axis of magnetization is $(\lambda, \mu, \nu)$ be at the origin of coordinates, the potential at a point $(x, y, z)$ distance $r$ from the origin is, by Art. 387,

$$-m\left(\lambda\frac{d}{dx} + \mu\frac{d}{dy} + \nu\frac{d}{dz}\right)\frac{1}{r};$$

$$\therefore \quad c = m\left(\lambda\frac{d^2}{dx\,dz} + \mu\frac{d^2}{dy\,dz} + \nu\frac{d^2}{dz^2}\right)\frac{1}{r},$$

which, by Laplace's equation, may be thrown into the form

$$m\frac{d}{dx}\left(\lambda\frac{d}{dz} - \nu\frac{d}{dx}\right)\frac{1}{r} - m\frac{d}{dy}\left(\nu\frac{d}{dy} - \mu\frac{d}{dz}\right)\frac{1}{r}.$$

The quantities $a$, $b$ may be dealt with in a similar manner.

Hence
$$F = m\left(\nu \frac{d}{dy} - \mu \frac{d}{dz}\right)\frac{1}{r},$$
$$= \frac{m(\mu z - \nu y)}{r^3}.$$

From this expression $G$ and $H$ may be found by symmetry. We thus see that the vector-potential at a given point, due to a magnetized particle placed at the origin, is numerically equal to the magnetic moment of the particle divided by the square of the radius vector and multiplied by the sine of the angle between the axis of magnetization and the radius vector, and the direction of the vector-potential is perpendicular to the plane of the axis of magnetization and the radius vector, and is such that to an eye looking in the positive direction along the axis of magnetization the vector-potential is drawn in the direction of rotation of the hands of a watch.

Hence, for a magnet of any form in which $A$, $B$, $C$ are the components of magnetization at the point $(x, y, z)$, the components of the vector-potential at the point $(\xi, \eta, \zeta)$, are

$$\left.\begin{aligned} F &= \iiint \left(B\frac{dp}{dz} - C\frac{dp}{dy}\right) dx\,dy\,dz, \\ G &= \iiint \left(C\frac{dp}{dx} - A\frac{dp}{dz}\right) dx\,dy\,dz, \\ H &= \iiint \left(A\frac{dp}{dy} - B\frac{dp}{dx}\right) dx\,dy\,dz; \end{aligned}\right\} \quad (22)$$

where $p$ is put, for conciseness, for the reciprocal of the distance between the points $(\xi, \eta, \zeta)$ and $(x, y, z)$, and the integrations are extended over the space occupied by the magnet.

406.] The scalar, or ordinary, potential of magnetic force, Art. 385, becomes when expressed in the same notation,

$$V = \iiint \left(A\frac{dp}{dx} + B\frac{dp}{dy} + C\frac{dp}{dz}\right) dx\,dy\,dz. \quad (23)$$

Remembering that $\frac{dp}{dx} = -\frac{dp}{d\xi}$, and that the integral

$$\iiint A\left(\frac{d^2p}{dx^2} + \frac{d^2p}{dy^2} + \frac{d^2p}{dz^2}\right) dx\,dy\,dz$$

has the value $-4\pi(A)$ when the point $(\xi, \eta, \zeta)$ is included within the limits of integration, and is zero when it is not so included, $(A)$ being the value of $A$ at the point $(\xi, \eta, \zeta)$, we find for the value of the $x$-component of the magnetic induction,

$$a = \frac{dH}{d\eta} - \frac{dG}{d\zeta}$$

$$= \iiint \left\{ A\left(\frac{d^2 p}{dy\, d\eta} + \frac{d^2 p}{dz\, d\zeta}\right) - B\frac{d^2 p}{dx\, d\eta} - C\frac{d^2 p}{dx\, d\zeta}\right\} dx\, dy\, dz$$

$$= -\frac{d}{d\xi}\iiint \left\{ A\frac{dp}{dx} + B\frac{dp}{dy} + C\frac{dp}{dz}\right\} dx\, dy\, dz$$

$$-\iiint A\left(\frac{d^2 p}{dx^2} + \frac{d^2 p}{dy^2} + \frac{d^2 p}{dz^2}\right) dx\, dy\, dz. \quad (24)$$

The first term of this expression is evidently $-\frac{dV}{d\xi}$, or, $\alpha$ the component of the magnetic force.

The quantity under the integral sign in the second term is zero for every element of volume except that in which the point $(\xi, \eta, \zeta)$ is included. If the value of $A$ at the point $(\xi, \eta, \zeta)$ is $(A)$, the value of the second term is easily proved to be $4\pi(A)$, where $(A)$ is evidently zero at all points outside the magnet.

We may now write the value of the $x$-component of the magnetic induction

$$a = \alpha + 4\pi(A), \quad (25)$$

an equation which is identical with the first of those given in Art. 400. The equations for $b$ and $c$ will also agree with those of Art. 400.

We have already seen that the magnetic force $\mathfrak{H}$ is derived from the scalar magnetic potential $V$ by the application of Hamilton's operator $\nabla$ so that we may write, as in Art. 17,

$$\mathfrak{H} = -\nabla V, \quad (26)$$

and that this equation is true both without and within the magnet.

It appears from the present investigation that the magnetic induction $\mathfrak{B}$ is derived from the vector-potential $\mathfrak{A}$ by the application of the same operator, and that the result is true within the magnet as well as without it.

The application of this operator to a vector-function produces, in general, a scalar quantity as well as a vector. The scalar part, however, which we have called the convergence of the vector-function, vanishes when the vector-function satisfies the solenoidal condition

$$\frac{dF}{d\xi} + \frac{dG}{d\eta} + \frac{dH}{d\zeta} = 0. \tag{27}$$

By differentiating the expressions for $F$, $G$, $H$ in equations (22), we find that this equation is satisfied by these quantities.

We may therefore write the relation between the magnetic induction and its vector-potential

$$\mathfrak{B} = \nabla\, \mathfrak{A},$$

which may be expressed in words by saying that the magnetic induction is the curl of its vector-potential. See Art. 25.

# CHAPTER III.

## MAGNETIC SOLENOIDS AND SHELLS *.

### *On Particular Forms of Magnets.*

407.] If a long narrow filament of magnetic matter like a wire is magnetized everywhere in a longitudinal direction, then the product of any transverse section of the filament into the mean intensity of the magnetization across it is called the strength of the magnet at that section. If the filament were cut in two at the section without altering the magnetization, the two surfaces, when separated, would be found to have equal and opposite quantities of superficial magnetization, each of which is numerically equal to the strength of the magnet at the section.

A filament of magnetic matter, so magnetized that its strength is the same at every section, at whatever part of its length the section be made, is called a Magnetic Solenoid.

If $m$ is the strength of the solenoid, $ds$ an element of its length, $s$ being measured from the negative to the positive pole of the magnet, $r$ the distance of that element from a given point, and $\epsilon$ the angle which $r$ makes with the axis of magnetization of the element, the potential at the given point due to the element is

$$\frac{m\,ds\cos\epsilon}{r^2} = -\frac{m}{r^2}\frac{dr}{ds}ds.$$

Integrating this expression with respect to $s$, so as to take into account all the elements of the solenoid, the potential is found to be

$$V = m\left(\frac{1}{r_1} - \frac{1}{r^2}\right),$$

$r_1$ being the distance of the positive end of the solenoid, and $r_2$ that of the negative end from the point where $V$ is measured.

* See Sir W. Thomson's 'Mathematical Theory of Magnetism,' *Phil. Trans.*, June 1849 and June 1850, or *Reprint* of Papers on Electrostatics and Magnetism, p. 340.

Hence the potential due to a solenoid, and consequently all its magnetic effects, depend only on its strength and the position of its ends, and not at all on its form, whether straight or curved, between these points.

Hence the ends of a solenoid may be called in a strict sense its poles.

If a solenoid forms a closed curve the potential due to it is zero at every point, so that such a solenoid can exert no magnetic action, nor can its magnetization be discovered without breaking it at some point and separating the ends.

If a magnet can be divided into solenoids, all of which either form closed curves or have their extremities in the outer surface of the magnet, the magnetization is said to be solenoidal, and, since the action of the magnet depends entirely upon that of the ends of the solenoids, the distribution of imaginary magnetic matter will be entirely superficial.

Hence the condition of the magnetization being solenoidal is

$$\frac{dA}{dx} + \frac{dB}{dy} + \frac{dC}{dz} = 0,$$

where $A$, $B$, $C$ are the components of the magnetization at any point of the magnet.

408.] A longitudinally magnetized filament, of which the strength varies at different parts of its length, may be conceived to be made up of a bundle of solenoids of different lengths, the sum of the strengths of all the solenoids which pass through a given section being the magnetic strength of the filament at that section. Hence any longitudinally magnetized filament may be called a Complex Solenoid.

If the strength of a complex solenoid at any section is $m$, then the potential due to its action is

$$V = -\int \frac{m}{r^2}\frac{dr}{ds}\,ds \text{ where } m \text{ is variable,}$$

$$= \frac{m_1}{r_1} - \frac{m_2}{r_2} - \int \frac{1}{r}\frac{dm}{ds}\,ds.$$

This shews that besides the action of the two ends, which may in this case be of different strengths, there is an action due to the distribution of imaginary magnetic matter along the filament with a linear density

$$\lambda = -\frac{dm}{ds}.$$

## *Magnetic Shells.*

409.] If a thin shell of magnetic matter is magnetized in a direction everywhere normal to its surface, the intensity of the magnetization at any place multiplied by the thickness of the shell at that place is called the Strength of the magnetic shell at that place.

If the strength of a shell is everywhere the same, it is called a Simple magnetic shell; if it varies from point to point it may be conceived to be made up of a number of simple shells superposed and overlapping each other. It is therefore called a Complex magnetic shell.

Let $dS$ be an element of the surface of the shell at $Q$, and $\Phi$ the strength of the shell, then the potential at any point, $P$, due to the element of the shell, is

$$dV = \Phi \frac{1}{r^2} dS \cos\epsilon,$$

where $\epsilon$ is the angle between the vector $QP$, or $r$, and the normal drawn outwards from the positive side of the shell.

But if $d\omega$ is the solid angle subtended by $dS$ at the point $P$

$$r^2 d\omega = dS \cos\epsilon,$$

whence
$$dV = \Phi\, d\omega,$$

and therefore in the case of a simple magnetic shell

$$V = \Phi\omega,$$

or, *the potential due to a magnetic shell at any point is the product of its strength into the solid angle subtended by its edge at the given point**.

410.] The same result may be obtained in a different way by supposing the magnetic shell placed in any field of magnetic force, and determining the potential energy due to the position of the shell.

If $V$ is the potential at the element $dS$, then the energy due to this element is

$$\Phi\left(l\frac{dV}{dx} + m\frac{dV}{dy} + n\frac{dV}{dz}\right) dS,$$

or, *the product of the strength of the shell into the part of the surface-integral of $dV/dv$ due to the element $dS$ of the shell.*

* This theorem is due to Gauss, *General Theory of Terrestrial Magnetism*, § 38.

Hence, integrating with respect to all such elements, the energy due to the position of the shell in the field is equal to the product of the strength of the shell and the surface-integral of the magnetic induction taken over the surface of the shell.

Since this surface-integral is the same for any two surfaces which have the same bounding edge and do not include between them any centre of force, the action of the magnetic shell depends only on the form of its edge.

Now suppose the field of force to be that due to a magnetic pole of strength $m$. We have seen (Art. 76, *Cor.*) that the surface-integral over a surface bounded by a given edge is the product of the strength of the pole and the solid angle subtended by the edge at the pole. Hence the energy due to the mutual action of the pole and the shell is

$$\Phi m \omega,$$

and this, by Green's theorem, is equal to the product of the strength of the pole into the potential due to the shell at the pole. The potential due to the shell is therefore $\Phi\omega$.

411.] If a magnetic pole $m$ starts from a point on the negative surface of a magnetic shell, and travels along any path in space so as to come round the edge to a point close to where it started but on the positive side of the shell, the solid angle will vary continuously, and will increase by $4\pi$ during the process. The work done by the pole will be $4\pi\Phi m$, and the potential at any point on the positive side of the shell will exceed that at the neighbouring point on the negative side by $4\pi\Phi$.

If a magnetic shell forms a closed surface, the potential outside the shell is everywhere zero, and that in the space within is everywhere $4\pi\Phi$, being positive when the positive side of the shell is inward. Hence such a shell exerts no action on any magnet placed either outside or inside the shell.

412.] If a magnet can be divided into simple magnetic shells, either closed or having their edges on the surface of the magnet, the distribution of magnetism is called Lamellar. If $\phi$ is the sum of the strengths of all the shells traversed by a point in passing from a given point to a point $(x, y, z)$ by a line drawn within the magnet, then the conditions of lamellar magnetization are

$$A = \frac{d\phi}{dx}, \quad B = \frac{d\phi}{dy}, \quad C = \frac{d\phi}{dz}.$$

The quantity, $\phi$, which thus completely determines the magnetization at any point may be called the Potential of Magnetization. It must be carefully distinguished from the Magnetic Potential.

413.] A magnet which can be divided into complex magnetic shells is said to have a complex lamellar distribution of magnetism. The condition of such a distribution is that the lines of magnetization must be such that a system of surfaces can be drawn cutting them at right angles. This condition is expressed by the well-known equation

$$A\left(\frac{dC}{dy}-\frac{dB}{dz}\right)+B\left(\frac{dA}{dz}-\frac{dC}{dx}\right)+C\left(\frac{dB}{dx}-\frac{dA}{dy}\right)=0.$$

*Forms of the Potentials of Solenoidal and Lamellar Magnets.*

414.] The general expression for the scalar potential of a magnet is

$$V=\iiint\left(A\frac{dp}{dx}+B\frac{dp}{dy}+C\frac{dp}{dz}\right)dx\,dy\,dz,$$

where $p$ denotes the potential at $(x, y, z)$, due to a unit magnetic pole placed at $(\xi, \eta, \zeta)$, or in other words, the reciprocal of the distance between $(\xi, \eta, \zeta)$, the point at which the potential is measured, and $(x, y, z)$, the position of the element of the magnet to which it is due.

This quantity may be integrated by parts, as in Arts. 96, 386,

$$V=\iint p(Al+Bm+Cn)\,dS-\iiint p\left(\frac{dA}{dx}+\frac{dB}{dy}+\frac{dC}{dz}\right)dx\,dy\,dz,$$

where $l, m, n$ are the direction-cosines of the normal drawn outwards from $dS$, an element of the surface of the magnet.

When the magnet is solenoidal the expression under the integral sign in the second term is zero for every point within the magnet, so that the triple integral is zero, and the scalar potential at any point, whether outside or inside the magnet, is given by the surface-integral in the first term.

The scalar potential of a solenoidal magnet is therefore completely determined when the normal component of the magnetization at every point of the surface is known, and it is independent of the form of the solenoids within the magnet.

415.] In the case of a lamellar magnet the magnetization is determined by $\phi$, the potential of magnetization, so that

$$A=\frac{d\phi}{dx},\qquad B=\frac{d\phi}{dy},\qquad C=\frac{d\phi}{dz}.$$

The expression for $V$ may therefore be written

$$V = \iiint \left(\frac{d\phi}{dx}\frac{dp}{dx} + \frac{d\phi}{dy}\frac{dp}{dy} + \frac{d\phi}{dz}\frac{dp}{dz}\right) dx\, dy\, dz.$$

Integrating this expression by parts, we find

$$V = \iint \phi \left(l\frac{dp}{dx} + m\frac{dp}{dy} + n\frac{dp}{dz}\right) dS - \iiint \phi \left(\frac{d^2p}{dx^2} + \frac{d^2p}{dy^2} + \frac{d^2p}{dz^2}\right) dx\, dy\, dz.$$

The second term is zero unless the point $(\xi, \eta, \zeta)$ is included in the magnet, in which case it becomes $4\pi(\phi)$, where $(\phi)$ is the value of $\phi$ at the point $(\xi, \eta, \zeta)$. The surface-integral may be expressed in terms of $r$, the line drawn from $(x, y, z)$ to $(\xi, \eta, \zeta)$, and $\theta$ the angle which this line makes with the normal drawn outwards from $dS$, so that the potential may be written

$$V = \iint \frac{1}{r^2}\phi \cos\theta\, dS + 4\pi(\phi),$$

where the second term is of course zero when the point $(\xi, \eta, \zeta)$ is not included in the substance of the magnet.

The potential, $V$, expressed by this equation, is continuous even at the surface of the magnet, where $\phi$ becomes suddenly zero, for if we write

$$\Omega = \iint \frac{1}{r^2}\phi \cos\theta\, dS,$$

and if $\Omega_1$ is the value of $\Omega$ at a point just within the surface, and $\Omega_2$ that at a point close to the first but outside the surface,

$$\Omega_2 = \Omega_1 + 4\pi(\phi),$$

or

$$V_2 = V_1.$$

The quantity $\Omega$ is not continuous at the surface of the magnet.

The components of magnetic induction are related to $\Omega$ by the equations

$$a = -\frac{d\Omega}{dx}, \quad b = -\frac{d\Omega}{dy}, \quad c = -\frac{d\Omega}{dz}.$$

416.] In the case of a lamellar distribution of magnetism we may also simplify the vector-potential of magnetic induction.

Its $x$-component may be written

$$F = \iiint \left(\frac{d\phi}{dy}\frac{dp}{dz} - \frac{d\phi}{dz}\frac{dp}{dy}\right) dx\, dy\, dz.$$

By integration by parts we may put this in the form of the surface-integral

$$F = \iint \phi \left(m\frac{dp}{dz} - n\frac{dp}{dy}\right) dS,$$

or

$$F = -\iint p \left(m\frac{d\phi}{dz} - n\frac{d\phi}{dy}\right) dS.$$

The other components of the vector-potential may be written down from these expressions by making the proper substitutions.

*On Solid Angles.*

417.] We have already proved that at any point $P$ the potential due to a magnetic shell is equal to the solid angle subtended by the edge of the shell multiplied by the strength of the shell. As we shall have occasion to refer to solid angles in the theory of electric currents, we shall now explain how they may be measured.

*Definition.* The solid angle subtended at a given point by a closed curve is measured by the area of a spherical surface whose centre is the given point and whose radius is unity, the outline of which is traced by the intersection of the radius vector with the sphere as it traces the closed curve. This area is to be reckoned positive or negative according as it lies on the left or the right-hand of the path of the radius vector as seen from the given point *.

Let $(\xi, \eta, \zeta)$ be the given point, and let $(x, y, z)$ be a point on the closed curve. The coordinates $x, y, z$ are functions of $s$, the length of the curve reckoned from a given point. They are periodic functions of $s$, recurring whenever $s$ is increased by the whole length of the closed curve.

We may calculate the solid angle $\omega$ directly from the definition thus. Using spherical coordinates with centre at $(\xi, \eta, \zeta)$, and putting

$$x-\xi = r\sin\theta\cos\phi, \quad y-\eta = r\sin\theta\sin\phi, \quad z-\zeta = r\cos\theta,$$

we find the area of any curve on the sphere by integrating

$$\omega = \int (1-\cos\theta)\, d\phi,$$

or, using the rectangular coordinates,

$$\omega = \int d\phi - \int_0^s \frac{z-\zeta}{r\{(x-\xi)^2+(y-\eta)^2\}}\left[(x-\xi)\frac{dy}{ds} - (y-\eta)\frac{dx}{ds}\right] ds,$$

the integration being extended round the curve $s$.

If the axis of $z$ passes once through the closed curve the first

* {If, while the point at which the solid angle subtended by a given curve is to be determined moves about, we suppose the extremity of the radius vector always to travel round the curve in the same direction, then the area on the sphere may be taken as positive if it is on that side of the sphere where the motion of the end of the radius vector looks clockwise when seen from the centre, negative if it is on the other side.}

term is $2\pi$. If the axis of $z$ does not pass through it this term is zero.

418.] This method of calculating a solid angle involves a choice of axes which is to some extent arbitrary, and it does not depend solely on the closed curve. Hence the following method, in which no surface is supposed to be constructed, may be stated for the sake of geometrical propriety.

As the radius vector from the given point traces out the closed curve, let a plane passing through the given point roll on the closed curve so as to be a tangent plane at each point of the curve in succession. Let a line of unit-length be drawn from the given point perpendicular to this plane. As the plane rolls round the closed curve the extremity of the perpendicular will trace a second closed curve. Let the length of the second closed curve be $\sigma$, then the solid angle subtended by the first closed curve is

$$\omega = 2\pi - \sigma.$$

This follows from the well-known theorem that the area of a closed curve on a sphere of unit radius, together with the circumference of the polar curve, is numerically equal to the circumference of a great circle of the sphere.

This construction is sometimes convenient for calculating the solid angle subtended by a rectilinear figure. For our own purpose, which is to form clear ideas of physical phenomena, the following method is to be preferred, as it employs no constructions which do not flow from the physical data of the problem.

419.] A closed curve $s$ is given in space, and we have to find the solid angle subtended by $s$ at a given point $P$.

If we consider the solid angle as the potential of a magnetic shell of unit strength whose edge coincides with the closed curve, we must define it as the work done by a unit magnetic pole against the magnetic force while it moves from an infinite distance to the point $P$. Hence, if $\sigma$ is the path of the pole as it approaches the point $P$, the potential must be the result of a line-integration along this path. It must also be the result of a line-integration along the closed curve $s$. The proper form of the expression for the solid angle must therefore be that of a double integration with respect to the two curves $s$ and $\sigma$.

When $P$ is at an infinite distance, the solid angle is evidently zero. As the point $P$ approaches, the closed curve, as seen from the moving point, appears to open out, and the whole solid angle may be conceived to be generated by the apparent motion of the different elements of the closed curve as the moving point approaches.

As the point $P$ moves from $P$ to $P'$ over the element $d\sigma$, the element $QQ'$ of the closed curve, which we denote by $ds$, will change its position relatively to $P$, and the line on the unit sphere corresponding to $QQ'$ will sweep over an area on the spherical surface, which we may write

$$d\omega = \Pi \, ds \, d\sigma. \tag{1}$$

To find $\Pi$ let us suppose $P$ fixed while the closed curve is moved parallel to itself through a distance $d\sigma$ equal to $PP'$ but in the opposite direction. The relative motion of the point $P$ will be the same as in the real case.

During this motion the element $QQ'$ will generate an area in the form of a parallelogram whose sides are parallel and equal to $QQ'$ and $PP'$. If we construct a pyramid on this parallelogram as base with its vertex at $P$, the solid angle of this pyramid will be the increment $d\omega$ which we are in search of.

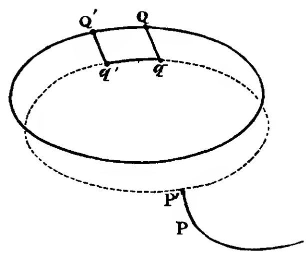

Fig. 3.

To determine the value of this solid angle, let $\theta$ and $\theta'$ be the angles which $ds$ and $d\sigma$ make with $PQ$ respectively, and let $\phi$ be the angle between the planes of these two angles, then the area of the projection of the parallelogram $ds \, . \, d\sigma$ on a plane perpendicular to $PQ$ or $r$ will be

$$ds \, d\sigma \sin\theta \sin\theta' \sin\phi,$$

and since this is equal to $r^2 d\omega$, we find

$$d\omega = \Pi \, ds \, d\sigma = \frac{1}{r^2} \sin\theta \sin\theta' \sin\phi \, ds \, d\sigma. \tag{2}$$

Hence $$\Pi = \frac{1}{r^2} \sin\theta \sin\theta' \sin\phi. \tag{3}$$

420.] We may express the angles $\theta$, $\theta'$, and $\phi$ in terms of $r$, and its differential coefficients with respect to $s$ and $\sigma$, for

$$\cos\theta = \frac{dr}{ds}, \quad \cos\theta' = \frac{dr}{d\sigma}, \quad \text{and} \quad \sin\theta \sin\theta' \cos\phi = r \frac{d^2 r}{ds \, d\sigma}. \tag{4}$$

We thus find the following value for $\Pi^2$,

$$\Pi^2 = \frac{1}{r^4}\left[1-\left(\frac{dr}{ds}\right)^2\right]\left[1-\left(\frac{dr}{d\sigma}\right)^2\right] - \frac{1}{r^2}\left(\frac{d^2r}{ds\,d\sigma}\right)^2. \tag{5}$$

A third expression for $\Pi$ in terms of rectangular coordinates may be deduced from the consideration that the volume of the pyramid whose solid angle is $d\omega$ and whose side is $r$ is

$$\tfrac{1}{3}r^3 d\omega = \tfrac{1}{3}r^3 \Pi\, ds\, d\sigma.$$

But the volume of this pyramid may also be expressed in terms of the projections of $r$, $ds$, and $d\sigma$ on the axes of $x$, $y$ and $z$, as a determinant formed by these nine projections, of which we must take the third part. We thus find as the value of $\Pi$,*

$$\Pi = -\frac{1}{r^3}\begin{vmatrix} \xi - x, & \eta - y, & \zeta - z, \\ \frac{d\xi}{d\sigma}, & \frac{d\eta}{d\sigma}, & \frac{d\zeta}{d\sigma}, \\ \frac{dx}{ds}, & \frac{dy}{ds}, & \frac{dz}{ds}. \end{vmatrix} \tag{6}$$

This expression gives the value of $\Pi$ free from the ambiguity of sign introduced by equation (5).

421.] The value of $\omega$, the solid angle subtended by the closed curve at the point $P$, may now be written

$$\omega = \iint \Pi\, ds\, d\sigma + \omega_0, \tag{7}$$

where the integration with respect to $s$ is to be extended completely round the closed curve, and that with respect to $\sigma$ from $A$ a fixed point on the curve to the point $P$. The constant $\omega_0$ is the value of the solid angle at the point $A$. It is zero if $A$ is at an infinite distance from the closed curve.

The value of $\omega$ at any point $P$ is independent of the form of the curve between $A$ and $P$ provided that it does not pass through the magnetic shell itself. If the shell be supposed infinitely thin, and if $P$ and $P'$ are two points close together, but $P$ on the positive and $P'$ on the negative surface of the shell, then the curves $AP$ and $AP'$ must lie on opposite sides of the edge of the shell, so that $PAP'$ is a line which with the infinitely short line $P'P$ forms a closed circuit embracing the

* {The sign of $\Pi$ is most easily got by considering a simple case, that of a circular disk magnetized at right angles to its plane is very convenient for this purpose.}

edge. The value of $\omega$ at $P$ exceeds that at $P'$ by $4\pi$, that is, by the surface of a sphere of radius unity.

Hence, if a closed curve be drawn so as to pass once through the shell, or in other words, if it be linked once with the edge of the shell, the value of the integral $\int\int \Pi\, ds\, d\sigma$ extended round both curves will be $4\pi$.

This integral therefore, considered as depending only on the closed curve $s$ and the arbitrary curve $AP$, is an instance of a function of multiple values, since, if we pass from $A$ to $P$ along different paths the integral will have different values according to the number of times which the curve $AP$ is twined round the curve $s$.

If one form of the curve between $A$ and $P$ can be transformed into another by continuous motion without intersecting the curve $s$, the integral will have the same value for both curves, but if during the transformation it intersects the closed curve $n$ times the values of the integral will differ by $4\pi n$.

If $s$ and $\sigma$ are any two closed curves in space, then, if they are not linked together, the integral extended once round both is zero.

If they are intertwined $n$ times in the same direction, the value of the integral is $4\pi n$. It is possible, however, for two curves to be intertwined alternately in opposite directions, so that they are inseparably linked together though the value of the integral is zero. See Fig. 4.

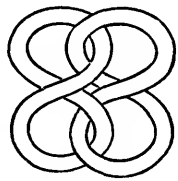

Fig. 4.

It was the discovery by Gauss of this very integral, expressing the work done on a magnetic pole while describing a closed curve in presence of a closed electric current, and indicating the geometrical connexion between the two closed curves, that led him to lament the small progress made in the Geometry of Position since the time of Leibnitz, Euler and Vandermonde. We have now, however, some progress to report, chiefly due to Riemann, Helmholtz, and Listing.

422.] Let us now investigate the result of integrating with respect to $s$ round the closed curve.

One of the terms of $\Pi$ in equation (7) is

$$-\frac{\xi - x}{r^3}\frac{d\eta}{d\sigma}\frac{dz}{ds} = \frac{d\eta}{d\sigma}\frac{d}{d\xi}\left(\frac{1}{r}\frac{dz}{ds}\right). \tag{8}$$

If we now write for brevity

$$F=\int\frac{1}{r}\frac{dx}{ds}ds, \quad G=\int\frac{1}{r}\frac{dy}{ds}ds, \quad H=\int\frac{1}{r}\frac{dz}{ds}ds, \tag{9}$$

the integrals being taken once round the closed curve $s$, this term of $\Pi$ may be written

$$\frac{d\eta}{d\sigma}\frac{d^2H}{d\xi\,ds},$$

and the corresponding term of $\int \Pi\, ds$ will be

$$\frac{d\eta}{d\sigma}\frac{dH}{d\xi}.$$

Collecting all the terms of $\Pi$, we may now write

$$-\frac{d\omega}{d\sigma}=-\int \Pi\, ds$$

$$=\left(\frac{dH}{d\eta}-\frac{dG}{d\zeta}\right)\frac{d\xi}{d\sigma}+\left(\frac{dF}{d\zeta}-\frac{dH}{d\xi}\right)\frac{d\eta}{d\sigma}+\left(\frac{dG}{d\xi}-\frac{dF}{d\eta}\right)\frac{d\zeta}{d\sigma}. \tag{10}$$

This quantity is evidently the rate of decrement of $\omega$, the magnetic potential, in passing along the curve $\sigma$, or in other words, it is the magnetic force in the direction of $d\sigma$.

By assuming $d\sigma$ successively in the direction of the axes of $x$, $y$ and $z$, we obtain for the values of the components of the magnetic force

$$\left.\begin{aligned}
\alpha &= -\frac{d\omega}{d\xi}=\frac{dH}{d\eta}-\frac{dG}{d\zeta},\\
\beta &= -\frac{d\omega}{d\eta}=\frac{dF}{d\zeta}-\frac{dH}{d\xi},\\
\gamma &= -\frac{d\omega}{d\zeta}=\frac{dG}{d\xi}-\frac{dF}{d\eta}.
\end{aligned}\right\} \tag{11}$$

The quantities $F$, $G$, $H$ are the components of the vector-potential of the magnetic shell whose strength is unity, and whose edge is the curve $s$. They are not, like the scalar potential $\omega$, functions having a series of values, but are perfectly determinate for every point in space.

The vector-potential at a point $P$ due to a magnetic shell bounded by a closed curve may be found by the following geometrical construction:

Let a point $Q$ travel round the closed curve with a velocity numerically equal to its distance from $P$, and let a second point

$R$ start from a fixed point $A$ and travel with a velocity the direction of which is always parallel to that of $Q$, but whose magnitude is unity. When $Q$ has travelled once round the closed curve join $AR$, then the line $AR$ represents in direction and in numerical magnitude the vector-potential due to the closed curve at $P$.

*Potential Energy of a Magnetic Shell placed in a Magnetic Field.*

423.] We have already shewn, in Art. 410, that the potential energy of a shell of strength $\phi$ placed in a magnetic field whose potential is $V$, is

$$M = \phi \iint \left(l\frac{dV}{dx} + m\frac{dV}{dy} + n\frac{dV}{dz}\right) dS, \qquad (12)$$

where $l$, $m$, $n$ are the direction-cosines of the normal to the shell drawn outwards from the positive side, and the surface-integral is extended over the shell.

Now this surface-integral may be transformed into a line-integral by means of the vector-potential of the magnetic field, and we may write

$$M = -\phi \int \left(F\frac{dx}{ds} + G\frac{dy}{ds} + H\frac{dz}{ds}\right) ds, \qquad (13)$$

where the integration is extended once round the closed curve $s$ which forms the edge of the magnetic shell, the direction of $ds$ being opposite to that of the hands of a watch when viewed from the positive side of the shell.

If we now suppose that the magnetic field is that due to a second magnetic shell whose strength is $\phi'$, we may determine the value of $F$ directly from the results of Art. 416 or from Art. 405. If $l'$, $m'$, $n'$ be the direction-cosines of the normal to the element $dS'$ of the second shell, we have

$$F = \phi' \iint \left(m'\frac{d}{dz'}\frac{1}{r} - n'\frac{d}{dy'}\frac{1}{r}\right) dS',$$

where $r$ is the distance between the element $dS'$ and a point on the boundary of the first shell.

Now this surface-integral may be converted into a line-integral round the boundary of the second shell; viz. it is

$$\phi' \int \frac{1}{r}\frac{dx'}{ds'} ds'. \qquad (14)$$

In like manner

$$G = \phi' \int \frac{1}{r} \frac{dy'}{ds'} ds',$$

$$H = \phi' \int \frac{1}{r} \frac{dz'}{ds'} ds'.$$

Substituting these values in the expression for $M$ we find

$$M = -\phi\phi' \iint \frac{1}{r} \left( \frac{dx}{ds} \frac{dx'}{ds'} + \frac{dy}{ds} \frac{dy'}{ds'} + \frac{dz}{ds} \frac{dz'}{ds'} \right) ds ds', \tag{15}$$

where the integration is extended once round $s$ and once round $s'$. This expression gives the potential energy due to the mutual action of the two shells, and is, as it ought to be, the same when $s$ and $s'$ are interchanged. This expression with its sign reversed, when the strength of each shell is unity, is called the potential of the two closed curves $s$ and $s'$. It is a quantity of great importance in the theory of electric currents. If we write $\epsilon$ for the angle between the directions of the elements $ds$ and $ds'$, the potential of $s$ and $s'$ may be written

$$\iint \frac{\cos \epsilon}{r} ds ds'. \tag{16}$$

It is evidently a quantity of the dimension of a line.

# CHAPTER IV.

## INDUCED MAGNETIZATION.

424.] We have hitherto considered the actual distribution of magnetization in a magnet as given explicitly among the data of the investigation. We have not made any assumption as to whether this magnetization is permanent or temporary, except in those parts of our reasoning in which we have supposed the magnet broken up into small portions, or small portions removed from the magnet in such a way as not to alter the magnetization of any part.

We have now to consider the magnetization of bodies with respect to the mode in which it may be produced and changed. A bar of iron held parallel to the direction of the earth's magnetic force is found to become magnetic, with its poles turned the opposite way from those of the earth, or the same way as those of a compass needle in stable equilibrium.

Any piece of soft iron placed in a magnetic field is found to exhibit magnetic properties. If it be placed in a part of the field where the magnetic force is great, as between the poles of a horse-shoe magnet, the magnetism of the iron becomes intense. If the iron is removed from the magnetic field, its magnetic properties are greatly weakened or disappear entirely. If the magnetic properties of the iron depend entirely on the magnetic force of the field in which it is placed, and vanish when it is removed from the field, it is called Soft iron. Iron which is soft in the magnetic sense is also soft in the literal sense. It is easy to bend it and give it a permanent set, and difficult to break it.

Iron which retains its magnetic properties when removed from the magnetic field is called Hard iron. Such iron does not take up the magnetic state so readily as soft iron. The operation of hammering, or any other kind of vibration, allows hard iron under the influence of magnetic force to assume the magnetic state more readily, and to part with it more readily when the magnetizing force is removed *. Iron which is magnetically hard is also more stiff to bend and more apt to break.

The processes of hammering, rolling, wire-drawing, and sudden cooling tend to harden iron, and that of annealing tends to soften it.

The magnetic as well as the mechanical differences between steel of hard and soft temper are much greater than those between hard and soft iron. Soft steel is almost as easily magnetized and demagnetized as iron, while the hardest steel is the best material for magnets which we wish to be permanent.

Cast iron, though it contains more carbon than steel, is not so retentive of magnetization.

If a magnet could be constructed so that the distribution of its magnetization is not altered by any magnetic force brought to act upon it, it might be called a rigidly magnetized body. The only known body which fulfils this condition is a conducting circuit round which a constant electric current is made to flow.

Such a circuit exhibits magnetic properties, and may therefore be called an electromagnet, but these magnetic properties are not affected by the other magnetic forces in the field. We shall return to this subject in Part IV.

All actual magnets, whether made of hardened steel or of loadstone, are found to be affected by any magnetic force which is brought to bear upon them.

It is convenient, for scientific purposes, to make a distinction between the permanent and the temporary magnetization, defining the permanent magnetization as that which exists independently of the magnetic force, and the temporary magnetization as that which depends on this force. We must observe, however, that this distinction is not founded on a knowledge of the intimate nature of the magnetizable substances: it is only the expression of an hypothesis introduced for the sake of bringing calculation to bear on the phenomena. We shall return to the physical theory of magnetization in Chapter VI.

* {Ewing (*Phil. Trans.*, Part ii. 1885) has shewn that soft iron free from vibrations and demagnetizing forces can retain a larger proportion of its magnetism than the hardest steel.}

425.] At present we shall investigate the temporary magnetization on the assumption that the magnetization of any particle of the substance depends solely on the magnetic force acting on that particle. This magnetic force may arise partly from external causes, and partly from the temporary magnetization of neighbouring particles.

A body thus magnetized in virtue of the action of magnetic force is said to be magnetized by induction, and the magnetization is said to be induced by the magnetizing force.

The magnetization induced by a given magnetizing force differs in different substances. It is greatest in the purest and softest iron, in which the ratio of the magnetization to the magnetic force may reach the value 32, or even 45*.

Other substances, such as the metals nickel and cobalt, are capable of an inferior degree of magnetization, and all substances when subjected to a sufficiently strong magnetic force are found to give indications of polarity.

When the magnetization is in the same direction as the magnetic force, as in iron, nickel, cobalt, &c., the substance is called Paramagnetic, Ferromagnetic, or more simply Magnetic. When the induced magnetization is in the direction opposite to the magnetic force, as in bismuth, &c., the substance is said to be Diamagnetic.

In all these diamagnetic substances the ratio of the magnetization to the magnetic force which produces it is exceedingly small, being only about $-\frac{1}{400000}$ in the case of bismuth, which is the most highly diamagnetic substance known.

In crystallized, strained, and organized substances the direction of the magnetization does not always coincide with that of the magnetic force which produces it. The relation between the components of magnetization, referred to axes fixed in the body, and those of the magnetic force, may be expressed by a system of three linear equations. Of the nine coefficients involved in these equations we shall shew that only six are independent. The phenomena of bodies of this kind are classed under the name of Magnecrystallic phenomena.

When placed in a field of magnetic force, crystals tend to set

* Thalén, *Nova Acta, Reg. Soc. Sc.*, Upsal, 1863. {Ewing (*loc. cit.*) has shewn that it may be as great as 279, and that if the wire be shaken while the magnetizing force is applied it may rise to as much as 1600.}

themselves so that the axis of greatest paramagnetic, or of least diamagnetic, induction is parallel to the lines of magnetic force. See Art. 436.

In soft iron, the direction of the magnetization coincides with that of the magnetic force at the point, and for small values of the magnetic force the magnetization is nearly proportional to it*. As the magnetic force increases, however, the magnetization increases more slowly, and it would appear from experiments described in Chap. VI, that there is a limiting value of the magnetization, beyond which it cannot pass, whatever be the value of the magnetic force.

In the following outline of the theory of induced magnetism, we shall begin by supposing the magnetization proportional to the magnetic force, and in the same line with it.

*Definition of the Coefficient of Induced Magnetization.*

426.] Let $\mathfrak{H}$ be the magnetic force, defined as in Art. 398, at any point of the body, and let $\mathfrak{I}$ be the magnetization at that point, then the ratio of $\mathfrak{I}$ to $\mathfrak{H}$ is called the Coefficient of Induced Magnetization.

Denoting this coefficient by $\kappa$, the fundamental equation of induced magnetism is

$$\mathfrak{I} = \kappa \mathfrak{H}. \tag{1}$$

The coefficient $\kappa$ is positive for iron and paramagnetic substances, and negative for bismuth and diamagnetic substances. It reaches the value {1600} in iron, and it is said to be large in the case of nickel and cobalt, but in all other cases it is a very small quantity, not greater than 0·00001.

The force $\mathfrak{H}$ arises partly from the action of magnets external to the body magnetized by induction, and partly from the induced magnetization of the body itself. Both parts satisfy the condition of having a potential.

427.] Let $V$ be the potential due to magnetism external to the body, and let $\Omega$ be that due to the induced magnetization, then if $U$ is the actual potential due to both causes

$$U = V + \Omega. \tag{2}$$

Let the components of the magnetic force $\mathfrak{H}$, resolved in the

* {Lord Rayleigh, *Phil. Mag.* 23, p. 225, 1887, has shewn that when the magnetizing force is less than $\frac{1}{10}$ of the earth's horizontal magnetic force, the magnetization is proportional to the magnetizing force, and that it ceases to be so when the force is greater.}

directions of $x$, $y$, $z$, be $\alpha$, $\beta$, $\gamma$, and let those of the magnetization $\mathfrak{I}$ be $A$, $B$, $C$, then by equation (1),

$$\left.\begin{aligned} A &= \kappa\alpha, \\ B &= \kappa\beta, \\ C &= \kappa\gamma. \end{aligned}\right\} \tag{3}$$

Multiplying these equations by $dx$, $dy$, $dz$ respectively, and adding, we find

$$A\,dx + B\,dy + C\,dz = \kappa\,(\alpha\,dx + \beta\,dy + \gamma\,dz).$$

But since $\alpha$, $\beta$ and $\gamma$ are derived from the potential $U$, we may write the second member $-\kappa\,dU$.

Hence, if $\kappa$ is constant throughout the substance, the first member must also be a complete differential of a function of $x$, $y$ and $z$, which we shall call $\phi$, and the equation becomes

$$d\phi = -\kappa\,dU, \tag{4}$$

where $$A = \frac{d\phi}{dx}, \quad B = \frac{d\phi}{dy}, \quad C = \frac{d\phi}{dz}. \tag{5}$$

The magnetization is therefore lamellar, as defined in Art. 412.

It was shewn in Art. 385 that if $\rho$ is the volume-density of free magnetism,

$$\rho = -\left(\frac{dA}{dx} + \frac{dB}{dy} + \frac{dC}{dz}\right),$$

which becomes in virtue of equations (3),

$$\rho = -\kappa\left(\frac{d\alpha}{dx} + \frac{d\beta}{dy} + \frac{d\gamma}{dz}\right).$$

But, by Art. 77,

$$\frac{d\alpha}{dx} + \frac{d\beta}{dy} + \frac{d\gamma}{dz} = -4\pi\rho.$$

Hence $$(1 + 4\pi\kappa)\,\rho = 0,$$

whence $$\rho = 0 \tag{6}$$

throughout the substance, and the magnetization is therefore solenoidal as well as lamellar. See Art. 407.

There is therefore no free magnetism except on the bounding surface of the body. If $\nu$ be the normal drawn inwards from the surface, the magnetic surface-density is

$$\sigma = -\frac{d\phi}{d\nu}. \tag{7}$$

The potential $\Omega$ due to this magnetization at any point may therefore be found from the surface-integral

$$\Omega = \iint \frac{\sigma}{r} dS. \tag{8}$$

The value of $\Omega$ will be finite and continuous everywhere, and will satisfy Laplace's equation at every point both within and without the surface. If we distinguish by an accent the value of $\Omega$ outside the surface, and if $\nu'$ be the normal drawn outwards, we have at the surface

$$\Omega' = \Omega; \tag{9}$$

$$\begin{aligned}\frac{d\Omega}{d\nu} + \frac{d\Omega'}{d\nu'} &= -4\pi\sigma, \text{ by Art. } 78b,\\ &= 4\pi\frac{d\phi}{d\nu}, \text{ by } (7),\\ &= -4\pi\kappa\frac{dU}{d\nu}, \text{ by } (4),\\ &= -4\pi\kappa\left(\frac{dV}{d\nu} + \frac{d\Omega}{d\nu}\right), \text{ by } (2).\end{aligned}$$

We may therefore write the second surface-condition

$$(1+4\pi\kappa)\frac{d\Omega}{d\nu} + \frac{d\Omega'}{d\nu'} + 4\pi\kappa\frac{dV}{d\nu} = 0. \tag{10}$$

Hence the determination of the magnetism induced in a homogeneous isotropic body, bounded by a surface $S$, and acted upon by external magnetic forces whose potential is $V$, may be reduced to the following mathematical problem.

We must find two functions $\Omega$ and $\Omega'$ satisfying the following conditions:

Within the surface $S$, $\Omega$ must be finite and continuous, and must satisfy Laplace's equation.

Outside the surface $S$, $\Omega'$ must be finite and continuous, it must vanish at an infinite distance, and must satisfy Laplace's equation.

At every point of the surface itself, $\Omega = \Omega'$, and the derivatives of $\Omega$, $\Omega'$ and $V$ with respect to the normal must satisfy equation (10).

This method of treating the problem of induced magnetism is due to Poisson. The quantity $k$ which he uses in his memoirs is not the same as $\kappa$, but is related to it as follows:

$$4\pi\kappa(k-1) + 3k = 0. \tag{11}$$

The coefficient $\kappa$ which we have here used was introduced by F. E. Neumann.

428.] The problem of induced magnetism may be treated in a different manner by introducing the quantity which we have called, with Faraday, the Magnetic Induction.

The relation between $\mathfrak{B}$, the magnetic induction, $\mathfrak{H}$, the magnetic force, and $\mathfrak{I}$, the magnetization, is expressed by the equation

$$\mathfrak{B} = \mathfrak{H} + 4\pi\mathfrak{I}. \tag{12}$$

The equation which expresses the induced magnetization in terms of the magnetic force is

$$\mathfrak{I} = \kappa\mathfrak{H}. \tag{13}$$

Hence, eliminating $\mathfrak{I}$, we find

$$\mathfrak{B} = (1 + 4\pi\kappa)\mathfrak{H} \tag{14}$$

as the relation between the magnetic induction and the magnetic force in substances whose magnetization is induced by magnetic force.

In the most general case $\kappa$ may be a function, not only of the position of the point in the substance, but of the direction of the vector $\mathfrak{H}$, but in the case which we are now considering $\kappa$ is a numerical quantity.

If we next write

$$\mu = 1 + 4\pi\kappa, \tag{15}$$

we may define $\mu$ as the ratio of the magnetic induction to the magnetic force, and we may call this ratio the magnetic inductive capacity of the substance, thus distinguishing it from $\kappa$, the coefficient of induced magnetization.

If we write $U$ for the total magnetic potential compounded of $V$, the potential due to external causes, and $\Omega$ that due to the induced magnetization, we may express $a$, $b$, $c$, the components of magnetic induction, and $\alpha$, $\beta$, $\gamma$, the components of magnetic force, as follows:

$$\left.\begin{aligned} a &= \mu\alpha = -\mu\frac{dU}{dx}, \\ b &= \mu\beta = -\mu\frac{dU}{dy}, \\ c &= \mu\gamma = -\mu\frac{dU}{dz}. \end{aligned}\right\} \tag{16}$$

The components $a$, $b$, $c$ satisfy the solenoidal condition

$$\frac{da}{dx} + \frac{db}{dy} + \frac{dc}{dz} = 0. \tag{17}$$

Hence, the potential $U$ must satisfy Laplace's equation

$$\frac{d^2 U}{dx^2} + \frac{d^2 U}{dy^2} + \frac{d^2 U}{dz^2} = 0 \tag{18}$$

at every point where $\mu$ is constant, that is, at every point within the homogeneous substance, or in empty space.

At the surface itself, if $\nu$ is a normal drawn towards the magnetic substance, and $\nu'$ one drawn outwards, and if the symbols of quantities outside the substance are distinguished by accents, the condition of continuity of the magnetic induction is

$$a\frac{dx}{d\nu} + b\frac{dy}{d\nu} + c\frac{dz}{d\nu} + a'\frac{dx}{d\nu'} + b'\frac{dy}{d\nu'} + c'\frac{dz}{d\nu'} = 0\,; \tag{19}$$

or, by equations (16),

$$\mu\frac{dU}{d\nu} + \mu'\frac{dU'}{d\nu'} = 0. \tag{20}$$

$\mu'$, the coefficient of induction outside the magnet, will be unity unless the surrounding medium be magnetic or diamagnetic.

If we substitute for $U$ its value in terms of $V$ and $\Omega$, and for $\mu$ its value in terms of $\kappa$, we obtain the same equation (10) as we arrived at by Poisson's method.

The problem of induced magnetism, when considered with respect to the relation between magnetic induction and magnetic force, corresponds exactly with the problem of the conduction of electric currents through heterogeneous media, as given in Art. 310.

The magnetic force is derived from the magnetic potential, precisely as the electric force is derived from the electric potential.

The magnetic induction is a quantity of the nature of a flux, and satisfies the same conditions of continuity as the electric current does.

In isotropic media the magnetic induction depends on the magnetic force in a manner which exactly corresponds with that in which the electric current depends on the electromotive force.

The specific magnetic inductive capacity in the one problem corresponds to the specific conductivity in the other. Hence Thomson, in his *Theory of Induced Magnetism* (*Reprint*, 1872, p. 484), has called this quantity the *permeability* of the medium.

We are now prepared to consider the theory of induced magnetism from what I conceive to be Faraday's point of view.

When magnetic force acts on any medium, whether magnetic or diamagnetic, or neutral, it produces within it a phenomenon called Magnetic Induction.

Magnetic induction is a directed quantity of the nature of a flux, and it satisfies the same conditions of continuity as electric currents and other fluxes do.

In isotropic media the magnetic force and the magnetic induction are in the same direction, and the magnetic induction is the product of the magnetic force into a quantity called the coefficient of induction, which we have expressed by $\mu$.

In empty space the coefficient of induction is unity. In bodies capable of induced magnetization the coefficient of induction is $1+4\pi\kappa = \mu$, where $\kappa$ is the quantity already defined as the coefficient of induced magnetization.

429.] Let $\mu, \mu'$ be the values of $\mu$ on opposite sides of a surface separating two media, then if $V$, $V'$ are the potentials in the two media, the magnetic forces towards the surface in the two media are $\dfrac{dV}{d\nu}$ and $\dfrac{dV'}{d\nu'}$.

The quantities of magnetic induction through the element of surface $dS$ are $\mu\dfrac{dV}{d\nu}dS$ and $\mu'\dfrac{dV'}{d\nu'}dS$ in the two media respectively reckoned towards $dS$.

Since the total flux towards $dS$ is zero,

$$\mu\frac{dV}{d\nu}+\mu'\frac{dV'}{d\nu'} = 0.$$

But by the theory of the potential near a surface of density $\sigma$,

$$\frac{dV}{d\nu}+\frac{dV'}{d\nu'}+4\pi\sigma = 0.$$

Hence
$$\frac{dV}{d\nu}\left(1-\frac{\mu}{\mu'}\right)+4\pi\sigma = 0.$$

If $\kappa_1$ is the ratio of the superficial magnetization to the normal force in the first medium whose coefficient is $\mu$, we have

$$4\pi\kappa_1 = \frac{\mu-\mu'}{\mu'}.$$

Hence $\kappa_1$ will be positive or negative according as $\mu$ is greater or less than $\mu'$. If we put $\mu = 4\pi\kappa+1$ and $\mu' = 4\pi\kappa'+1$,

$$\kappa_1 = \frac{\kappa-\kappa'}{4\pi\kappa'+1}.$$

In this expression $\kappa$ and $\kappa'$ are the coefficients of induced magnetization of the first and second media deduced from experiments made in air, and $\kappa_1$ is the coefficient of induced magnetization of the first medium when surrounded by the second medium.

If $\kappa'$ is greater than $\kappa$, then $\kappa_1$ is negative, or the apparent magnetization of the first medium is in the opposite direction to the magnetizing force.

Thus, if a vessel containing a weak aqueous solution of a paramagnetic salt of iron is suspended in a stronger solution of the same salt, and acted on by a magnet, the vessel moves as if it were magnetized in the opposite direction from that in which a magnet would set itself if suspended in the same place.

This may be explained by the hypothesis that the solution in the vessel is really magnetized in the same direction as the magnetic force, but that the solution which surrounds the vessel is magnetized more strongly in the same direction. Hence the vessel is like a weak magnet placed between two strong ones all magnetized in the same direction, so that opposite poles are in contact. The north pole of the weak magnet points in the same direction as those of the strong ones, but since it is in contact with the south pole of a stronger magnet, there is an excess of south magnetism in the neighbourhood of its north pole, which causes the weak magnet to appear oppositely magnetized.

In some substances, however, the apparent magnetization is negative even when they are suspended in what is called a vacuum.

If we assume $\kappa = 0$ for a vacuum, it will be negative for these substances. No substance, however, has been discovered for which $\kappa$ has a negative value numerically greater than $\frac{1}{4\pi}$, and therefore for all known substances $\mu$ is positive.

Substances for which $\kappa$ is negative, and therefore $\mu$ less than unity, are called Diamagnetic substances. Those for which $\kappa$ is positive, and $\mu$ greater than unity, are called Paramagnetic, Ferromagnetic, or simply magnetic, substances.

We shall consider the physical theory of the diamagnetic and paramagnetic properties when we come to electromagnetism, Arts. 832–845.

430.] The mathematical theory of magnetic induction was first given by Poisson *. The physical hypothesis on which he founded his theory was that of two magnetic fluids, an hypothesis which has the same mathematical advantages and physical difficulties as the theory of two electric fluids. In order, however, to explain the fact that, though a piece of soft iron can be magnetized by induction, it cannot be charged with unequal quantities of the two kinds of magnetism, he supposes that the substance in general is a non-conductor of these fluids, and that only certain small portions of the substance contain the fluids under circumstances in which they are free to obey the forces which act on them. These small magnetic elements of the substance contain each precisely equal quantities of the two fluids, and within each element the fluids move with perfect freedom, but the fluids can never pass from one magnetic element to another.

The problem therefore is of the same kind as that relating to a number of small conductors of electricity disseminated through a dielectric insulating medium. The conductors may be of any form provided they are small and do not touch each other.

If they are elongated bodies all turned in the same general direction, or if they are crowded more in one direction than another, the medium, as Poisson himself shews, will not be isotropic. Poisson therefore, to avoid useless intricacy, examines the case in which each magnetic element is spherical, and the elements are disseminated without regard to axes. He supposes that the whole volume of all the magnetic elements in unit of volume of the substance is $k$.

We have already considered in Art. 314 the electric conductivity of a medium in which small spheres of another medium are distributed.

If the conductivity of the medium is $\mu_1$, and that of the spheres $\mu_2$, we have found that the conductivity of the composite system is

$$\mu = \mu_1 \frac{2\mu_1 + \mu_2 + 2k(\mu_2 - \mu_1)}{2\mu_1 + \mu_2 - k(\mu_2 - \mu_1)}.$$

Putting $\mu_1 = 1$ and $\mu_2 = \infty$, this becomes

$$\mu = \frac{1+2k}{1-k}.$$

* *Mémoires de l'Institut*, 1824, p. 247.

This quantity $\mu$ is the electric conductivity of a medium consisting of perfectly conducting spheres disseminated through a medium of conductivity unity, the aggregate volume of the spheres in unit of volume being $k$.

The symbol $\mu$ also represents the coefficient of magnetic induction of a medium, consisting of spheres for which the permeability is infinite, disseminated through a medium for which it is unity.

The symbol $k$, which we shall call Poisson's Magnetic Coefficient, represents the ratio of the volume of the magnetic elements to the whole volume of the substance.

The symbol $\kappa$ is known as Neumann's Coefficient of Magnetization by Induction. It is more convenient than Poisson's.

The symbol $\mu$ we shall call the Coefficient of Magnetic Induction. Its advantage is that it facilitates the transformation of magnetic problems into problems relating to electricity and heat.

The relations of these three symbols are as follows:

$$k = \frac{4\pi\kappa}{4\pi\kappa+3}, \qquad k = \frac{\mu-1}{\mu+2},$$

$$\kappa = \frac{\mu-1}{4\pi}, \qquad \kappa = \frac{3k}{4\pi(1-k)},$$

$$\mu = \frac{1+2k}{1-k}, \qquad \mu = 4\pi\kappa+1.$$

If we put $\kappa = 32$, the value given by Thalén's * experiments on soft iron, we find $k = \frac{134}{135}$. This, according to Poisson's theory, is the ratio of the volume of the magnetic molecules to the whole volume of the iron. It is impossible to pack a space with equal spheres so that the ratio of their volume to the whole space shall be so nearly unity, and it is exceedingly improbable that so large a proportion of the volume of iron is occupied by solid molecules, whatever be their form. This is one reason why we must abandon Poisson's hypothesis. Others will be stated in Chapter VI. Of course the value of Poisson's mathematical investigations remains unimpaired, as they do not rest on his hypothesis, but on the experimental fact of induced magnetization.

* *Recherches sur les propriétés magnétiques du fer*, *Nova Acta*, Upsal, 1863.

# CHAPTER V.

## PARTICULAR PROBLEMS IN MAGNETIC INDUCTION.

### *A Hollow Spherical Shell.*

431.] THE first example of the complete solution of a problem in magnetic induction was that given by Poisson for the case of a hollow spherical shell acted on by any magnetic forces whatever.

For simplicity we shall suppose the origin of the magnetic forces to be in the space outside the shell.

If $V$ denotes the potential due to the external magnetic system, we may expand $V$ in a series of solid harmonics of the form

$$V = C_0 S_0 + C_1 S_1 r + \&c. + C_i S_i r^i + \dots, \tag{1}$$

where $r$ is the distance from the centre of the shell, $S_i$ is a surface harmonic of order $i$, and $C_i$ is a coefficient.

This series will be convergent provided $r$ is less than the distance of the nearest magnet of the system which produces this potential. Hence, for the hollow spherical shell and the space within it, this expansion is convergent.

Let the external radius of the shell be $a_2$ and the inner radius $a_1$, and let the potential due to its induced magnetism be $\Omega$. The form of the function $\Omega$ will in general be different in the hollow space, in the substance of the shell, and in the space beyond. If we expand these functions in harmonic series, then, confining our attention to those terms which involve the surface harmonic $S_i$, we shall find that if $\Omega_1$ is that which corresponds to the hollow space within the shell, the expansion of $\Omega_1$ must be in positive harmonics of the form $A_1 S_i r^i$, because the potential must not become infinite within the sphere whose radius is $a_1$.

In the substance of the shell, where $r$ lies between $a_1$ and $a_2$, the series may contain both positive and negative powers of $r$, of the form

$$A_2 S_i r^i + B_2 S_i r^{-(i+1)}.$$

Outside the shell, where $r$ is greater than $a_2$, since the series must be convergent however great $r$ may be, we must have only negative powers of $r$, of the form

$$B_3 S_i r^{-(i+1)}.$$

The conditions which must be satisfied by the function $\Omega$ are: It must be $1^0$ finite, and $2^0$ continuous, and $3^0$ must vanish at an infinite distance, and it must $4^0$ everywhere satisfy Laplace's equation.

On account of $1^0$, $B_1 = 0$.

On account of $2^0$, when $r = a_1$,

$$(A_1 - A_2)\, a_1^{2i+1} - B_2 = 0, \tag{2}$$

and when $r = a_2$,

$$(A_2 - A_3) a_2^{2i+1} + B_2 - B_3 = 0. \tag{3}$$

On account of $3^0$, $A_3 = 0$, and the condition $4^0$ is satisfied everywhere, since the functions are harmonic.

But, besides these, there are other conditions to be satisfied at the inner and outer surfaces in virtue of equation (10), Art. 427.

At the inner surface where $r = a_1$,

$$(1 + 4\pi\kappa)\frac{d\Omega_2}{dr} - \frac{d\Omega_1}{dr} + 4\pi\kappa\frac{dV}{dr} = 0, \tag{4}$$

and at the outer surface where $r = a_2$,

$$-(1 + 4\pi\kappa)\frac{d\Omega_2}{dr} + \frac{d\Omega_3}{dr} - 4\pi\kappa\frac{dV}{dr} = 0. \tag{5}$$

From these conditions we obtain the equations

$$(1 + 4\pi\kappa)\,\{iA_2 a_1^{2i+1} - (i+1) B_2\} - iA_1 a_1^{2i+1} + 4\pi\kappa i C_i a_1^{2i+1} = 0, \tag{6}$$

$$(1 + 4\pi\kappa)\,\{iA_2 a_2^{2i+1} - (i+1) B_2\} + (i+1) B_3 + 4\pi\kappa i C_i a_2^{2i+1} = 0; \tag{7}$$

and if we put

$$N_i = \frac{1}{(1 + 4\pi\kappa)(2i+1)^2 + (4\pi\kappa)^2 i(i+1)\left(1 - \left(\frac{a_1}{a_2}\right)^{2i+1}\right)}, \tag{8}$$

we find

$$A_1 = -(4\pi\kappa)^2 i(i+1)\left(1 - \left(\frac{a_1}{a_2}\right)^{2i+1}\right) N_i C_i, \tag{9}$$

$$A_2 = -4\pi\kappa i\left[2i + 1 + 4\pi\kappa(i+1)\left(1 - \left(\frac{a_1}{a_2}\right)^{2i+1}\right)\right] N_i C_i, \tag{10}$$

$$B_2 = 4\pi\kappa i(2i+1)\, a_1^{2i+1} N_i C_i, \tag{11}$$

$$B_3 = -4\pi\kappa i\,\{2i + 1 + 4\pi\kappa(i+1)\}\,(a_2^{2i+1} - a_1^{2i+1}) N_i C_i. \tag{12}$$

These quantities being substituted in the harmonic expansions give the part of the potential due to the magnetization of the shell. The quantity $N_i$ is always positive, since $1 + 4\pi\kappa$ can never be negative. Hence $A_1$ is always negative, or in other words, the action of the magnetized shell on a point within it is always opposed to that of the external magnetic force, whether the shell be paramagnetic or diamagnetic. The actual value of the resultant potential within the shell is

$$(C_i + A_1) S_i r^i,$$

$$\text{or} \qquad (1 + 4\pi\kappa)(2i+1)^2 N_i C_i S_i r^i. \tag{13}$$

432.] When $\kappa$ is a large number, as it is in the case of soft iron, then, unless the shell is very thin, the magnetic force within it is but a small fraction of the external force.

In this way Sir W. Thomson has rendered his marine galvanometer independent of external magnetic force by enclosing it in a tube of soft iron.

433.] The case of greatest practical importance is that in which $i = 1$. In this case

$$N_1 = \frac{1}{9(1+4\pi\kappa) + 2(4\pi\kappa)^2 \left(1 - \left(\frac{a_1}{a_2}\right)^3\right)}, \tag{14}$$

$$\left.\begin{aligned}
A_1 &= -2(4\pi\kappa)^2 \left(1 - \left(\frac{a_1}{a_2}\right)^3\right) N_1 C_1, \\
A_2 &= -4\pi\kappa \left[3 + 8\pi\kappa \left(-1 \left(\frac{a_1}{a_2}\right)^3\right)\right] N_1 C_1, \\
B_2 &= 12\pi\kappa a_1^3 N_1 C_1, \\
B_3 &= -4\pi\kappa(3+8\pi\kappa)(a_2^3 - a_1^3) N_1 C_1.
\end{aligned}\right\} \tag{15}$$

The magnetic force within the hollow shell is in this case uniform and equal in magnitude to

$$C_1 + A_1 = \frac{9(1+4\pi\kappa)}{9(1+4\pi\kappa) + 2(4\pi\kappa)^2 \left(1 - \left(\frac{a_1}{a_2}\right)^3\right)} C_1. \tag{16}$$

If we wish to determine $\kappa$ by measuring the magnetic force within a hollow shell and comparing it with the external magnetic force, the best value of the thickness of the shell may be found from the equation

$$1 - \frac{a_1^3}{a_2^3} = \frac{9}{2} \frac{1+4\pi\kappa}{(4\pi\kappa)^2}. \tag{17}$$

{This value of $\frac{a_1}{a_2}$ makes $\frac{d}{d\kappa}\left\{1 + \frac{A_1}{C_1}\right\}$ a maximum, so that for a given error in $\frac{(C_1 + A_1)}{C_1}$ the corresponding error in $\kappa$ is as small as possible.} The magnetic force inside the shell is then half of its value outside.

Since, in the case of iron, $\kappa$ is a number between 20 and 30, the thickness of the shell ought to be about the two hundredth part of its radius. This method is applicable only when the value of $\kappa$ is large. When it is very small the value of $A_1$ becomes insensible, since it depends on the square of $\kappa$.

For a nearly solid sphere with a very small spherical hollow

$$\left.\begin{aligned} A_1 &= -\frac{2\,(4\,\pi\,\kappa)^2}{(3+4\pi\kappa)\,(3+8\,\pi\kappa)}\,C_1, \\ A_2 &= -\frac{4\,\pi\,\kappa}{3+4\,\pi\,\kappa}\,C_1, \\ B_3 &= -\frac{4\,\pi\,\kappa}{3+4\,\pi\,\kappa}\,C_1 a_2^3. \end{aligned}\right\} \tag{18}$$

The whole of this investigation might have been deduced directly from that of conduction through a spherical shell, as given in Art. 312, by putting $k_1 = (1+4\,\pi\kappa)\,k_2$ in the expressions there given, remembering that $A_1$ and $A_2$ in the problem of conduction are equivalent to $C_1 + A_1$ and $C_1 + A_2$ in the problem of magnetic induction.

434.] The corresponding solution in two dimensions is graphically represented in Fig. XV, at the end of this volume. The lines of induction, which at a distance from the centre of the figure are nearly horizontal, are represented as disturbed by a cylindric rod magnetized transversely and placed in its position of stable equilibrium. The lines which cut this system at right angles represent the equipotential surfaces, one of which is a cylinder. The large dotted circle represents the section of a cylinder of a paramagnetic substance, and the dotted horizontal straight lines within it, which are continuous with the external lines of induction, represent the lines of induction within the substance. The dotted vertical lines represent the internal equipotential surfaces, and are continuous with the external system. It will be observed that the lines of induction are drawn nearer together within the substance, and the equipotential surfaces are separated farther apart by the paramagnetic cylinder, which, in the language of Faraday, conducts the lines of induction better than the surrounding medium.

If we consider the system of vertical lines as lines of induction, and the horizontal system as equipotential surfaces, we have, in the first place, the case of a cylinder magnetized transversely and placed in the position of unstable equilibrium among the lines of force, which it causes to diverge. In the second place, considering the large dotted circle as the section of a diamagnetic cylinder, the dotted straight lines within it, together with the lines external to it, represent the effect of a diamagnetic substance in separating the lines of induction and drawing together the equipotential surfaces, such a substance being a worse conductor of magnetic induction than the surrounding medium.

### *Case of a Sphere in which the Coefficients of Magnetization are Different in Different Directions.*

435.] Let $\alpha$, $\beta$, $\gamma$ be the components of magnetic force, and $A$, $B$, $C$ those of the magnetization at any point, then the most general linear relation between these quantities is given by the equations

$$\left.\begin{aligned} A &= r_1\alpha + p_3\beta + q_2\gamma, \\ B &= q_3\alpha + r_2\beta + p_1\gamma, \\ C &= p_2\alpha + q_1\beta + r_3\gamma, \end{aligned}\right\} \tag{1}$$

where the coefficients $r$, $p$, $q$ are the nine coefficients of magnetization.

Let us now suppose that these are the conditions of magnetization within a sphere of radius $a$, and that the magnetization at every point of the substance is uniform and in the same direction, having the components $A$, $B$, $C$.

Let us also suppose that the external magnetizing force is also uniform and parallel to one direction, and has for its components $X$, $Y$, $Z$.

The value of $V$ is therefore

$$V = -(Xx + Yy + Zz), \tag{2}$$

and that of $\Omega'$, the potential outside the sphere of the magnetization, is by Art. 391,

$$\Omega' = \frac{4\pi a^3}{3\;r^3}(Ax + By + Cz). \tag{3}$$

The value of $\Omega$, the potential within the sphere of the magnetization, is

$$\Omega = \frac{4\pi}{3}(Ax + By + Cz). \tag{4}$$

The actual potential within the sphere is $V + \Omega$, so that we shall have for the components of the magnetic force within the sphere

$$\left.\begin{aligned} \alpha &= X - \tfrac{4}{3}\pi A, \\ \beta &= Y - \tfrac{4}{3}\pi B, \\ \gamma &= Z - \tfrac{4}{3}\pi C. \end{aligned}\right\} \qquad (5)$$

Hence

$$\left.\begin{aligned} (1+\tfrac{4}{3}\pi r_1)A + \tfrac{4}{3}\pi p_3 B + \tfrac{4}{3}\pi q_2 C &= r_1 X + p_3 Y + q_2 Z, \\ \tfrac{4}{3}\pi q_3 A + (1+\tfrac{4}{3}\pi r_2)B + \tfrac{4}{3}\pi p_1 C &= q_3 X + r_2 Y + p_1 Z, \\ \tfrac{4}{3}\pi p_2 A + \tfrac{4}{3}\pi q_1 B + (1+\tfrac{4}{3}\pi r_3)C &= p_2 X + q_1 Y + r_3 Z. \end{aligned}\right\} \qquad (6)$$

Solving these equations, we find

$$\left.\begin{aligned} A &= r_1'X + p_3'Y + q_2'Z, \\ B &= q_3'X + r_2'Y + p_1'Z, \\ C &= p_2'X + q_1'Y + r_3'Z, \end{aligned}\right\} \qquad (7)$$

where
$$\left.\begin{aligned} D'r_1' &= r_1 + \tfrac{4}{3}\pi(r_3 r_1 - p_2 q_2 + r_1 r_2 - p_3 q_3) + (\tfrac{4}{3}\pi)^2 D, \\ D'p_1' &= p_1 - \tfrac{4}{3}\pi(q_2 q_3 - p_1 r_1), \\ D'q_1' &= q_1 - \tfrac{4}{3}\pi(p_2 p_3 - q_1 r_1), \\ &\text{\&c.}, \end{aligned}\right\} \qquad (8)$$

where $D$ is the determinant of the coefficients on the right side of equations (6), and $D'$ that of the coefficients on the left.

The new system of coefficients $p'$, $q'$, $r'$ will be symmetrical only when the system $p$, $q$, $r$ is symmetrical, that is, when the coefficients of the form $p$ are equal to the corresponding ones of the form $q$.

436.] *The moment of the couple tending to turn the sphere about the axis of $x$ from $y$ towards $z$ is found by considering the couples arising from an elementary volume and taking the sum of the moments for the whole sphere. The result is

$$\begin{aligned} L &= \tfrac{4}{3}\pi a^3(\gamma B - \beta C) \\ &= \tfrac{4}{3}\pi a^3 \{p_1'Z^2 - q_1'Y^2 + (r_2' - r_3')YZ + X(q_3'Z - p_2'Y)\}. \end{aligned} \qquad (9)$$

* [The equality of the coefficients $p$ and $q$ may be shewn as follows: Let the forces acting on the sphere turn it about a diameter whose direction-cosines are $\lambda, \mu, \nu$ through an angle $\delta\theta$; then, if $W$ denote the energy of the sphere, we have, by Art. 436,

$$-\delta W = \tfrac{4}{3}\pi a^3 \{(ZB - YC)\lambda + (XC - ZA)\mu + (YA - XB)\nu\}\delta\theta.$$

But if the axes of coordinates be fixed in the sphere we have in consequence of the rotation

$$\delta X = (Y\nu - Z\mu)\delta\theta, \text{ etc.}$$

Hence we may put

$$-\delta W = \tfrac{4}{3}\pi a^3(A\delta X + B\delta Y + C\delta Z).$$

That the revolving sphere may not become a source of energy, the expression on the right-hand of the last equation must be a perfect differential. Hence, since $A$, $B$, $C$ are linear functions of $X$, $Y$, $Z$, it follows that $W$ is a quadratic function of $X$, $Y$, $Z$, and the required result is at once deduced.

See also Sir W. Thomson's Reprint of *Papers on Electrostatics and Magnetism*, pp. 480–481.]

If we make

$$X = 0, \qquad Y = F\cos\theta, \qquad Z = F\sin\theta,$$

this corresponds to a magnetic force $F$ in the plane of $yz$, and inclined to $y$ at an angle $\theta$. If we now turn the sphere while this force remains constant the work done in turning the sphere will be $\int_0^{2\pi} L\,d\theta$ in each complete revolution. But this is equal to

$$\tfrac{4}{3}\pi^2 a^3 F^2 (p_1' - q_1'). \tag{10}$$

Hence, in order that the revolving sphere may not become an inexhaustible source of energy, $p_1' = q_1'$, and similarly $p_2' = q_2'$ and $p_3' = q_3'$.

These conditions shew that in the original equations the coefficient of $B$ in the third equation is equal to that of $C$ in the second, and so on. Hence, the system of equations is symmetrical, and the equations become when referred to the principal axes of magnetization,

$$\left.\begin{aligned} A &= \frac{r_1}{1 + \frac{4}{3}\pi r_1} X, \\ B &= \frac{r_2}{1 + \frac{4}{3}\pi r_2} Y, \\ C &= \frac{r_3}{1 + \frac{4}{3}\pi r_3} Z. \end{aligned}\right\} \tag{11}$$

The moment of the couple tending to turn the sphere round the axis of $x$ is

$$L = \tfrac{4}{3}\pi a^3 \frac{r_2 - r_3}{(1 + \frac{4}{3}\pi r_2)(1 + \frac{4}{3}\pi r_3)} YZ. \tag{12}$$

In most cases the differences between the coefficients of magnetization in different directions are very small, so that we may put, if $r$ represents the mean value of the coefficients,

$$L = \tfrac{2}{3}\pi a^3 \frac{r_2 - r_3}{(1 + \frac{4}{3}\pi r)^2} F^2 \sin 2\theta. \tag{13}$$

This is the force tending to turn a crystalline sphere about the axis of $x$ from $y$ towards $z$. It always tends to place the axis of greatest magnetic coefficient (or least diamagnetic coefficient) parallel to the line of magnetic force.

The corresponding case in two dimensions is represented in Fig. XVI.

If we suppose the upper side of the figure to be towards the north, the figure represents the lines of force and equipotential surfaces as disturbed by a transversely magnetized cylinder placed with the north side eastwards. The resultant force tends to turn the cylinder from east to north. The large dotted circle represents a section of a cylinder of a crystalline substance which has a larger coefficient of induction along an axis from north-east to south-west than along an axis from north-west to south-east. The dotted lines within the circle represent the lines of induction and the equipotential surfaces, which in this case are not at right angles to each other. The resultant force on the cylinder tends evidently to turn it from east to north.

437.] The case of an ellipsoid placed in a field of uniform and parallel magnetic force has been solved in a very ingenious manner by Poisson.

If $V$ is the potential at the point $(x, y, z)$, due to the gravitation of a body of any form of uniform density $\rho$, then $-\dfrac{dV}{dx}$ is the potential of the magnetism of the same body if uniformly magnetized in the direction of $x$ with the intensity $I = \rho$.

For the value of $-\dfrac{dV}{dx}\delta x$ at any point is the excess of the value of $V$, the potential of the body, above $V'$, the value of the potential when the body is moved $-\delta x$ in the direction of $x$.

If we supposed the body shifted through the distance $-\delta x$, and its density changed from $\rho$ to $-\rho$ (that is to say, made of repulsive instead of attractive matter), then $-\dfrac{dV}{dx}\delta x$ would be the potential due to the two bodies.

Now consider any elementary portion of the body containing a volume $\delta v$. Its quantity is $\rho\delta v$, and corresponding to it there is an element of the shifted body whose quantity is $-\rho\delta v$ at a distance $-\delta x$. The effect of these two elements is equivalent to that of a magnet of strength $\rho\delta v$ and length $\delta x$. The intensity of magnetization is found by dividing the magnetic moment of an element by its volume. The result is $\rho\delta x$.

Hence $-\dfrac{dV}{dx}\delta x$ is the magnetic potential of the body magnetized with the intensity $\rho\delta x$ in the direction of $x$, and $-\dfrac{dV}{dx}$ is that of the body magnetized with intensity $\rho$.

This potential may be also considered in another light. The body was shifted through the distance $-\delta x$ and made of density

$-\rho$. Throughout that part of space common to the body in its two positions the density is zero, for, as far as attraction is concerned, the two equal and opposite densities annihilate each other. There remains therefore a shell of positive matter on one side and of negative matter on the other, and we may regard the resultant potential as due to these. The thickness of the shell at a point where the normal drawn outwards makes an angle $\epsilon$ with the axis of $x$ is $\delta x \cos\epsilon$ and its density is $\rho$. The surface-density is therefore $\rho\,\delta x\cos\epsilon$, and, in the case in which the potential is $-\dfrac{dV}{dx}$, the surface-density is $\rho\cos\epsilon$.

In this way we can find the magnetic potential of any body uniformly magnetized parallel to a given direction. Now if this uniform magnetization is due to magnetic induction, the magnetizing force at all points within the body must also be uniform and parallel.

This force consists of two parts, one due to external causes, and the other due to the magnetization of the body. If therefore the external magnetic force is uniform and parallel, the magnetic force due to the magnetization must also be uniform and parallel for all points within the body.

Hence, in order that this method may lead to a solution of the problem of magnetic induction, $\dfrac{dV}{dx}$ must be a linear function of the coordinates $x, y, z$ within the body, and therefore $V$ must be a quadratic function of the coordinates.

Now the only cases with which we are acquainted in which $V$ is a quadratic function of the coordinates within the body are those in which the body is bounded by a complete surface of the second degree, and the only case in which such a body is of finite dimensions is when it is an ellipsoid. We shall therefore apply the method to the case of an ellipsoid.

Let
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \tag{1}$$
be the equation of the ellipsoid, and let $\Phi_0$ denote the definite integral
$$\int_0^\infty \frac{d(\phi^2)}{\sqrt{(a^2+\phi^2)(b^2+\phi^2)(c^2+\phi^2)}}*. \tag{2}$$

* See Thomson and Tait's *Natural Philosophy*, § 525, 2nd Edition.

Then if we make

$$L = 4\pi abc\frac{d\Phi_0}{d(a^2)},\qquad M = 4\pi abc\frac{d\Phi_0}{d(b^2)},\qquad N = 4\pi abc\frac{d\Phi_0}{d(c^2)},\tag{3}$$

the value of the potential within the ellipsoid will be

$$V_0 = -\frac{\rho}{2}(Lx^2 + My^2 + Nz^2) + \text{const.}\tag{4}$$

If the ellipsoid is magnetized with uniform intensity $I$ in a direction making angles whose cosines are $l, m, n$ with the axes of $x, y, z$, so that the components of magnetization are

$$A = Il,\qquad B = Im,\qquad C = In,$$

the potential due to this magnetization within the ellipsoid will be

$$\Omega = -I(Llx + Mmy + Nnz).\tag{5}$$

If the external magnetizing force is $\mathfrak{H}$, and if its components are $X, Y, Z$, its potential will be

$$V = -(Xx + Yy + Zz).\tag{6}$$

The components of the actual magnetizing force at any point within the body are therefore

$$X + AL,\qquad Y + BM,\qquad Z + CN.\tag{7}$$

The most general relations between the magnetization and the magnetizing force are given by three linear equations, involving nine coefficients. It is necessary, however, in order to fulfil the condition of the conservation of energy, that in the case of magnetic induction three of these should be equal respectively to other three, so that we should have

$$\left.\begin{aligned} A &= \kappa_1(X + AL) + \kappa'_3(Y + BM) + \kappa'_2(Z + CN),\\ B &= \kappa'_3(X + AL) + \kappa_2(Y + BM) + \kappa'_1(Z + CN),\\ C &= \kappa'_2(X + AL) + \kappa'_1(Y + BM) + \kappa_3(Z + CN).\end{aligned}\right\}\tag{8}$$

From these equations we may determine $A$, $B$ and $C$ in terms of $X, Y, Z$, and this will give the most general solution of the problem.

The potential outside the ellipsoid will then be that due to the magnetization of the ellipsoid together with that due to the external magnetic force.

438.] The only case of practical importance is that in which

$$\kappa'_1 = \kappa'_2 = \kappa'_3 = 0.\tag{9}$$

We have then
$$\left.\begin{aligned} A &= \frac{\kappa_1}{1-\kappa_1 L}X, \\ B &= \frac{\kappa_2}{1-\kappa_2 M}Y, \\ C &= \frac{\kappa_3}{1-\kappa_3 N}Z. \end{aligned}\right\} \tag{10}$$

If the ellipsoid has two axes equal, and is of the planetary or flattened form,
$$b = c = \frac{a}{\sqrt{1-e^2}}; \tag{11}$$

$$\left.\begin{aligned} L &= -4\pi\left(\frac{1}{e^2} - \frac{\sqrt{1-e^2}}{e^3}\sin^{-1}e\right), \\ M = N &= -2\pi\left(\frac{\sqrt{1-e^2}}{e^3}\sin^{-1}e - \frac{1-e^2}{e^2}\right). \end{aligned}\right\} \tag{12}$$

If the ellipsoid is of the ovary or elongated form,
$$a = b = \sqrt{1-e^2}\,c; \tag{13}$$

$$\left.\begin{aligned} L = M &= -2\pi\left(\frac{1}{e^2} - \frac{1-e^2}{2e^3}\log\frac{1+e}{1-e}\right), \\ N &= -4\pi\left(\frac{1}{e^2}-1\right)\left(\frac{1}{2e}\log\frac{1+e}{1-e} - 1\right). \end{aligned}\right\} \tag{14}$$

In the case of a sphere, when $e = 0$,
$$L = M = N = -\tfrac{4}{3}\pi. \tag{15}$$

In the case of a very flattened planetoid $L$ becomes in the limit equal to $-4\pi$, and $M$ and $N$ become $-\pi^2\frac{a}{c}$.

In the case of a very elongated ovoid $L$ and $M$ approximate to the value $-2\pi$, while $N$ approximates to the form
$$-4\pi\frac{a^2}{c^2}\left(\log\frac{2c}{a} - 1\right),$$
and vanishes when $e = 1$.

It appears from these results that—

(1) When $\kappa$, the coefficient of magnetization, is very small, whether positive or negative, the induced magnetization is nearly equal to the magnetizing force multiplied by $\kappa$, and is almost independent of the form of the body.

(2) When $\kappa$ is a large positive quantity, the magnetization depends principally on the form of the body, and is almost independent of the precise value of $\kappa$, except in the case of a

longitudinal force acting on an ovoid so elongated that $N\kappa$ is a small quantity though $\kappa$ is large.

(3) If the value of $\kappa$ could be negative and equal to $\frac{1}{4\pi}$ we should have an infinite value of the magnetization in the case of a magnetizing force acting normally to a flat plate or disk. The absurdity of this result confirms what we said in Art. 428.

Hence, experiments to determine the value of $\kappa$ may be made on bodies of any form, provided $\kappa$ is very small, as it is in the case of all diamagnetic bodies, and all magnetic bodies except iron, nickel and cobalt.

If, however, as in the case of iron, $\kappa$ is a large number, experiments made on spheres or flattened figures are not suitable to determine $\kappa$; for instance, in the case of a sphere the ratio of the magnetization to the magnetizing force is as 1 to 4·22 if $\kappa = 30$, as it is in some kinds of iron, and if $\kappa$ were infinite the ratio would be as 1 to 4·19, so that a very small error in the determination of the magnetization would introduce a very large one in the value of $\kappa$.

But if we make use of a piece of iron in the form of a very elongated ovoid, then, as long as $N\kappa$ is of moderate value compared with unity, we may deduce the value of $\kappa$ from a determination of the magnetization, and the smaller the value of $N$ the more accurate will be the value of $\kappa$.

In fact, if $N\kappa$ be made small enough, a small error in the value of $N$ itself will not introduce much error, so that we may use any elongated body, such as a wire or long rod, instead of an ovoid*.

We must remember, however, that it is only when the product $N\kappa$ is small compared with unity that this substitution is allowable. In fact the distribution of magnetism on a long cylinder with flat ends does not resemble that on a long ovoid, for the free magnetism is very much concentrated towards the ends of the cylinder, whereas it varies directly as the distance from the equator in the case of the ovoid.

The distribution of electricity on a cylinder, however, is really comparable with that on an ovoid, as we have already seen, Art. 152.

These results also enable us to understand why the magnetic

* {If wires are used their length should be at least 300 times their diameter.}

moment of a permanent magnet can be made so much greater when the magnet has an elongated form. If we were to magnetize a disk with intensity $I$ in a direction normal to its surface, and then leave it to itself, the interior particles would experience a constant demagnetizing force equal to $4\pi I$, and this, if not sufficient of itself to destroy part of the magnetization, would soon do so if aided by vibrations or changes of temperature *.

If we were to magnetize a cylinder transversely the demagnetizing force would be only $2\pi I$.

If the magnet were a sphere the demagnetizing force would be $\frac{4}{3}\pi I$.

In a disk magnetized transversely the demagnetizing force is $\pi^2 \frac{a}{c} I$, and in an elongated ovoid magnetized longitudinally it is least of all, being $4\pi \frac{a^2}{c^2} I \log \frac{2c}{a}$.

Hence an elongated magnet is less likely to lose its magnetism than a short thick one.

The moment of the force acting on an ellipsoid having different magnetic coefficients for the three axes which tends to turn it about the axis of $x$, is

$$\tfrac{4}{3}\pi abc\,(BZ - CY) = \tfrac{4}{3}\pi abc\,YZ\,\frac{\kappa_2 - \kappa_3 + \kappa_2\kappa_3(M-N)}{(1-\kappa_2 M)(1-\kappa_3 N)}.$$

Hence, if $\kappa_2$ and $\kappa_3$ are small, this force will depend principally on the crystalline quality of the body and not on its shape, provided its dimensions are not very unequal, but if $\kappa_2$ and $\kappa_3$ are considerable, as in the case of iron, the force will depend principally on the shape of the body, and it will turn so as to set its longer axis parallel to the lines of force.

If a sufficiently strong, yet uniform, field of magnetic force could be obtained, an elongated isotropic diamagnetic body

* {The magnetic force in the disk $= X + AL$

$$= \frac{X}{1-\kappa L};$$

and since $L = -4\pi$ in this case, the magnetic force is

$$\frac{X}{1+4\pi\kappa}.$$

Thus the magnetic induction through the disk is $X$, the value it would have in the air if the disk were removed.}

would also set itself with its longest dimension parallel to the lines of magnetic force *.

439.] The question of the distribution of the magnetization of an ellipsoid of revolution under the action of any magnetic forces has been investigated by J. Neumann †. Kirchhoff ‡ has extended the method to the case of a cylinder of infinite length acted on by any force.

Green, in the 17th section of his Essay, has given an investigation of the distribution of magnetism in a cylinder of finite length acted on by a uniform external force $X$ parallel to its axis. Though some of the steps of this investigation are not very rigorous, it is probable that the result represents roughly the actual magnetization in this most important case. It certainly expresses very fairly the transition from the case of a cylinder for which $\kappa$ is a large number to that in which it is very small, but it fails entirely in the case in which $\kappa$ is negative, as in diamagnetic substances.

Green finds that the linear density of free magnetism at a distance $x$ from the middle of a cylinder whose radius is $a$ and whose length is $2l$, is

$$\lambda = \pi\kappa X pa \frac{e^{\frac{px}{a}} - e^{-\frac{px}{a}}}{e^{\frac{pl}{a}} + e^{-\frac{pl}{a}}},$$

where $p$ is a numerical quantity to be found from the equation

$$0{\cdot}231863 - 2\log_e p + 2p = \frac{1}{\pi\kappa p^2}.$$

The following are a few of the corresponding values of $p$ and $\kappa$.

| $\kappa$ | $p$ | $\kappa$ | $p$ |
|---|---|---|---|
| $\infty$ | 0 | 11·802 | 0·07 |
| 336·4 | 0·01 | 9·137 | 0·08 |
| 62·02 | 0·02 | 7·517 | 0·09 |
| 48·416 | 0·03 | 6·319 | 0·10 |
| 29·475 | 0·04 | 0·1427 | 1·00 |
| 20·185 | 0·05 | 0·0002 | 10·00 |
| 14·794 | 0·06 | 0·0000 | $\infty$ |
| | | negative | imaginary. |

* {This effect depends on the square of $\kappa$, the forces investigated in § 440 depend upon the first power of $\kappa$, thus since $\kappa$ is very small for diamagnetic bodies the latter forces will, except in exceptional cases, over-power the tendency investigated in this Art.}

† *Crelle*, bd. xxxvii (1848).

‡ *Crelle*, bd. xlviii (1854).

When the length of the cylinder is great compared with its radius, the whole quantity of free magnetism on either side of the middle of the cylinder is, as it ought to be,

$$M = \pi a^2 \kappa X.$$

Of this $\frac{1}{2}pM$ is on the flat end of the cylinder*, and the distance of the centre of gravity of the whole quantity $M$ from the end of the cylinder is $\frac{a}{p}$.

When $\kappa$ is very small $p$ is large, and nearly the whole free magnetism is on the ends of the cylinder. As $\kappa$ increases $p$ diminishes, and the free magnetism is spread over a greater distance from the ends. When $\kappa$ is infinite the free magnetism at any point of the cylinder is simply proportional to its distance from the middle point, the distribution being similar to that of free electricity on a conductor in a field of uniform force.

440.] In all substances except iron, nickel, and cobalt, the coefficient of magnetization is so small that the induced magnetization of the body produces only a very slight alteration of the forces in the magnetic field. We may therefore assume, as a first approximation, that the actual magnetic force within the body is the same as if the body had not been there. The superficial magnetization of the body is therefore, as a first approximation, $\kappa \frac{dV}{d\nu}$, where $\frac{dV}{d\nu}$ is the rate of increase of the magnetic potential due to the external magnet along a normal to the surface drawn inwards. If we now calculate the potential due to this superficial distribution, we may use it in proceeding to a second approximation.

To find the mechanical energy due to the distribution of

* {The quantity of free magnetism on the curved surface on the positive side of the cylinder

$$= \int_0^l \lambda\, dx = \pi a^2 \kappa X \left(1 - \operatorname{sech} \frac{pl}{a}\right).$$

The quantity on the flat end, supposing the density to be the same as on the curved surface when $x = l$, is

$$\frac{\pi \kappa X p a}{2 \pi a} \tanh \frac{pl}{a} \cdot \pi a^2.$$

Thus the total quantity of free magnetism is

$$\pi a^2 \kappa X \left(1 - \operatorname{sech} \frac{pl}{a} + \frac{p}{2} \tanh \frac{pl}{a}\right)$$

When $pl/a$ is large this is equal to

$$M\left(1 + \frac{p}{2}\right)\Big\}.$$

magnetism on this first approximation we must find the surface-integral

$$E = \tfrac{1}{2}\iint \kappa V \frac{dV}{d\nu} dS$$

taken over the whole surface of the body. Now we have shewn in Art. 100 that this is equal to the volume-integral

$$E = -\tfrac{1}{2}\iiint \kappa \left(\overline{\frac{dV}{dx}}\bigg|^2 + \overline{\frac{dV}{dy}}\bigg|^2 + \overline{\frac{dV}{dz}}\bigg|^2\right) dx\, dy\, dz$$

taken through the whole space occupied by the body, or, if $R$ is the resultant magnetic force,

$$E = -\tfrac{1}{2}\iiint \kappa R^2 dx\, dy\, dz.$$

Now since the work done by the magnetic force on the body during a displacement $\delta x$ is $X\delta x$ where $X$ is the mechanical force in the direction of $x$, and since

$$\int X\delta x + E = \text{constant},$$

$$X = -\frac{dE}{dx} = \tfrac{1}{2}\frac{d}{dx}\iiint \kappa R^2 dx\, dy\, dz = \tfrac{1}{2}\iiint \kappa \frac{d.R^2}{dx} dx\, dy\, dz,$$

which shews that the force acting on the body is as if every part of it tended to move from places where $R^2$ is less to places where it is greater, with a force which on every unit of volume is

$$\tfrac{1}{2}\kappa \frac{d.R^2}{dx}.$$

If $\kappa$ is negative, as in diamagnetic bodies, this force is, as Faraday first shewed, from stronger to weaker parts of the magnetic field. Most of the actions observed in the case of diamagnetic bodies depend on this property.

*Ship's Magnetism.*

441.] Almost every part of magnetic science finds its use in navigation. The directive action of the earth's magnetism on the compass-needle is the only method of ascertaining the ship's course when the sun and stars are hid. The declination of the needle from the true meridian seemed at first to be a hindrance to the application of the compass to navigation, but after this difficulty had been overcome by the construction of magnetic charts it appeared likely that the declination itself would assist the mariner in determining his ship's place.

The greatest difficulty in navigation had always been to ascertain the longitude; but since the declination is different at different points on the same parallel of latitude, an observation of the declination together with a knowledge of the latitude would enable the mariner to find his position on the magnetic chart.

But in recent times iron is so largely used in the construction of ships that it has become impossible to use the compass at all without taking into account the action of the ship, as a magnetic body, on the needle.

To determine the distribution of magnetism in a mass of iron of any form under the influence of the earth's magnetic force, even though not subjected to mechanical strain or other disturbances, is, as we have seen, a very difficult problem.

In this case, however, the problem is simplified by the following considerations.

The compass is supposed to be placed with its centre at a fixed point of the ship, and so far from any iron that the magnetism of the needle does not induce any perceptible magnetism in the ship. The size of the compass-needle is supposed so small that we may regard the magnetic force at every point of the needle as the same.

The iron of the ship is supposed to be of two kinds only.

(1) Hard iron, magnetized in a constant manner.

(2) Soft iron, the magnetization of which is induced by the earth or other magnets.

In strictness we must admit that the hardest iron is not only capable of induction but that it may lose part of its so-called permanent magnetization in various ways.

The softest iron is capable of retaining what is called residual magnetization. The actual properties of iron cannot be accurately represented by supposing it compounded of the hard iron and the soft iron above defined. But it has been found that when a ship is acted on only by the earth's magnetic force, and not subjected to any extraordinary stress of weather, the supposition that the magnetism of the ship is due partly to permanent magnetization and partly to induction leads to sufficiently accurate results when applied to the correction of the compass.

The equations on which the theory of the variation of the compass is founded were given by Poisson in the fifth volume of the *Mémoires de l'Institut*, p. 533 (1824).

The only assumption relative to induced magnetism which is involved in these equations is, that if a magnetic force $X$ due to external magnetism produces in the iron of the ship an induced magnetization, and if this induced magnetization exerts on the compass needle a disturbing force whose components are $X'$, $Y'$, $Z'$, then, if the external magnetic force is altered in a given ratio, the components of the disturbing force will be altered in the same ratio.

It is true that when the magnetic force acting on iron is very great the induced magnetization is no longer proportional to the external magnetic force, but this want of proportionality is insensible for magnetic forces of the magnitude of those due to the earth's action.

Hence, in practice we may assume that if a magnetic force whose value is unity produces through the intervention of the iron of the ship a disturbing force at the compass-needle whose components are $a$ in the direction of $x$, $d$ in that of $y$, and $g$ in that of $z$, the components of the disturbing force due to a force $X$ in the direction of $x$ will be $aX$, $dX$, and $gX$.

If therefore we assume axes fixed in the ship, so that $x$ is towards the ship's head, $y$ to the starboard side, and $z$ towards the keel, and if $X$, $Y$, $Z$ represent the components of the earth's magnetic force in these directions, and $X'$, $Y'$, $Z'$ the components of the combined magnetic force of the earth and ship on the compass-needle,

$$\left.\begin{aligned} X' &= X + aX + bY + cZ + P, \\ Y' &= Y + dX + eY + fZ + Q, \\ Z' &= Z + gX + hY + kZ + R. \end{aligned}\right\} \qquad (1)$$

In these equations $a$, $b$, $c$, $d$, $e$, $f$, $g$, $h$, $k$ are nine constant coefficients depending on the amount, the arrangement, and the capacity for induction of the soft iron of the ship.

$P$, $Q$, and $R$ are constant quantities depending on the permanent magnetization of the ship.

It is evident that these equations are sufficiently general if magnetic induction is a linear function of magnetic force, for they are neither more nor less than the most general expression of a vector as a linear function of another vector.

It may also be shewn that they are not too general, for, by a proper arrangement of iron, any one of the coefficients may be made to vary independently of the others.

Thus, a long thin rod of iron under the action of a longitudinal magnetic force acquires poles, the strength of each of which is numerically equal to the cross-section of the rod multiplied by the magnetizing force and by the coefficient of induced magnetization. A magnetic force transverse to the rod produces a much feebler magnetization, the effect of which is almost insensible at a distance of a few diameters.

If a long iron rod be placed fore and aft with one end at a distance $x$ from the compass-needle, measured towards the ship's head, then, if the section of the rod is $A$, and its coefficient of magnetization $\kappa$, the strength of the pole will be $A\kappa X$, and, if $A = \frac{ax^2}{\kappa}$, the force exerted by this pole on the compass-needle will be $aX$. The rod may be supposed so long that the effect of the other pole on the compass may be neglected.

We have thus obtained the means of giving any required value to the coefficient $a$.

If we place another rod of section $B$ with one extremity at the same point, distant $x$ from the compass toward the head of the vessel, and extending to starboard to such a distance that the distant pole produces no sensible effect on the compass, the disturbing force due to this rod will be in the direction of $x$, and equal to $\frac{B\kappa Y}{x^2}$, or if $B = \frac{bx^2}{\kappa}$, the force will be $bY$.

This rod therefore introduces the coefficient $b$.

A third rod extending downwards from the same point will introduce the coefficient $c$.

The coefficients $d$, $e$, $f$ may be produced by three rods extending to head, to starboard, and downward from a point to starboard of the compass, and $g$, $h$, $k$ by three rods in parallel directions from a point below the compass.

Hence each of the nine coefficients can be separately varied by means of iron rods properly placed.

The quantities $P$, $Q$, $R$ are simply the components of the force on the compass arising from the permanent magnetization of the ship together with that part of the induced magnetization which is due to the action of this permanent magnetization.

A complete discussion of the equations (1), and of the relation between the true magnetic course of the ship and the course as indicated by the compass, is given by Mr. Archibald Smith in the Admiralty *Manual of the Deviation of the Compass.*

A valuable graphic method of investigating the problem is there given. Taking a fixed point as origin, a line is drawn from this point representing in direction and magnitude the horizontal part of the actual magnetic force on the compass-needle. As the ship is swung round so as to bring her head into different azimuths in succession, the extremity of this line describes a curve, each point of which corresponds to a particular azimuth.

Such a curve, by means of which the direction and magnitude of the force on the compass is given in terms of the magnetic course of the ship, is called a Dygogram.

There are two varieties of the Dygogram. In the first, the curve is traced on a plane fixed in space as the ship turns round. In the second kind, the curve is traced on a plane fixed with respect to the ship.

The dygogram of the first kind is the Limaçon of Pascal, that of the second kind is an ellipse. For the construction and use of these curves, and for many theorems as interesting to the mathematician as they are important to the navigator, the reader is referred to the Admiralty *Manual of the Deviation of the Compass.*

# CHAPTER VI.

## WEBER'S THEORY OF INDUCED MAGNETISM.

442.] We have seen that Poisson supposed the magnetization of iron to consist in a separation of the magnetic fluids within each magnetic molecule. If we wish to avoid the assumption of the existence of magnetic fluids, we may state the same theory in another form, by saying that each molecule of the iron, when the magnetizing force acts on it, becomes a magnet.

Weber's theory differs from this in assuming that the molecules of the iron are always magnets, even before the application of the magnetizing force, but that in ordinary iron the magnetic axes of the molecules are turned indifferently in every direction, so that the iron as a whole exhibits no magnetic properties.

When a magnetic force acts on the iron it tends to turn the axes of the molecules all in one direction, and so to cause the iron, as a whole, to become a magnet.

If the axes of all the molecules were set parallel to each other, the iron would exhibit the greatest intensity of magnetization of which it is capable. Hence Weber's theory implies the existence of a limiting intensity of magnetization, and the experimental evidence that such a limit exists is therefore necessary to the theory. Experiments shewing an approach to a limiting value of magnetization have been made by Joule *, J. Müller †, and Ewing and Low ‡.

The experiments of Beetz § on electrotype iron deposited

* *Annals of Electricity*, iv. p. 131, 1839; *Phil. Mag.* [4] iii. p. 32.

† Pogg., *Ann.* lxxix. p. 337, 1850.

‡ Phil. Trans. 1889. A. p. 221.

§ Pogg. cxi. 1860.

under the action of magnetic force furnish the most complete evidence of this limit:—

A silver wire was varnished, and a very narrow line on the metal was laid bare by making a fine longitudinal scratch on the varnish. The wire was then immersed in a solution of a salt of iron, and placed in a magnetic field with the scratch in the direction of a line of magnetic force. By making the wire the cathode of an electric current through the solution, iron was deposited on the narrow exposed surface of the wire, molecule by molecule. The filament of iron thus formed was then examined magnetically. Its magnetic moment was found to be very great for so small a mass of iron, and when a powerful magnetizing force was made to act in the same direction the increase of temporary magnetization was found to be very small, and the permanent magnetization was not altered. A magnetizing force in the reverse direction at once reduced the filament to the condition of iron magnetized in the ordinary way.

Weber's theory, which supposes that in this case the magnetizing force placed the axis of each molecule in the same direction during the instant of its deposition, agrees very well with what is observed.

Beetz found that when the electrolysis is continued under the action of the magnetizing force the intensity of magnetization of the subsequently deposited iron diminishes. The axes of the molecules are probably deflected from the line of magnetizing force when they are being laid down side by side with the molecules already deposited, so that an approximation to parallelism can be obtained only in the case of a very thin filament of iron.

If, as Weber supposes, the molecules of iron are already magnets, any magnetic force sufficient to render their axes parallel as they are electrolytically deposited will be sufficient to produce the highest intensity of magnetization in the deposited filament.

If, on the other hand, the molecules of iron are not magnets, but are only capable of magnetization, the magnetization of the deposited filament will depend on the magnetizing force in the same way in which that of soft iron in general depends on it. The experiments of Beetz leave no room for the latter hypothesis.

443.] We shall now assume, with Weber, that in every unit of volume of the iron there are $n$ magnetic molecules, and that the magnetic moment of each is $m$. If the axes of all the molecules were placed parallel to one another, the magnetic moment of the unit of volume would be

$$M = nm,$$

and this would be the greatest intensity of magnetization of which the iron is capable.

In the unmagnetized state of ordinary iron Weber supposes the axes of its molecules to be placed indifferently in all directions.

To express this, we may suppose a sphere to be described, and a radius drawn from the centre parallel to the direction of the axis of each of the $n$ molecules. The distribution of the extremities of these radii will represent that of the axes of the molecules. In the case of ordinary iron these $n$ points are equally distributed over every part of the surface of the sphere, so that the number of molecules whose axes make an angle less than $a$ with the axis of $x$ is

$$\frac{n}{2}(1-\cos a),$$

and the number of molecules whose axes make angles with that of $x$ between $a$ and $a+da$ is therefore

$$\frac{n}{2}\sin a\,da.$$

This is the arrangement of the molecules in a piece of iron which has never been magnetized.

Let us now suppose that a magnetic force $X$ is made to act on the iron in the direction of the axis of $x$, and let us consider a molecule whose axis was originally inclined $a$ to the axis of $x$.

If this molecule is perfectly free to turn, it will place itself with its axis parallel to the axis of $x$, and if all the molecules did so, the very slightest magnetizing force would be found sufficient to develope the very highest degree of magnetization. This, however, is not the case.

The molecules do not turn with their axes parallel to $x$, and this is either because each molecule is acted on by a force tending to preserve it in its original direction, or because an equivalent effect is produced by the mutual action of the entire system of molecules.

Weber adopts the former of these suppositions as the simplest, and supposes that each molecule, when deflected, tends to return to its original position with a force which is the same as that which a magnetic force $D$, acting in the original direction of its axis, would produce.

The position which the axis actually assumes is therefore in the direction of the resultant of $X$ and $D$.

Let $APB$ represent a section of a sphere whose radius represents, on a certain scale, the force $D$.

Let the radius $OP$ be parallel to the axis of a particular molecule in its original position.

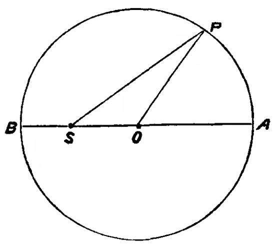


Fig. 5.

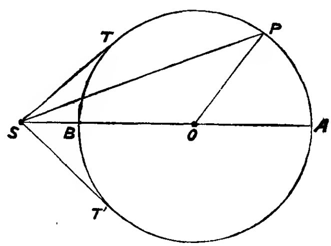


Fig. 6.

Let $SO$ represent on the same scale the magnetizing force $X$ which is supposed to act from $S$ towards $O$. Then, if the molecule is acted on by the force $X$ in the direction $SO$, and by a force $D$ in a direction parallel to $OP$, the original direction of its axis, its axis will set itself in the direction $SP$, that of the resultant of $X$ and $D$.

Since the axes of the molecules are originally in all directions, $P$ may be at any point of the sphere indifferently. In Fig. 5, in which $X$ is less than $D$, $SP$, the final position of the axis, may be in any direction whatever, but not indifferently, for more of the molecules will have their axes turned towards $A$ than towards $B$. In Fig. 6, in which $X$ is greater than $D$, the axes of the molecules will be all confined within the cone $TST'$ touching the sphere.

Hence there are two different cases according as $X$ is less or greater than $D$.

Let $\alpha = AOP$, the original inclination of the axis of a molecule to the axis of $x$.

$\theta = ASP$, the inclination of the axis when deflected by the force $X$.

$\beta = SPO$, the angle of deflexion.

$SO = X$ *, the magnetizing force.

$OP = D$, the force tending towards the original position.

$SP = R$, the resultant of $X$ and $D$.

$m =$ magnetic moment of the molecule.

Then the moment of the statical couple due to $X$, tending to diminish the angle $\theta$, is

$$mL = mX\sin\theta,$$

and the moment of the couple due to $D$, tending to increase $\theta$, is

$$mL = mD\sin\beta.$$

Equating these values, and remembering that $\beta = \alpha - \theta$, we find

$$\tan\theta = \frac{D\sin\alpha}{X + D\cos\alpha} \tag{1}$$

to determine the direction of the axis after deflexion.

We have next to find the intensity of magnetization produced in the mass by the force $X$, and for this purpose we must resolve the magnetic moment of every molecule in the direction of $x$, and add all these resolved parts.

The resolved part of the moment of a molecule in the direction of $x$ is

$$m\cos\theta.$$

The number of molecules whose original inclinations lay between $\alpha$ and $\alpha + d\alpha$ is

$$\frac{n}{2}\sin\alpha d\alpha.$$

We have therefore to integrate

$$I = \int_0^\pi \frac{mn}{2}\cos\theta\sin\alpha d\alpha, \tag{2}$$

remembering that $\theta$ is a function of $\alpha$.

* {The force acting on a magnetic pole inside a magnet is indefinite, depending on the shape of the cavity in which the pole is placed. The force $X$ is thus indefinite, for since we know nothing about the shape or disposition of these molecular magnets there does not seem any reason for assuming that the force is that in a cavity of one shape rather than another. Thus it would seem that unless further assumptions are made we ought to put $X = X_0 + pI$, where $X_0$ is the external magnetic force and $p$ a constant, of which all we can say is that it must lie between 0 and $4\pi$. This uncertainty about the value of $X$ is the more embarrassing from the fact that in iron $I$ is very much greater than $X_0$, so that the term about which there is the uncertainty may be much the more important of the two.}

We may express both $\theta$ and $a$ in terms of $R$, and the expression to be integrated becomes

$$-\frac{mn}{4X^2D}(R^2+X^2-D^2)dR, \tag{3}$$

the general integral of which is

$$-\frac{mnR}{12X^2D}(R^2+3X^2-3D^2)+C. \tag{4}$$

In the first case, that in which $X$ is less than $D$, the limits of integration are from $R=D+X$ to $R=D-X$. In the second case, in which $X$ is greater than $D$, the limits are from $R=X+D$ to $R=X-D$.

When $X$ is less than $D$, $$I=\frac{2}{3}\frac{mn}{D}X. \tag{5}$$

When $X$ is equal to $D$, $$I=\frac{2}{3}mn. \tag{6}$$

When $X$ is greater than $D$, $$I=mn\left(1-\frac{1}{3}\frac{D^2}{X^2}\right); \tag{7}$$

and when $X$ becomes infinite, $$I=mn. \tag{8}$$

According to this form of the theory, which is that adopted by Weber*, as the magnetizing force increases from 0 to $D$, the magnetization increases in the same proportion. When the magnetizing force attains the value $D$, the magnetization is two-thirds of its limiting value. When the magnetizing force is further increased, the magnetization, instead of increasing indefinitely, tends towards a finite limit.

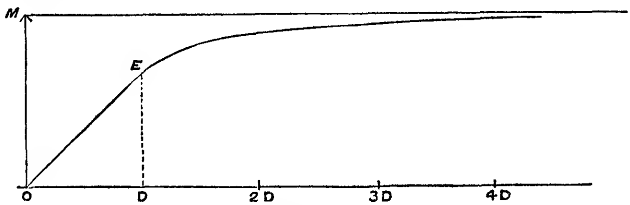


Fig. 7.

The law of magnetization is expressed in Fig. 7, where the magnetizing force is reckoned from $O$ towards the right, and the

* There is some mistake in the formula given by Weber, Abhandlungen der Kg. Sächs-Gesellschaft der Wissens. i. p. 572 (1852), or Pogg., *Ann.*, lxxxvii. p. 167 (1852), as the result of this integration, the steps of which are not given by him. His formula

$$I=mn\frac{X}{\sqrt{X^2+D^2}}\cdot\frac{X^4+\frac{7}{6}X^2D^2+\frac{2}{3}D^4}{X^4+X^2D^2+D^4}.$$

magnetization is expressed by the vertical ordinates. Weber's own experiments give results in satisfactory accordance with this law. It is probable, however, that the value of $D$ is not the same for all the molecules of the same piece of iron, so that the transition from the straight line from $O$ to $E$ to the curve beyond $E$ may not be so abrupt as is here represented.

444.] The theory in this form gives no account of the residual magnetization which is found to exist after the magnetizing force is removed. I have therefore thought it desirable to examine the results of making a further assumption relating to the conditions under which the position of equilibrium of a molecule may be permanently altered.

Let us suppose that the axis of a magnetic molecule, if deflected through any angle $\beta$ less than $\beta_0$, will return to its original position when the deflecting force is removed, but that if the deflexion $\beta$ exceeds $\beta_0$, then, when the deflecting force is removed, the axis will not return to its original position, but will be permanently deflected through an angle $\beta - \beta_0$, which may be called the permanent *set* of the molecule *.

This assumption with respect to the law of molecular deflexion is not to be regarded as founded on any exact knowledge of the intimate structure of bodies, but is adopted, in our ignorance of the true state of the case, as an assistance to the imagination in following out the speculation suggested by Weber.

Let $$L = D \sin \beta_0, \tag{9}$$
then, if the moment of the couple acting on a molecule is less than $mL$, there will be no permanent deflexion, but if it exceeds $mL$ there will be a permanent change of the position of equilibrium.

To trace the results of this supposition, describe a sphere whose centre is $O$ and radius $OL = L$.

As long as $X$ is less than $L$ everything will be the same as in the case already considered, but as soon as $X$ exceeds $L$ it will begin to produce a permanent deflexion of some of the molecules.

Let us take the case of Fig. 8, in which $X$ is greater than $L$ but less than $D$. Through $S$ as vertex draw a double cone

* {The assumption really made by Maxwell seems not to be that in this paragraph, but that enunciated in the foot-note to Art. 445.}

touching the sphere $L$. Let this cone meet the sphere $D$ in $P$ and $Q$. Then if the axis of a molecule in its original position lies between $OA$ and $OP$, or between $OB$ and $OQ$, it will be

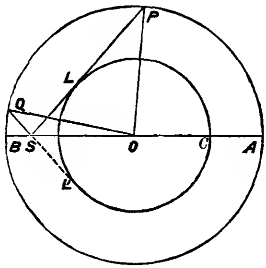


Fig. 8.

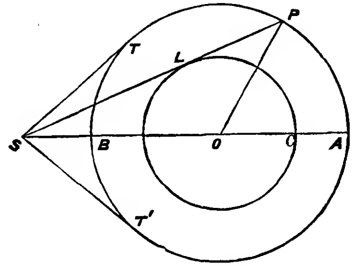


Fig. 9.

deflected through an angle less than $\beta_0$, and will not be permanently deflected. But if the axis of the molecule lies originally between $OP$ and $OQ$, then a couple whose moment is greater than $L$ will act upon it and will deflect it into the position $SP$, and when the force $X$ ceases to act it will not resume its original direction, but will be permanently set in the direction $OP$.

Let us put

$$L = X \sin \theta_0 \quad \text{where} \quad \theta_0 = PSA \text{ or } QSB,$$

then all those molecules whose axes, on the former hypothesis, would have values of $\theta$ between $\theta_0$ and $\pi - \theta_0$ will be made to have the value $\theta_0$ during the action of the force $X$.

During the action of the force $X$, therefore, those molecules whose axes when deflected lie within either sheet of the double cone whose semivertical angle is $\theta_0$ will be arranged as in the former case, but all those whose axes on the former theory would lie outside of these sheets will be permanently deflected, so that their axes will form a dense fringe round that sheet of the cone which lies towards $A$.

As $X$ increases, the number of molecules belonging to the cone about $B$ continually diminishes, and when $X$ becomes equal to $D$ all the molecules have been wrenched out of their former positions of equilibrium, and have been forced into the fringe of the cone round $A$, so that when $X$ becomes greater than $D$ all the molecules form part of the cone round $A$ or of its fringe.

When the force $X$ is removed, then in the case in which $X$ is less than $L$ everything returns to its primitive state. When $X$ is between $L$ and $D$, then there is a cone round $A$ whose angle

$$AOP = \theta_0 + \beta_0,$$

and another cone round $B$ whose angle

$$BOQ = \theta_0 - \beta_0.$$

Within these cones the axes of the molecules are distributed uniformly. But all the molecules, the original direction of whose axes lay outside of both these cones, have been wrenched from their primitive positions and form a fringe round the cone about $A$.

If $X$ is greater than $D$, then the cone round $B$ is completely dispersed, and all the molecules which formed it are converted into the fringe round $A$, and are inclined at the angle $\theta_0 + \beta_0$.

445.] Treating this case in the same way as before *, we find

* [The results given in the text may be obtained, with one slight exception, by the processes given below, the statement of the modified theory of Art. 444 being as follows: The axis of a magnetic molecule, if deflected through an angle $\beta$ less than $\beta_0$, will return to its original position when the deflecting force is removed; but when the deflexion exceeds $\beta_0$ the force tending to oppose the deflexion gives way and permits the molecule to be deflected into the same direction as those whose deflexion is $\beta_0$, and when the deflecting force is removed the molecule takes up a direction parallel to that of the molecule whose deflexion was $\beta_0$. This direction may be called the permanent set of the molecules.

In the case $X > L < D$, the expression $I$ for the magnetic moment consists of two parts, the first of which is due to the molecules within the cones $AOP$, $BOQ$, and is to be found precisely as in Art. 443, due regard being had to the limits of integration. Referring to Fig. 8 we find for the second part, according to the above statement of the theory,

$$\tfrac{1}{2}\, mn \cos ASP \times \frac{\text{Projection of } QP \text{ on } BA}{OP}.$$

The two parts together when reduced give the result in the text.

When $X > D$, the integral again consists of two parts, one of which is to be taken over the cone $AOP$ as in Art. 443. The second part is, (Fig. 9),

$$\tfrac{1}{2}\, mn \cos ASP \times \frac{\text{Projection of } BP \text{ on } BA}{OP}.$$

The value of $I$ in this case, when reduced, differs from the value given in the text in the third term, viz.: we have then $-\dfrac{1}{6}\dfrac{D^2}{X^2}$ instead of $-\dfrac{1}{6}\dfrac{D}{X}$. The effect of this change on the table of numerical values given in the text will be that when $X = 6$, 7, 8, the corresponding values of $I$ will be 887, 917, 936. These changes do not alter the general character of the curve of Temporary Magnetization given in Fig. 10.

The value of $I'$ in the case of Fig. 8 is

$$\tfrac{1}{2}\, mn \left\{ \int_0^{AOP} \sin\alpha \cos\alpha\, d\alpha + \int_{AOQ}^{\pi} \sin\alpha \cos\alpha\, d\alpha \right.$$
$$\left. + \cos AOP \times \frac{\text{Projection of } QP \text{ on } BA}{OP} \right\}.$$

The value of $I'$ in the case of Fig. 9 may be found in like manner.]

for the intensity of the temporary magnetization during the action of the force $X$, which is supposed to act on iron which has never before been magnetized,

When $X$ is less than $L$, $\quad I = \frac{2}{3} M \frac{X}{D}$.

When $X$ is equal to $L$, $\quad I = \frac{2}{3} M \frac{L}{D}$.

When $X$ is between $L$ and $D$,

$$I = M\left\{\frac{2}{3}\frac{X}{D} + \left(1 - \frac{L^2}{X^2}\right)\left[\sqrt{1 - \frac{L^2}{D^2}} - \frac{2}{3}\sqrt{\frac{X^2}{D^2} - \frac{L^2}{D^2}}\right]\right\}.$$

When $X$ is equal to $D$,

$$I = M\left\{\frac{2}{3} + \frac{1}{3}\left(1 - \frac{L^2}{D^2}\right)^{\frac{3}{2}}\right\}.$$

When $X$ is greater than $D$,

$$I = M\left\{\frac{1}{3}\frac{X}{D} + \frac{1}{2} - \frac{1}{6}\frac{D}{X} + \frac{(D^2 - L^2)^{\frac{3}{2}}}{6\,X^2 D} - \frac{\sqrt{X^2 - L^2}}{6\,X^2 D}(2\,X^2 - 3\,XD + L^2)\right\}.$$

When $X$ is infinite, $\quad I = M$.

When $X$ is less than $L$ the magnetization follows the former law, and is proportional to the magnetizing force. As soon as $X$ exceeds $L$ the magnetization assumes a more rapid rate of increase on account of the molecules beginning to be transferred from the one cone to the other. This rapid increase, however, soon comes to an end as the number of molecules forming the negative cone diminishes, and at last the magnetization reaches the limiting value $M$.

If we were to assume that the values of $L$ and of $D$ are different for different molecules, we should obtain a result in which the different stages of magnetization are not so distinctly marked.

The residual magnetization, $I'$, produced by the magnetizing force $X$, and observed after the force has been removed, is as follows:

When $X$ is less than $L$, $\quad$ No residual magnetization.

When $X$ is between $L$ and $D$,

$$I' = M\left(1 - \frac{L^2}{D^2}\right)\left(1 - \frac{L^2}{X^2}\right).$$

When $X$ is equal to $D$,

$$I' = M\left(1 - \frac{L^2}{D^2}\right)^2.$$

When $X$ is greater than $D$,

$$I' = \frac{1}{4} M \left\{ 1 - \frac{L^2}{XD} + \sqrt{1 - \frac{L^2}{D^2}} \sqrt{1 - \frac{L^2}{X^2}} \right\}^2.$$

When $X$ is infinite,

$$I' = \frac{1}{4} M \left\{ 1 + \sqrt{1 - \frac{L^2}{D^2}} \right\}^2.$$

If we make

$$M = 1000, \qquad L = 3, \qquad D = 5,$$

we find the following values of the temporary and the residual magnetization :—

| Magnetizing Force. $X$ | Temporary Magnetization. $I$ | Residual Magnetization. $I'$ |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 133 | 0 |
| 2 | 267 | 0 |
| 3 | 400 | 0 |
| 4 | 729 | 280 |
| 5 | 837 | 410 |
| 6 | 864 | 485 |
| 7 | 882 | 537 |
| 8 | 897 | 575 |
| $\infty$ | 1000 | 810 |

These results are laid down in Fig. 10.

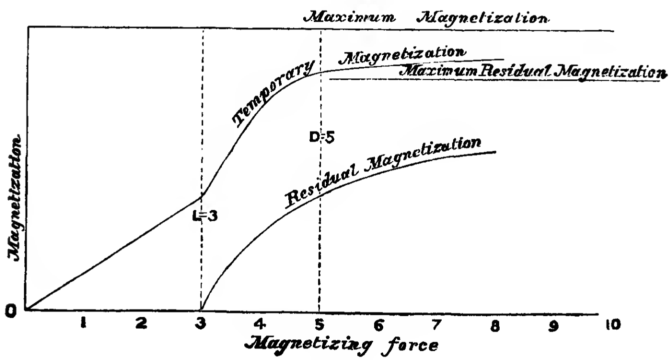


Fig. 10.

The curve of temporary magnetization is at first a straight line from $X = 0$ to $X = L$. It then rises more rapidly till $X = D$, and as $X$ increases it approaches its horizontal asymptote.

The curve of residual magnetization begins when $X = L$, and approaches an asymptote whose ordinate $= \cdot 81\ M$.

It must be remembered that the residual magnetism thus found corresponds to the case in which, when the external force is removed, there is no demagnetizing force arising from the distribution of magnetism in the body itself. The calculations are therefore applicable only to very elongated bodies magnetized longitudinally. In the case of short thick bodies the residual magnetism will be diminished by the reaction of the free magnetism in the same way as if an external reversed magnetizing force were made to act upon it *.

446.] The scientific value of a theory of this kind, in which we make so many assumptions, and introduce so many adjustable constants, cannot be estimated merely by its numerical agreement with certain sets of experiments. If it has any value it is because it enables us to form a mental image of what takes place in a piece of iron during magnetization. To test the theory, we shall apply it to the case in which a piece of iron, after being subjected to a magnetizing force $X_0$, is again subjected to a magnetizing force $X_1$.

If the new force $X_1$ acts in the same direction as that in which $X_0$ acted, which we shall call the positive direction, then $X_1$, if less than $X_0$, will produce no permanent set of the molecules, and when $X_1$ is removed the residual magnetization will be the same as that produced by $X_0$. If $X_1$ is greater than $X_0$, then it will produce exactly the same effect as if $X_0$ had not acted.

But let us suppose $X_1$ to act in the negative direction, and let us suppose $$X_0 = L \operatorname{cosec} \theta_0, \quad \text{and} \quad X_1 = -L \operatorname{cosec} \theta_1.$$

* {Consider the case of a piece of iron subjected to a magnetic force in the positive direction which increases from zero to a value $X_0$ sufficient to produce permanent magnetization, then let the magnetic force diminish again to zero, it is evident that on the preceding theory the intensity of magnetization will in consequence of the permanent set given to some of the molecular magnets be greater for a given value of the magnetizing force when this force is decreasing than when it was increasing. Thus the behaviour of the iron in the magnetic field will depend upon its previous treatment. This effect has been called hysteresis by Ewing and has been very fully investigated by him (see *Phil. Trans.* Part II, 1885). The theory given in Art. 445 will not however explain all the phenomena discovered by Ewing, for if in the above case after decreasing the magnetic force we increase it again, the value of the intensity of magnetization for a value $X_1 < X_0$ of the magnetic force ought to be the same as when the force was first decreased to $X_1$. Ewing's researches shew however that it is not so. A short account of these and similar researches will be given in the Supplementary Volume.}

As $X_1$ increases numerically, $\theta_1$ diminishes. The first molecules on which $X_1$ will produce a permanent deflexion are those which form the fringe of the cone round $A$*, and these have an inclination when undeflected of $\theta_0+\beta_0$.

As soon as $\theta_1-\beta_0$ becomes less than $\theta_0+\beta_0$ the process of demagnetization will commence. Since, at this instant, $\theta_1=\theta_0+2\beta_0$, $X_1$, the force required to begin the demagnetization, is less than $X_0$, the force which produced the magnetization.

If the values of $D$ and of $L$ were the same for all the molecules, the slightest increase of $X_1$ would wrench the whole of the fringe of molecules whose axes have the inclination $\theta_0+\beta_0$ into a position in which their axes are inclined $\theta_1+\beta_0$ to the negative axis $OB$.

Though the demagnetization does not take place in a manner so sudden as this, it takes place so rapidly as to afford some confirmation of this mode of explaining the process.

Let us now suppose that by giving a proper value to the reverse force $X_1$ we have on the removal of $X_1$ exactly demagnetized the piece of iron.

The axes of the molecules will not now be arranged indifferently in all directions, as in a piece of iron which has never been magnetized, but will form three groups.

(1) Within a cone of semiangle $\theta_1-\beta_0$ surrounding the positive pole, the axes of the molecules remain in their primitive positions.

(2) The same is the case within a cone of semiangle $\theta_0-\beta_0$ surrounding the negative pole.

(3) The directions of the axes of all the other molecules form a conical sheet surrounding the negative pole, and are at an inclination $\theta_1+\beta_0$.

When $X_0$ is greater than $D$ the second group is absent. When $X_1$ is greater than $D$ the first group is also absent.

The state of the iron, therefore, though apparently demagnetized, is different from that of a piece of iron which has never been magnetized.

To shew this, let us consider the effect of a magnetizing force $X_2$ acting in either the positive or the negative direction. The first permanent effect of such a force will be on the third group

* {This assumes that in figs. 8 and 9 $P$ is to the right of $C$.}

of molecules, whose axes make angles $= \theta_1 + \beta_0$ with the negative axis.

If the force $X_2$ acts in the negative direction it will begin to produce a permanent effect as soon as $\theta_2 + \beta_0$ becomes less than $\theta_1 + \beta_0$, that is, as soon as $X_2$ becomes greater than $X_1$. But if $X_2$ acts in the positive direction it will begin to remagnetize the iron as soon as $\theta_2 - \beta_0$ becomes less than $\theta_1 + \beta_0$, that is, when $\theta_2 = \theta_1 + 2\beta_0$, or while $X_2$ is still much less than $X_1$.

It appears therefore from our hypothesis that—

When a piece of iron is magnetized by means of a force $X_0$, its residual magnetism cannot be increased without the application of a force greater than $X_0$. A reverse force, less than $X_0$, is sufficient to diminish its residual magnetization.

If the iron is exactly demagnetized by the reversed force $X_1$, then it cannot be magnetized in the reversed direction without the application of a force greater than $X_1$, but a positive force less than $X_1$ is sufficient to begin to remagnetize the iron in its original direction.

These results are consistent with what has been actually observed by Ritchie *, Jacobi †, Marianini ‡, and Joule §.

A very complete account of the relations of the magnetization of iron and steel to magnetic forces and to mechanical strains is given by Wiedemann in his *Galvanismus*. By a detailed comparison of the effects of magnetization with those of torsion, he shews that the ideas of elasticity and plasticity which we derive from experiments on the temporary and permanent torsion of wires can be applied with equal propriety to the temporary and permanent magnetization of iron and steel.

447.] Matteucci || found that the extension of a hard iron bar during the action of the magnetizing force increases its temporary magnetism ¶. This has been confirmed by Wertheim. In the case of soft iron bars the magnetism is diminished by extension.

The permanent magnetism of an iron bar increases when it is extended, and diminishes when it is compressed.

* *Phil. Mag.* 3, 1833. † Pogg., *Ann.*, 31, 367, 1834.

‡ *Ann. de Chimie et de Physique*, 16, pp. 436 and 448, 1846.

§ *Phil. Trans.*, 1856, p. 287. || *Ann. de Chimie et de Physique*, 53, p. 385, 1858.

¶ {Villari shewed that this is only true when the magnetizing force is less than a certain critical value, but when it exceeds this value an extension produces a diminution on the intensity of magnetization; Pogg., *Ann.* 126, p. 87, 1865.

The statement in the text as to the behaviour of soft iron bars does not hold for small strains and low magnetic fields.}

Hence, if a piece of iron is first magnetized in one direction, and then extended in another direction, the direction of magnetization will tend to approach the direction of extension. If it be compressed, the direction of magnetization will tend to become normal to the direction of compression.

This explains the result of an experiment of Wiedemann's. A current was passed downward through a vertical wire. If, either during the passage of the current or after it has ceased, the wire be twisted in the direction of a right-handed screw, the lower end becomes a north pole.

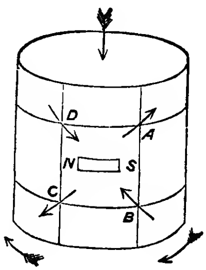


Fig. 11.

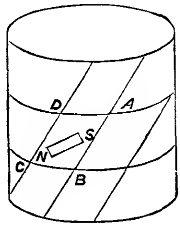


Fig. 12.

Here the downward current magnetizes every part of the wire in a tangential direction, as indicated by the letters *NS*.

The twisting of the wire in the direction of a right-handed screw causes the portion *ABCD* to be extended along the diagonal *AC* and compressed along the diagonal *BD*. The direction of magnetization therefore tends to approach *AC* and to recede from *BD*, and thus the lower end becomes a north pole and the upper end a south pole.

*Effect of Magnetization on the Dimensions of the Magnet.*

448.] Joule *, in 1842, found that an iron bar becomes lengthened when it is rendered magnetic by an electric current in a coil which surrounds it. He afterwards † shewed, by placing the bar in water within a glass tube, that the volume of the iron is not augmented by this magnetization, and concluded that its transverse dimensions were contracted.

Finally, he passed an electric current through the axis of an

* Sturgeon's *Annals of Electricity*, vol. viii. p. 219.
† *Phil. Mag.*, xxx. 1847.

iron tube, and back outside the tube, so as to make the tube into a closed magnetic solenoid, the magnetization being at right angles to the axis of the tube. The length of the axis of the tube was found in this case to be shortened.

He found that an iron rod under longitudinal pressure is also elongated when it is magnetized. When, however, the rod is under considerable longitudinal tension, the effect of magnetization is to shorten it.

This was the case with a wire of a quarter of an inch diameter when the tension exceeded 600 pounds weight.

In the case of a hard steel wire the effect of the magnetizing force was in every case to shorten the wire, whether the wire was under tension or pressure. The change of length lasted only as long as the magnetizing force was in action, no alteration of length was observed due to the permanent magnetization of the steel.

Joule found the elongation of iron wires to be nearly proportional to the square of the actual magnetization, so that the first effect of a demagnetizing current was to shorten the wire*.

On the other hand, he found that the shortening effect on wires under tension, and on steel, varied as the product of the magnetization and the magnetizing current.

Wiedemann found that if a vertical wire is magnetized with its south end uppermost, and if a current is then passed downwards through the wire, the lower end of the wire, if free, twists in the direction of the hands of a watch as seen from above, or, in other words, the wire becomes twisted like a right-handed screw if the relation between the longitudinal current and the magnetizing current is right-handed.

In this case the resultant magnetization due to the action of the current and the previously existing magnetization is in the direction of a right-handed screw round the wire. Hence the twisting would indicate that when the iron is magnetized it expands in the direction of magnetization and contracts in directions at right angles to the magnetization. This agrees with Joule's results.

For further developments of the theory of magnetization, see Arts. 832–845.

* {Shelford Bidwell has shewn that when the magnetizing force is very great, the length of the magnet diminishes as the magnetizing force increases. *Proc. Roy. Soc.* xl. p. 109.}

# CHAPTER VII.

## MAGNETIC MEASUREMENTS.

449.] THE principal magnetic measurements are the determination of the magnetic axis and magnetic moment of a magnet, and that of the direction and intensity of the magnetic force at a given place.

Since these measurements are made near the surface of the earth, the magnets are always acted on by gravity as well as by terrestrial magnetism, and since the magnets are made of steel their magnetism is partly permanent and partly induced. The permanent magnetism is altered by changes of temperature, by strong induction, and by violent blows; the induced magnetism varies with every variation of the external magnetic force.

The most convenient way of observing the force acting on a magnet is by making the magnet free to turn about a vertical axis. In ordinary compasses this is done by balancing the magnet on a vertical pivot. The finer the point of the pivot the smaller is the moment of the friction which interferes with the action of the magnetic force. For more refined observations the magnet is suspended by a thread composed of a silk fibre without twist, either single, or doubled on itself a sufficient number of times, and so formed into a thread of parallel fibres, each of which supports as nearly as possible an equal part of the weight. The force of torsion of such a thread is much less than that of a metal wire of equal strength, and it may be calculated in terms of the observed azimuth of the magnet, which is not the case with the force arising from the friction of a pivot.

The suspension fibre can be raised or lowered by turning a horizontal screw which works in a fixed nut. The fibre is wound round the thread of the screw, so that when the screw is turned the suspension fibre always hangs in the same vertical line.

The suspension fibre carries a small horizontal divided circle called the Torsion-circle, and a stirrup with an index, which can be placed so that the index coincides with any given division of the torsion circle. The stirrup is so shaped that the magnet bar can be fitted into it with its axis horizontal, and with any one of its four sides uppermost.

To ascertain the zero of torsion a non-magnetic body of the same weight as the magnet is placed in the stirrup, and the position of the torsion circle when in equilibrium ascertained.

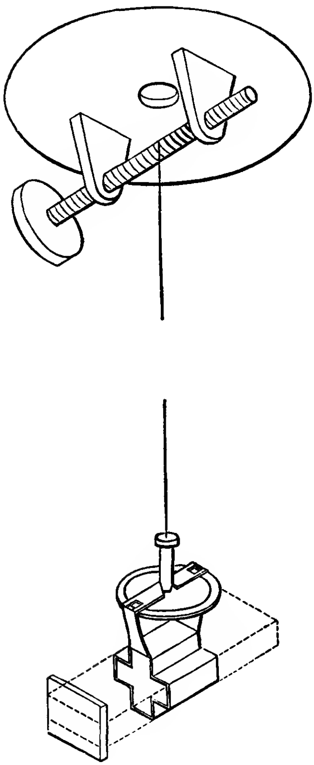

Fig. 13.

The magnet itself is a piece of hard-tempered steel. According to Gauss and Weber its length ought to be at least eight times its greatest transverse dimension. This is necessary when permanence of the direction of the magnetic axis within the magnet is the most important consideration. Where promptness of motion is required the magnet should be shorter, and it may even be advisable in observing sudden alterations in magnetic force to use a bar magnetized transversely and suspended with its longest dimension vertical *.

450.] The magnet is provided with an arrangement for ascertaining its angular position. For ordinary purposes its ends are pointed, and a divided circle is placed below the ends, by which their positions are read off by an eye placed in a plane through the suspension thread and the point of the needle.

For more accurate observations a plane mirror is fixed to the magnet, so that the normal to the mirror coincides as nearly as

* Joule, *Proc. Phil. Soc., Manchester*, Nov. 29, 1864.

possible with the axis of magnetization. This is the method adopted by Gauss and Weber.

Another method is to attach to one end of the magnet a lens and to the other end a scale engraved on glass, the distance of the lens from the scale being equal to the principal focal length of the lens. The straight line joining the zero of the scale with the optical centre of the lens ought to coincide as nearly as possible with the magnetic axis.

As these optical methods of ascertaining the angular position of suspended apparatus are of great importance in many physical researches, we shall here consider once for all their mathematical theory.

*Theory of the Mirror Method.*

We shall suppose that the apparatus whose angular position is to be determined is capable of revolving about a vertical axis. This axis is in general a fibre or wire by which it is suspended. The mirror should be truly plane, so that a scale of millimetres may be seen distinctly by reflexion at a distance of several metres from the mirror.

The normal through the middle of the mirror should pass through the axis of suspension, and should be accurately horizontal. We shall refer to this normal as the line of collimation of the apparatus.

Having roughly ascertained the mean direction of the line of collimation during the experiments which are to be made, a telescope is erected at a convenient distance in front of the mirror, and a little above the level of the mirror.

The telescope is capable of motion in a vertical plane, it is directed towards the suspension-fibre just above the mirror, and a fixed mark is erected in the line of vision, at a horizontal distance from the object-glass equal to twice the distance of the mirror from the object-glass. The apparatus should, if possible, be so arranged that this mark is on a wall or other fixed object. In order to see the mark and the suspension-fibre at the same time through the telescope, a cap may be placed over the object-glass having a slit along a vertical diameter. This should be removed for the other observations. The telescope is then adjusted so that the mark is seen distinctly to coincide with the vertical wire at the focus of the telescope. A plumb-line is then adjusted so as to pass close in front of the optical centre of the object-glass and to hang below the telescope. Below the telescope and just behind the plumb-line a scale of equal parts is placed so as to be bisected at right angles by the plane through the mark, the suspension-fibre, and the plumb-line. The sum of the heights of the scale and the object-glass from the floor should be equal to twice the height of the mirror. The telescope being now directed towards the mirror, the observer will see in it the reflexion of the scale. If the part of the scale where the plumb-line crosses it appears to coincide with the vertical wire of the telescope, then the line of collimation of the mirror coincides with the plane through the mark and the optical centre of the object-glass. If the vertical wire coincides with any other division of the scale, the angular position of the line of collimation is to be found as follows:—

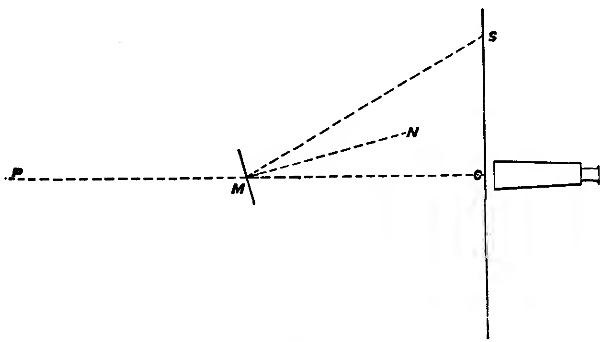


Fig. 14.

Let the plane of the paper be horizontal, and let the various points be projected on this plane. Let $O$ be the centre of the object-glass of the telescope, $P$ the fixed mark: $P$ and the vertical wire of the telescope are conjugate foci with respect to the object-glass. Let $M$ be the point where $OP$ cuts the plane of the mirror. Let $MN$ be the normal to the mirror; then $OMN = \theta$ is the angle which the line of collimation makes with the fixed plane. Let $MS$ be a line in the plane of $OM$ and $MN$, such that $NMS = OMN$, then $S$ will be the part of the scale which will be seen by reflexion to coincide with the vertical wire of the telescope. Now, since $MN$ is horizontal, the projected angles $OMN$ and $NMS$ in the figure are equal, and $OMS = 2\theta$. Hence $OS = OM \tan 2\theta$.

We have therefore to measure $OM$ in terms of the divisions of the scale; then, if $s_0$ is the division of the scale which coincides with the plumb-line, and $s$ the observed division,

$$s - s_0 = OM \tan 2\theta,$$

whence $\theta$ may be found. In measuring $OM$ we must remember that if the mirror is of glass, silvered at the back, the virtual reflecting surface is at a distance behind the front surface of the glass $= \frac{t}{\mu}$, where $t$ is the thickness of the glass, and $\mu$ is the index of refraction.

We must also remember that if the line of suspension does not pass through the point of reflexion, the position of $M$ will alter with $\theta$. Hence, when it is possible, it is advisable to make the centre of the mirror coincide with the line of suspension.

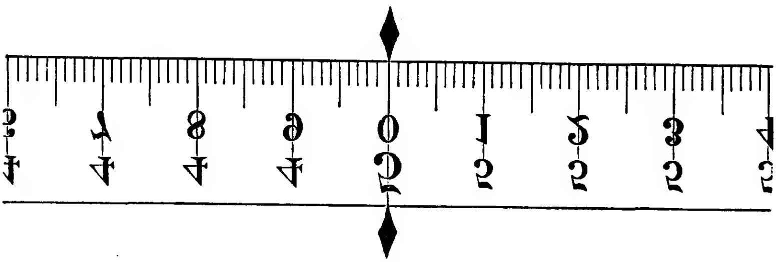

Fig. 15.

It is also advisable, especially when large angular motions have to be observed, to make the scale in the form of a concave cylindric surface, whose axis is the line of suspension. The angles are then observed at once in circular measure without reference to a table of tangents. The scale should be carefully adjusted, so that the axis of the cylinder coincides with the suspension-fibre. The numbers on the scale should always run from the one end to the other in the same direction so as to avoid negative readings. Fig. 15 represents the middle portion of a scale to be used with a mirror and an inverting telescope.

This method of observation is the best when the motions are slow. The observer sits at the telescope and sees the image of the scale moving to right or to left past the vertical wire of the telescope. With a clock beside him he can note the instant at which a given division of the scale passes the wire, or the division of the scale which is passing at a given tick of the clock, and he can also record the extreme limits of each oscillation.

When the motion is more rapid it becomes impossible to read the divisions of the scale except at the instants of rest at the extremities of an oscillation. A conspicuous mark may be placed at a known division of the scale, and the instant of transit of this mark may be noted.

When the apparatus is very light, and the forces variable, the motion is so prompt and swift that observation through a telescope would be useless. In this case the observer looks at the scale directly, and observes the motions of the image of the vertical wire thrown on the scale by a lamp.

It is manifest that since the image of the scale reflected by the mirror and refracted by the object-glass coincides with the vertical wire, the image of the vertical wire, if sufficiently illuminated, will coincide with the scale. To observe this the room is darkened, and the concentrated rays of a lamp are thrown on the vertical wire towards the object-glass. A bright patch of light crossed by the shadow of the wire is seen on the scale. Its motions can be followed by the eye, and the division of the scale at which it comes to rest can be fixed on by the eye and read off at leisure. If it be desired to note the instant of the passage of the bright spot past a given point on the scale, a pin or a bright metal wire may be placed there so as to flash out at the time of passage.

By substituting a small hole in a diaphragm for the cross-wire the image becomes a small illuminated dot moving to right or left on the scale, and by substituting for the scale a cylinder revolving by clock-work about a horizontal axis and covered with photographic paper, the spot of light traces out a curve which can be afterwards rendered visible. Each abscissa of this curve corresponds to a particular time, and the ordinate indicates the angular position of the mirror at that time. In this way an automatic system of continuous registration of all the elements of terrestrial magnetism has been established at Kew and other observatories.

In some cases the telescope is dispensed with, a vertical wire is illuminated by a lamp placed behind it, and the mirror is a concave one, which forms the image of the wire on the scale as a dark line across a patch of light.

451.] In the Kew portable apparatus, the magnet is made in the form of a tube, having at one end a lens, and at the other a glass scale, so adjusted as to be at the principal focus of the lens. Light is admitted from behind the scale, and after passing through the lens it is viewed by means of a telescope.

Since the scale is at the principal focus of the lens, rays from any division of the scale emerge from the lens parallel, and if the telescope is adjusted for celestial objects, it will shew the scale in optical coincidence with the cross-wires of the telescope. If a given division of the scale coincides with the intersection of the cross-wires, then the line joining that division with the optical centre of the lens must be parallel to the line of collimation of the telescope. By fixing the magnet and moving the telescope, we may ascertain the angular value of the divisions of the scale, and then, when the magnet is suspended and the position of the telescope known, we may determine the position of the magnet at any instant by reading off the division of the scale which coincides with the cross-wires.

The telescope is supported on an arm which is centred in the line of the suspension-fibre, and the position of the telescope is read off by verniers on the azimuth circle of the instrument.

This arrangement is suitable for a small portable magnetometer in which the whole apparatus is supported on one tripod, and in which the oscillations due to accidental disturbances rapidly subside.

*Determination of the Direction of the Axis of the Magnet, and of the Direction of Terrestrial Magnetism.*

452.] Let a system of axes be drawn in a magnet, of which the axis of $z$ is in the direction of the length of the bar, and $x$ and $y$ perpendicular to the sides of the bar supposed a parallelopiped.

Let $l$, $m$, $n$ and $\lambda$, $\mu$, $\nu$ be the angles which the magnetic axis and the line of collimation make with these axes respectively.

Let $M$ be the magnetic moment of the magnet, let $H$ be the horizontal component of terrestrial magnetism, let $Z$ be the vertical component, and let $\delta$ be the azimuth in which $H$ acts, reckoned from the north towards the west.

Let $\zeta$ be the observed azimuth of the line of collimation, let $a$ be the azimuth of the stirrup, and $\beta$ the reading of the index of the torsion circle, then $\alpha-\beta$ is the azimuth of the lower end of the suspension-fibre.

Let $\gamma$ be the value of $\alpha-\beta$ when there is no torsion, then the moment of the force of torsion tending to diminish $\alpha$ will be

$$\tau(\alpha-\beta-\gamma),$$

where $\tau$ is a coefficient of torsion depending on the nature of the fibre.

To determine $\lambda_x$, the angle between the axis of $x$ and the projection of the line of collimation on the plane of $xz$, fix the stirrup so that $y$ is vertical and upwards, $z$ to the north and $x$ to the west, and observe the azimuth $\zeta$ of the line of collimation. Then remove the magnet, turn it through an angle $\pi$ about the axis of $z$ and replace it in this inverted position, and observe the azimuth $\zeta'$ of the line of collimation when $y$ is downwards and $x$ to the east,

$$\zeta = \alpha + \frac{\pi}{2} - \lambda_x, \tag{1}$$

$$\zeta' = \alpha - \frac{\pi}{2} + \lambda_x. \tag{2}$$

Hence

$$\lambda_x = \frac{\pi}{2} + \frac{1}{2}(\zeta' - \zeta). \tag{3}$$

Next, hang the stirrup to the suspension-fibre, and place the magnet in it, adjusting it carefully so that $y$ may be vertical and upwards, then the moment of the force tending to increase $\alpha$ is

$$MH \sin m \sin\left(\delta - \alpha - \frac{\pi}{2} + l_x\right) - \tau(\alpha-\beta-\gamma); \tag{4}$$

where $l_x$ is the angle between the axis of $x$ and the projection of the magnetic axis on the plane of $xz$.

But if $\zeta$ is the observed azimuth of the line of collimation

$$\zeta = \alpha + \frac{\pi}{2} - \lambda_x, \tag{5}$$

so that the force may be written

$$MH \sin m \sin(\delta - \zeta + l_x - \lambda_x) - \tau\left(\zeta + \lambda_x - \frac{\pi}{2} - \beta - \gamma\right). \tag{6}$$

When the apparatus is in equilibrium this quantity is zero for a particular value of $\zeta$.

When the apparatus never comes to rest, but must be observed in a state of vibration, the value of $\zeta$ corresponding to the position of equilibrium may be calculated by a method which will be described in Art. 735.

When the force of torsion is small compared with the moment of the magnetic force, we may put $\delta-\zeta+l_x-\lambda_x$ for the sine of that angle.

If we give to $\beta$, the reading of the torsion circle, two different values, $\beta_1$ and $\beta_2$, and if $\zeta_1$ and $\zeta_2$ are the corresponding values of $\zeta$,

$$MH(\zeta_2-\zeta_1)\sin m = \tau(\zeta_1-\zeta_2-\beta_1+\beta_2), \tag{7}$$

or, if we put

$$\frac{\zeta_2-\zeta_1}{\zeta_1-\zeta_2-\beta_1+\beta_2}=\tau', \quad \text{then} \quad \tau=\tau' MH\sin m, \tag{8}$$

and equation (6) becomes, dividing by $MH\sin m$,

$$\delta-\zeta+l_x-\lambda_x-\tau'\left(\zeta+\lambda_x-\frac{\pi}{2}-\beta-\gamma\right)=0. \tag{9}$$

If we now reverse the magnet so that $y$ is downwards, and adjust the apparatus till $y$ is exactly vertical, and if $\zeta'$ is the new value of the azimuth, and $\delta'$ the corresponding declination,

$$\delta'-\zeta'-l_x+\lambda_x-\tau'\left(\zeta'-\lambda_x+\frac{\pi}{2}-\beta-\gamma\right)=0, \tag{10}$$

whence

$$\frac{\delta+\delta'}{2}=\tfrac{1}{2}(\zeta+\zeta')+\tfrac{1}{2}\tau'\{\zeta+\zeta'-2(\beta+\gamma)\}. \tag{11}$$

The reading of the torsion circle should now be adjusted, so that the coefficient of $\tau'$ may be as nearly as possible zero. For this purpose we must determine $\gamma$, the value of $a-\beta$ when there is no torsion. This may be done by placing a non-magnetic bar of the same weight as the magnet in the stirrup, and determining $a-\beta$ when there is equilibrium. Since $\tau'$ is small, great accuracy is not required. Another method is to use a torsion bar of the same weight as the magnet, containing within it a very small magnet whose magnetic moment is $\frac{1}{n}$ of that of the principal magnet. Since $\tau$ remains the same, $\tau'$ will become $n\tau'$, and if $\zeta_1$ and $\zeta_1'$ are the values of $\zeta$ as found by the torsion bar,

$$\frac{\delta+\delta'}{2}=\tfrac{1}{2}(\zeta_1+\zeta_1')+\tfrac{1}{2}n\tau'\{\zeta_1+\zeta_1'-2(\beta+\gamma)\}. \tag{12}$$

Subtracting this equation from (11),

$$2(n-1)(\beta+\gamma)=\left(n+\frac{1}{\tau'}\right)(\zeta_1+\zeta_1')-\left(1+\frac{1}{\tau'}\right)(\zeta+\zeta'). \tag{13}$$

Having found the value of $\beta+\gamma$ in this way, $\beta$, the reading of the torsion circle, should be altered till

$$\zeta+\zeta'-2(\beta+\gamma)=0, \tag{14}$$

as nearly as possible in the ordinary position of the apparatus.

Then, since $\tau'$ is a very small numerical quantity, and since its coefficient is very small, the value of the second term in the expression for $\delta$ will not vary much for small errors in the values of $\tau'$ and $\gamma$, which are the quantities whose values are least accurately known.

The value of $\delta$, the magnetic declination, may be found in this way with considerable accuracy, provided it remains constant during the experiments, so that we may assume $\delta' = \delta$.

When great accuracy is required it is necessary to take account of the variations of $\delta$ during the experiment. For this purpose observations of another suspended magnet should be made at the same instants that the different values of $\zeta$ are observed, and if $\eta$, $\eta'$ are the observed azimuths of the second magnet corresponding to $\zeta$ and $\zeta'$, and if $\delta$ and $\delta'$ are the corresponding values of $\delta$, then

$$\delta' - \delta = \eta' - \eta. \tag{15}$$

Hence, to find the value of $\delta$ we must add to (11) a correction

$$\tfrac{1}{2}(\eta - \eta').$$

The declination at the time of the first observation is therefore

$$\delta = \tfrac{1}{2}(\zeta + \zeta' + \eta - \eta') + \tfrac{1}{2}\tau'(\zeta + \zeta' - 2\beta - 2\gamma). \tag{16}$$

To find the direction of the magnetic axis within the magnet subtract (10) from (9) and add (15),

$$l_x = \lambda_x + \tfrac{1}{2}(\zeta - \zeta') - \tfrac{1}{2}(\eta - \eta') + \tfrac{1}{2}\tau'(\zeta - \zeta' + 2\lambda_x - \pi). \tag{17}$$

By repeating the experiments with the bar on its two edges, so that the axis of $x$ is vertically upwards and downwards, we can find the value of $m$. If the axis of collimation is capable of adjustment it ought to be made to coincide with the magnetic axis as nearly as possible, so that the error arising from the magnet not being exactly inverted may be as small as possible*.

### *On the Measurement of Magnetic Forces.*

453.] The most important measurements of magnetic force are those which determine $M$, the magnetic moment of a magnet, and $H$, the intensity of the horizontal component of terrestrial magnetism. This is generally done by combining the results of two experiments, one of which determines the ratio and the other th eproduct of these two quantities.

The intensity of the magnetic force due to an infinitely small

* See a Paper on 'Imperfect Inversion,' by W. Swan. *Trans. R. S. Edin.*, vol. xxi (1855), p. 349.

magnet whose magnetic moment is $M$, at a point distant $r$ from the centre of the magnet in the positive direction of the axis of the magnet, is

$$R = 2\frac{M}{r^3} \tag{1}$$

and is in the direction of $r$. If the magnet is of finite size but spherical, and magnetized uniformly in the direction of its axis, this value of the force will still be exact. If the magnet is a solenoidal bar magnet of length $2L$,

$$R = 2\frac{M}{r^3}\left(1 + 2\frac{L^2}{r^2} + 3\frac{L^4}{r^4} + \&c.\right). \tag{2}$$

If the magnet be of any kind, provided its dimensions are all small, compared with $r$,

$$R = 2\frac{M}{r^3}\left(1 + A_1\frac{1}{r} + A_2\frac{1}{r^2} + \&c.\right), \tag{3}$$

where $A_1$, $A_2$, &c. are coefficients depending on the distribution of the magnetization of the bar.

Let $H$ be the intensity of the horizontal part of terrestrial magnetism at any place. $H$ is directed towards magnetic north. Let $r$ be measured towards magnetic west, then the magnetic force at the extremity of $r$ will be $H$ towards the north and $R$ towards the west. The resultant force will make an angle $\theta$ with the magnetic meridian, measured towards the west, and such that

$$R = H \tan\theta. \tag{4}$$

Hence, to determine $\frac{R}{H}$ we proceed as follows:—

The direction of the magnetic north having been ascertained, a magnet, whose dimensions should not be too great, is suspended as in the former experiments, and the deflecting magnet $M$ is placed so that its centre is at a distance $r$ from that of the suspended magnet, in the same horizontal plane, and due magnetic east.

The axis of $M$ is carefully adjusted so as to be horizontal and in the direction of $r$.

The suspended magnet is observed before $M$ is brought near and also after it is placed in position. If $\theta$ is the observed deflexion, we have, if we use the approximate formula (1),

$$\frac{M}{H} = \frac{r^3}{2}\tan\theta; \tag{5}$$

or, if we use the formula (3),

$$\frac{1}{2}\frac{H}{M}r^3\tan\theta = 1 + A_1\frac{1}{r} + A_2\frac{1}{r^2} + \&c. \tag{6}$$

Here we must bear in mind that though the deflexion $\theta$ can be observed with great accuracy, the distance $r$ between the centres of the magnets is a quantity which cannot be precisely determined, unless both magnets are fixed and their centres defined by marks.

This difficulty is overcome thus:

The magnet $M$ is placed on a divided scale which extends east and west on both sides of the suspended magnet. The middle point between the ends of $M$ is reckoned the centre of the magnet. This point may be marked on the magnet and its position observed on the scale, or the positions of the ends may be observed and the arithmetical mean taken. Call this $s_1$, and let the line of the suspension-fibre of the suspended magnet when produced cut the scale at $s_0$, then $r_1 = s_1 - s_0$, where $s_1$ is known accurately and $s_0$ approximately. Let $\theta_1$ be the deflexion observed in this position of $M$.

Now reverse $M$, that is, place it on the scale with its ends reversed, then $r_1$ will be the same, but $M$ and $A_1$, $A_3$, &c. will have their signs changed, so that if $\theta_2$ is the deflexion to the west,

$$-\frac{1}{2}\frac{H}{M}r_1^3 \tan\theta_2 = 1 - A_1\frac{1}{r_1} + A_2\frac{1}{r_1^2} - \text{\&c.} \tag{7}$$

Taking the arithmetical mean of (6) and (7),

$$\frac{1}{4}\frac{H}{M}r_1^3(\tan\theta_1 - \tan\theta_2) = 1 + A_2\frac{1}{r_1^2} + A_4\frac{1}{r_1^4} + \text{\&c.} \tag{8}$$

Now remove $M$ to the west side of the suspended magnet, and place it with its centre at the point marked $2s_0 - s_1$ on the scale. Let the deflexion when the axis is in the first position be $\theta_3$, and when it is in the second $\theta_4$, then, as before,

$$\frac{1}{4}\frac{H}{M}r_2^3(\tan\theta_3 - \tan\theta_4) = 1 + A_2\frac{1}{r_2^2} + A_4\frac{1}{r_2^4} + \text{\&c.} \tag{9}$$

Let us suppose that the true position of the centre of the suspended magnet is not $s_0$ but $s_0 + \sigma$, then

$$r_1 = r - \sigma, \qquad r_2 = r + \sigma, \tag{10}$$

and
$$\frac{1}{2}(r_1^n + r_2^n) = r^n\left\{1 + \frac{n(n-1)}{2}\frac{\sigma^2}{r^2} + \text{\&c.}\right\}; \tag{11}$$

and since $\frac{\sigma^2}{r^2}$ may be neglected if the measurements are carefully made, we are sure that we may take the arithmetical mean of $r_1^n$ and $r_2^n$ for $r^n$.

Hence, taking the arithmetical mean of (8) and (9),

$$\frac{1}{8}\frac{H}{M}r^3(\tan\theta_1-\tan\theta_2+\tan\theta_3-\tan\theta_4) = 1+A_2\frac{1}{r^2}+\&c., \quad (12)$$

or, making $$\frac{1}{4}(\tan\theta_1-\tan\theta_2+\tan\theta_3-\tan\theta_4) = D, \quad (13)$$

$$\frac{1}{2}\frac{H}{M}Dr^3 = 1+A_2\frac{1}{r^2}+\&c.$$

454.] We may now regard $D$ and $r$ as capable of exact determination.

The quantity $A_2$ can in no case exceed $2L^2$, where $L$ is half the length of the magnet, so that when $r$ is considerable compared with $L$ we may neglect the term in $A_2$ and determine the ratio of $H$ to $M$ at once. We cannot, however, assume that $A_2$ is equal to $2L^2$, for it may be less, and may even be negative for a magnet whose largest dimensions are transverse to the axis. The term in $A_4$ and all higher terms may safely be neglected.

To eliminate $A_2$, repeat the experiment, using distances $r_1$, $r_2$, $r_3$, &c., and let the values of $D$ be $D_1$, $D_2$, $D_3$, &c., then

$$D_1 = \frac{2M}{H}\left(\frac{1}{r_1^{\,3}}+\frac{A_2}{r_1^{\,5}}\right), \quad D_2 = \frac{2M}{H}\left(\frac{1}{r_2^{\,3}}+\frac{A_2}{r_2^{\,5}}\right), \quad \&c.\ \&c.$$

If we suppose that the probable errors of these equations are equal, as they will be if they depend on the determination of $D$ only, and if there is no uncertainty about $r$, then, by multiplying each equation by $r^{-3}$ and adding the results, we obtain one equation, and by multiplying each equation by $r^{-5}$ and adding we obtain another, according to the general rule in the theory of the combination of fallible measurements when the probable error of each equation is supposed the same.

Let us write

$$\Sigma(Dr^{-3}) \text{ for } D_1 r_1^{\,-3}+D_2 r_2^{\,-3}+D_3 r_3^{\,-3}+\&c.,$$

and use similar expressions for the sums of other groups of symbols, then the two resultant equations may be written

$$\Sigma(Dr^{-3}) = \frac{2M}{H}\{\Sigma(r^{-6})+A_2\Sigma(r^{-8})\},$$

$$\Sigma(Dr^{-5}) = \frac{2M}{H}\{\Sigma(r^{-8})+A_2\Sigma(r^{-10})\},$$

whence

$$\frac{2M}{H}\{\Sigma(r^{-6})\Sigma(r^{-10})-[\Sigma(r^{-8})]^2\} = \Sigma(Dr^{-3})\Sigma(r^{-10})-\Sigma(Dr^{-5})\Sigma(r^{-8}),$$

and $$A_2\{\Sigma(Dr^{-3})\Sigma(r^{-10})-\Sigma(Dr^{-5})\Sigma(r^{-8})\}$$
$$= \Sigma(Dr^{-5})\Sigma(r^{-6})-\Sigma(Dr^{-3})\Sigma(r^{-8}).$$

The value of $A_2$ derived from these equations ought to be less than half the square of the length of the magnet $M$. If it is not we may suspect some error in the observations. This method of observation and reduction was given by Gauss in the 'First Report of the Magnetic Association.'

When the observer can make only two series of experiments at distances $r_1$ and $r_2$, the values of $\frac{2M}{H}$ and $A_2$ derived from these experiments are

$$Q = \frac{2M}{H} = \frac{D_1 r_1^5 - D_2 r_2^5}{r_1^2 - r_2^2}, \qquad A_2 = \frac{D_2 r_2^3 - D_1 r_1^3}{D_1 r_1^5 - D^2 r_2^5} r_1^2 r_2^2.$$

If $\delta D_1$ and $\delta D_2$ are the actual errors of the observed deflexions $D_1$ and $D_2$, the actual error of the calculated result $Q$ will be

$$\delta D = \frac{r_1^5 \delta D_1 - r_2^5 \delta D_2}{r_1^2 - r_2^2}.$$

If we suppose the errors $\delta D_1$ and $\delta D_2$ to be independent, and that the probable value of either is $\delta D$, then the probable value of the error in the calculated value of $Q$ will be $\delta Q$, where

$$(\delta Q)^2 = \frac{r_1^{10} + r_2^{10}}{(r_1^2 - r_2^2)^2} (\delta D)^2.$$

If we suppose that one of these distances, say the smaller, is given, the value of the greater distance may be determined so as to make $\delta Q$ a minimum. This condition leads to an equation of the fifth degree in $r_1^2$, which has only one real root greater than $r_2^2$. From this the best value of $r_1$ is found to be

$$r_1 = 1{\cdot}3189 r_2.*$$

If one observation only is taken the best distance is when

$$\frac{\delta D}{D} = \sqrt{3} \frac{\delta r}{r}, \dagger$$

where $\delta D$ is the probable error of a measurement of deflexion, and $\delta r$ is the probable error of a measurement of distance.

* See Airy's *Magnetism*.

† {In this case neglecting the term in $A_2$ we have

$$(\delta Q)^2 = (\delta D)^2 r^6 + 9 \frac{Q^2}{r^2} (\delta r)^2,$$

and this is a minimum when

$$\frac{\delta D}{D} = \sqrt{3} \frac{\delta r}{r} \Big\}.$$

*Method of Sines.*

455.] The method which we have just described may be called the Method of Tangents, because the tangent of the deflexion is a measure of the magnetic force.

If the line $r_1$, instead of being measured east or west, is adjusted till it is at right angles with the axis of the deflected magnet, then $R$ is the same as before, but in order that the suspended magnet may remain perpendicular to $r$, the resolved part of the force $H$ in the direction of $r$ must be equal and opposite to $R$. Hence, if $\theta$ is the deflexion, $R = H \sin \theta$.

This method is called the Method of Sines. It can be applied only when $R$ is less than $H$.

In the Kew portable apparatus this method is employed. The suspended magnet hangs from a part of the apparatus which revolves along with the telescope and the arm for the deflecting magnet, and the rotation of the whole is measured on the azimuth circle.

The apparatus is first adjusted so that the axis of the telescope coincides with the mean position of the line of collimation of the magnet in its undisturbed state. If the magnet is vibrating, the true azimuth of magnetic north is found by observing the extremities of the oscillation of the transparent scale and making the proper correction of the reading of the azimuth circle.

The deflecting magnet is then placed upon a straight rod which passes through the axis of the revolving apparatus at right angles to the axis of the telescope, and is adjusted so that the axis of the deflecting magnet is in a line passing through the centre of the suspended magnet.

The whole of the revolving apparatus is then moved till the line of collimation of the suspended magnet again coincides with the axis of the telescope, and the new azimuth reading is corrected, if necessary, by the mean of the scale readings at the extremities of an oscillation.

The difference of the corrected azimuths gives the deflexion, after which we proceed as in the method of tangents, except that in the expression for $D$ we put $\sin \theta$ instead of $\tan \theta$.

In this method there is no correction for the torsion of the suspending fibre, since the relative position of the fibre, telescope, and magnet is the same at every observation.

The axes of the two magnets remain always at right angles in this method, so that the correction for length can be more accurately made.

456.] Having thus measured the ratio of the moment of the deflecting magnet to the horizontal component of terrestrial magnetism, we have next to find the product of these quantities, by determining the moment of the couple with which terrestrial magnetism tends to turn the same magnet when its axis is deflected from the magnetic meridian.

There are two methods of making this measurement, the dynamical, in which the time of vibration of the magnet under the action of terrestrial magnetism is observed, and the statical, in which the magnet is kept in equilibrium between a measureable statical couple and the magnetic force.

The dynamical method requires simpler apparatus and is more accurate for absolute measurements, but takes up a considerable time; the statical method admits of almost instantaneous measurement, and is therefore useful in tracing the changes of the intensity of the magnetic force, but requires more delicate apparatus, and is not so accurate for absolute measurement.

*Method of Vibrations.*

The magnet is suspended with its magnetic axis horizontal, and is set in vibration in small arcs. The vibrations are observed by means of any of the methods already described.

A point on the scale is chosen corresponding to the middle of the arc of vibration. The instant of passage through this point of the scale in the positive direction is observed. If there is sufficient time before the return of the magnet to the same point, the instant of passage through the point in the negative direction is also observed, and the process is continued till $n+1$ positive and $n$ negative passages have been observed. If the vibrations are too rapid to allow of every consecutive passage being observed, every third or every fifth passage is observed, care being taken that the observed passages are alternately positive and negative.

Let the observed times of passage be $T_1$, $T_2$, $T_{2n+1}$, then if we put

$$\frac{1}{n}\left(\tfrac{1}{2}T_1 + T_3 + T_5 + \&\text{c.} + T_{2n-1} + \tfrac{1}{2}T_{2n+1}\right) = T_{n+1},$$

$$\frac{1}{n}\left(T_2 + T_4 + \&\text{c.} \qquad + T_{2n-2} \qquad + T_{2n}\right) = T'_{n+1};$$

then $T_{n+1}$ is the mean time of the positive passages, and ought to agree with $T'_{n+1}$, the mean time of the negative passages, if the point has been properly chosen. The mean of these results is to be taken as the mean time of the middle passage.

After a large number of vibrations have taken place, but before the vibrations have ceased to be distinct and regular, the observer makes another series of observations, from which he deduces the mean time of the middle passage of the second series.

By calculating the period of vibration either from the first series of observations or from the second, he ought to be able to be certain of the number of whole vibrations which have taken place in the interval between the time of middle passage in the two series. Dividing the interval between the mean times of middle passage in the two series by this number of vibrations, the mean time of vibration is obtained.

The observed time of vibration is then to be reduced to the time of vibration in infinitely small arcs by a formula of the same kind as that used in pendulum observations, and if the vibrations are found to diminish rapidly in amplitude, there is another correction for resistance, see Art. 740. These corrections, however, are very small when the magnet hangs by a fibre, and when the arc of vibration is only a few degrees.

The equation of motion of the magnet is

$$A\frac{d^2\theta}{dt^2} + MH\sin\theta + HM\tau'(\theta-\gamma) = 0,$$

where $\theta$ is the angle between the magnetic axis and the direction of the force $H$, $A$ is the moment of inertia of the magnet and suspended apparatus, $M$ is the magnetic moment of the magnet, $H$ the intensity of the horizontal magnetic force, and $MH\tau'$ the coefficient of torsion : $\tau'$ is determined as in Art. 452, and is a very small quantity. The value of $\theta$ for equilibrium is

$$\theta_0 = \frac{\tau'\gamma}{1+\tau'}, \text{ a very small angle,}$$

and the solution of the equation for small values of the amplitude is

$$\theta = C\cos\left(2\pi\frac{t}{T} + a\right) + \theta_0,$$

where $T$ is the periodic time, $a$ a constant, $C$ the amplitude, and

$$T^2 = \frac{4\pi^2 A}{MH(1+\tau')};$$

whence we find the value of $MH$,

$$MH = \frac{4\pi^2 A}{T^2(1+\tau')}.$$

Here $T$ is the time of a complete vibration determined from observation. $A$, the moment of inertia, is found once for all for the magnet, either by weighing and measuring it if it is of a regular figure, or by a dynamical process of comparison with a body whose moment of inertia is known.

Combining this value of $MH$ with that of $\frac{M}{H}$ formerly obtained,

we get $$M^2 = (MH)\left(\frac{M}{H}\right) = \frac{2\pi^2 A}{T^2(1+\tau')} Dr^3,$$

and $$H^2 = (MH)\left(\frac{H}{M}\right) = \frac{8\pi^2 A}{T^2(1+\tau')Dr^3}.$$

457.] We have supposed that $H$ and $M$ continue constant during the two series of experiments. The fluctuations of $H$ may be ascertained by simultaneous observations of the bifilar magnetometer to be presently described, and if the magnet has been in use for some time, and is not exposed during the experiments to changes of temperature or to concussion, the part of $M$ which depends on permanent magnetism may be assumed to be constant. All steel magnets, however, are capable of induced magnetism depending on the action of external magnetic force.

Now the magnet when employed in the deflexion experiments is placed with its axis east and west, so that the action of terrestrial magnetism is transverse to the magnet, and does not tend to increase or diminish $M$. When the magnet is made to vibrate, its axis is north and south, so that the action of terrestrial magnetism tends to magnetize it in the direction of the axis, and therefore to increase its magnetic moment by a quantity $kH$, where $k$ is a coefficient to be found by experiments on the magnet.

There are two ways in which this source of error may be avoided without calculating $k$, the experiments being arranged so that the magnet shall be in the same condition when employed in deflecting another magnet and when itself swinging.

We may place the deflecting magnet with its axis pointing north, at a distance $r$ from the centre of the suspended magnet, the line $r$ making an angle whose cosine is $\sqrt{\frac{1}{3}}$ with the magnetic meridian. The action of the deflecting magnet on the suspended one is then at right angles to its own direction, and is equal to

$$R = \sqrt{2}\frac{M}{r^3}.$$

Here $M$ is the magnetic moment when the axis points north, as in the experiment of vibration, so that no correction has to be made for induction.

This method, however, is extremely difficult, owing to the large errors which would be introduced by a slight displacement of the deflecting magnet, and as the correction by reversing the deflecting magnet is not applicable here, this method is not to be followed except when the object is to determine the coefficient of induction.

The following method, in which the magnet while vibrating is freed from the inductive action of terrestrial magnetism, is due to Dr. J. P. Joule*.

Two magnets are prepared whose magnetic moments are as nearly equal as possible. In the deflexion experiments these magnets are used separately, or they may be placed simultaneously on opposite sides of the suspended magnet to produce a greater deflexion. In these experiments the inductive force of terrestrial magnetism is transverse to the axis.

Let one of these magnets be suspended, and let the other be placed parallel to it with its centre exactly below that of the suspended magnet, and with its axis in the same direction. The force which the fixed magnet exerts on the suspended one is in the opposite direction from that of terrestrial magnetism. If the fixed magnet be gradually brought nearer to the suspended one the time of vibration will increase, till at a certain point the equilibrium will cease to be stable, and beyond this point the suspended magnet will make oscillations in the reverse position. By experimenting in this way a position of the fixed magnet is found at which it exactly neutralizes the effect of terrestrial magnetism on the suspended one. The two magnets are fastened together so as to be parallel, with their axes turned the same way, and at the distance just found by

* *Proc. Phil. S., Manchester*, March 19, 1867.

experiment. They are then suspended in the usual way and made to vibrate together through small arcs.

The lower magnet exactly neutralizes the effect of terrestrial magnetism on the upper one, and since the magnets are of equal moment, the upper one neutralizes the inductive action of the earth on the lower one.

The value of $M$ is therefore the same in the experiment of vibration as in the experiment of deflexion, and no correction for induction is required.

458.] The most accurate method of ascertaining the intensity of the horizontal magnetic force is that which we have just described. The whole series of experiments, however, cannot be performed with sufficient accuracy in much less than an hour, so that any changes in the intensity which take place in periods of a few minutes would escape observation. Hence a different method is required for observing the intensity of the magnetic force at any instant.

The statical method consists in deflecting the magnet by means of a statical couple acting in a horizontal plane. If $L$ be the moment of this couple, $M$ the magnetic moment of the magnet, $H$ the horizontal component of terrestrial magnetism, and $\theta$ the deflexion,

$$MH \sin\theta = L.$$

Hence, if $L$ is known in terms of $\theta$, $MH$ can be found.

The couple $L$ may be generated in two ways, by the torsional elasticity of a wire, as in the ordinary torsion balance, or by the weight of the suspended apparatus, as in the bifilar suspension.

In the torsion balance the magnet is fastened to the end of a vertical wire, the upper end of which can be turned round, and its rotation measured by means of a torsion circle.

We have then

$$L = \tau(a - a_0 - \theta) = MH \sin\theta.$$

Here $a_0$ is the value of the reading of the torsion circle when the axis of the magnet coincides with the magnetic meridian, and $a$ is the actual reading. If the torsion circle is turned so as to bring the magnet nearly perpendicular to the magnetic meridian, so that

$$\theta = \frac{\pi}{2} - \theta', \quad \text{then} \quad \tau\left(a - a_0 - \frac{\pi}{2} + \theta'\right) = MH\left(1 - \tfrac{1}{2}\theta'^2\right),$$

$$\text{or} \quad MH = \tau\left(1 + \tfrac{1}{2}\theta'^2\right)\left(a - a_0 - \frac{\pi}{2} + \theta'\right).$$

By observing $\theta'$, the deflexion of the magnet when in equilibrium, we can calculate $MH$ provided we know $\tau$.

If we only wish to know the relative value of $H$ at different times it is not necessary to know either $M$ or $\tau$.

We may easily determine $\tau$ in absolute measure by suspending a non-magnetic body from the same wire and observing its time of oscillation, then if $A$ is the moment of inertia of this body, and $T$ the time of a complete vibration,

$$\tau = \frac{4\pi^2 A}{T^2}.$$

The chief objection to the use of the torsion balance is that the zero-reading $a_0$ is liable to change. Under the constant twisting force, arising from the tendency of the magnet to turn to the north, the wire gradually acquires a permanent twist, so that it becomes necessary to determine the zero-reading of the torsion circle afresh at short intervals of time.

### *Bifilar Suspension.*

459.] The method of suspending the magnet by two wires or fibres was introduced by Gauss and Weber. As the bifilar suspension is used in many electrical instruments, we shall investigate it more in detail. The general appearance of the suspension is shewn in Fig. 16, and Fig. 17 represents the projection of the wires on a horizontal plane.

$AB$ and $A'B'$ are the projections of the two wires.

$AA'$ and $BB'$ are the lines joining the upper and the lower ends of the wires.

$a$ and $b$ are the lengths of the lines $AA'$ and $BB'$.

$\alpha$ and $\beta$ their azimuths.

$W$ and $W'$ the vertical components of the tensions of the wires.

$Q$ and $Q'$ their horizontal components.

$h$ the vertical distance between $AA'$ and $BB'$.

The forces which act on the magnet are—its weight, the couple arising from terrestrial magnetism, the torsion (if any) of the wires and their tensions. Of these the effects of magnetism and of torsion are of the nature of couples. Hence the resultant of the tensions must consist of a vertical force, equal to the weight of the magnet, together with a couple. The resultant of the vertical components of the tensions is therefore along the line whose projection is $O$, the intersection of $AA'$ and $BB'$, and either of these lines is divided in $O$ in the ratio of $W'$ to $W$.

The horizontal components of the tensions form a couple, and are therefore equal in magnitude and parallel in direction. Calling either of them $Q$, the moment of the couple which they form is

$$L = Q\,.\,PP', \tag{1}$$

where $PP'$ is the distance between the parallel lines $AB$ and $A'B'$.

To find the value of $L$ we have the equations of moments

$$Qh = W.AB = W'.\,A'B', \tag{2}$$

and the geometrical equation

$$(AB + A'B')\,PP' = ab \sin(a - \beta), \tag{3}$$

whence we obtain,

$$L = Q.PP' = \frac{ab}{h}\frac{WW'}{W+W'}\sin(a-\beta). \tag{4}$$

If $m$ is the mass of the suspended apparatus, and $g$ the intensity of gravity,

$$W + W' = mg. \tag{5}$$

If we also write

$$W - W' = nmg, \tag{6}$$

we find

$$L = \frac{1}{4}(1 - n^2)\,mg\frac{ab}{h}\sin(a-\beta). \tag{7}$$

The value of $L$ is therefore a maximum with respect to $n$ when $n$ is zero, that is, when the weight of the suspended mass is equally borne by the two wires.

We may adjust the tensions of the wires to equality by observing the time of vibration, and making it a minimum, or we may obtain a self-acting adjustment by attaching the ends of the wires, as in Fig. 16, to a pulley, which turns on its axis till the tensions are equal.

The distance between the upper ends of the suspension wires is regulated by means of two other pulleys. The distance between the lower ends of the wires is also capable of adjustment.

By this adjustment of the tension, the couple arising from the tensions of the wires becomes

$$L = \frac{1}{4}\frac{ab}{h}mg\sin(a-\beta).$$

The moment of the couple arising from the torsion of the wires is of the form

$$\tau(\gamma - \beta),$$

where $\tau$ is the sum of the coefficients of torsion of the wires.

The wires ought to be without torsion when $a = \beta$, we may then make $\gamma = a$.

The moment of the couple arising from the horizontal magnetic force is of the form

$$MH \sin(\delta - \theta),$$

where $\delta$ is the magnetic declination, and $\theta$ is the azimuth of the

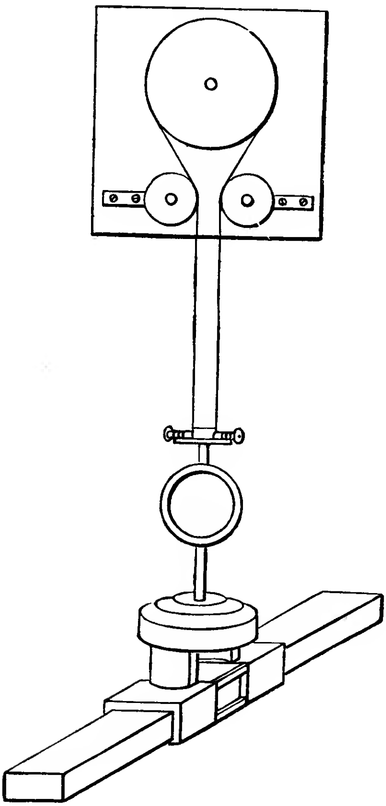

Fig. 16.

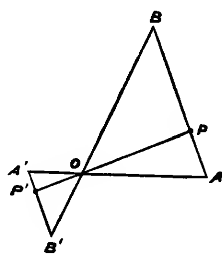


Fig. 17.

axis of the magnet. We shall avoid the introduction of unnecessary symbols without sacrificing generality if we assume that the axis of the magnet is parallel to $BB'$, or that $\beta = \theta$.

The equation of motion then becomes

$$A \frac{d^2\theta}{dt^2} = MH \sin(\delta - \theta) + \frac{1}{4} \frac{ab}{h} mg \sin(a - \theta) + \tau(a - \theta). \qquad (8)$$

There are three principal positions of this apparatus.

(1) When $a$ is nearly equal to $\delta$. If $T_1$ is the time of a complete oscillation in this position, then

$$\frac{4\pi^2 A}{T_1^2} = \frac{1}{4}\frac{ab}{h} mg + \tau + MH. \tag{9}$$

(2) When $a$ is nearly equal to $\delta + \pi$. If $T_2$ is the time of a complete oscillation in this position, the north end of the magnet being now turned towards the south,

$$\frac{4\pi^2 A}{T_2^2} = \frac{1}{4}\frac{ab}{h} mg + \tau - MH. \tag{10}$$

The quantity on the right-hand of this equation may be made as small as we please by diminishing $a$ or $b$, but it must not be made negative, or the equilibrium of the magnet will become unstable. The magnet in this position forms an instrument by which small variations in the *direction* of the magnetic force may be rendered sensible.

For when $\theta - \delta$ is nearly equal to $\pi$, $\sin(\delta - \theta)$ is nearly equal to $\theta - \delta - \pi$, and we find

$$\theta = a - \frac{MH}{\frac{1}{4}\frac{ab}{h} mg + \tau - MH}(\delta + \pi - a). \tag{11}$$

By diminishing the denominator of the fraction in the last term we may make the variation of $\theta$ very large compared with that of $\delta$. We should notice that the coefficient of $\delta$ in this expression is negative, so that when the direction of the magnetic force turns in one direction the magnet turns in the opposite direction.

(3) In the third position the upper part of the suspension-apparatus is turned round till the axis of the magnet is nearly perpendicular to the magnetic meridian.

If we make

$$\theta - \delta = \frac{\pi}{2} + \theta', \quad \text{and} \quad a - \theta = \beta - \theta', \tag{12}$$

the equation of motion may be written

$$A\frac{d^2\theta'}{dt^2} = -MH\cos\theta' + \frac{1}{4}\frac{ab}{h} mg \sin(\beta - \theta') + \tau(\beta - \theta'). \tag{13}$$

If there is equilibrium when $H = H_0$ and $\theta' = 0$,

$$-MH_0 + \frac{1}{4}\frac{ab}{h} mg \sin\beta + \beta\tau = 0, \tag{14}$$

and if $H$ is the value of the horizontal force corresponding to a small angle $\theta'$,

$$H = H_0\left(1 - \frac{\frac{1}{4}\frac{ab}{h}mg\cos\beta + \tau}{\frac{1}{4}\frac{ab}{h}mg\sin\beta + \tau\beta}\theta'\right). \tag{15}$$

In order that the magnet may be in stable equilibrium it is necessary that the numerator of the fraction in the second member should be positive, but the more nearly it approaches zero, the more sensitive will be the instrument in indicating changes in the value of the intensity of the horizontal component of terrestrial magnetism.

The statical method of estimating the intensity of the force depends upon the action of an instrument which of itself assumes different positions of equilibrium for different values of the force. Hence, by means of a mirror attached to the magnet and throwing a spot of light upon a photographic surface moved by clock-work, a curve may be traced, from which the intensity of the force at any instant may be determined according to a scale, which we may for the present consider an arbitrary one.

460.] In an observatory, where a continuous system of registration of declination and intensity is kept up either by eye-observation or by the automatic photographic method, the absolute values of the declination and of the intensity, as well as the position and moment of the magnetic axis of a magnet, may be determined to a great degree of accuracy.

For the declinometer gives the declination at every instant affected by a constant error, and the bifilar magnetometer gives the intensity at every instant multiplied by a constant coefficient. In the experiments we substitute for $\delta$, $\delta' + \delta_0$, where $\delta'$ is the reading of the declinometer at the given instant, and $\delta_0$ is the unknown but constant error, so that $\delta' + \delta_0$ is the true declination at that instant.

In like manner for $H$, we substitute $CH'$, where $H'$ is the reading of the magnetometer on its arbitrary scale, and $C$ is an unknown but constant multiplier which converts these readings into absolute measure, so that $CH'$ is the horizontal force at a given instant.

The experiments to determine the absolute values of the quantities must be conducted at a sufficient distance from the declinometer and magnetometer, so that the different magnets may not sensibly disturb each other. The time of every observation must be noted and the corresponding values of $\delta'$ and $H'$ inserted. The equations are then to be treated so as to find $\delta_0$, the constant error of the declinometer, and $C$ the coefficient to be applied to the reading of the magnetometer. When these are found the readings of both instruments may be expressed in absolute measure. The absolute measurements, however, must be frequently repeated in order to take account of changes which may occur in the magnetic axis and magnetic moment of the magnets.

461.] The methods of determining the vertical component of the terrestrial magnetic force have not been brought to the same degree of precision. The vertical force must act on a magnet which turns about a horizontal axis. Now a body which turns about a horizontal axis cannot be made so sensitive to the action of small forces as a body which is suspended by a fibre and turns about a vertical axis. Besides this, the weight of a magnet is so large compared with the magnetic force exerted upon it that a small displacement of the centre of inertia by unequal dilatation, &c. produces a greater effect on the position of the magnet than a considerable change of the magnetic force.

Hence the measurement of the vertical force, or the comparison of the vertical and the horizontal forces, is the least perfect part of the system of magnetic measurements.

The vertical part of the magnetic force is generally deduced from the horizontal force by determining the direction of the total force.

If $i$ be the angle which the total force makes with its horizontal component, $i$ is called the magnetic Dip or Inclination, and if $H$ is the horizontal force already found, then the vertical force is $H \tan i$, and the total force is $H \sec i$.

The magnetic dip is found by means of the Dip Needle.

The theoretical dip-needle is a magnet with an axis which passes through its centre of inertia perpendicular to the magnetic axis of the needle. The ends of its axis are made in the form of cylinders of small radius, the axes of which are coincident with the line passing through the centre of inertia. These cylindrical ends rest on two horizontal planes and are free to roll on them.

When the axis is placed magnetic east and west, the needle is free to rotate in the plane of the magnetic meridian, and if the instrument is in perfect adjustment, the magnetic axis will set itself in the direction of the total magnetic force.

It is, however, practically impossible to adjust a dip-needle so that its weight does not influence its position of equilibrium, because its centre of inertia, even if originally in the line joining the centres of the rolling sections of the cylindrical ends, will cease to be in this line when the needle is imperceptibly bent or unequally expanded. Besides, the determination of the true centre of inertia of a magnet is a very difficult operation, owing to the interference of the magnetic force with that of gravity.

Let us suppose one end of the needle and one end of the pivot to be marked. Let a line, real or imaginary, be drawn on the needle, which we shall call the Line of Collimation. The position of this line is read off on a vertical circle. Let $\theta$ be the angle which this line makes with the radius to zero, which we shall suppose to be horizontal. Let $\lambda$ be the angle which the magnetic axis makes with the line of collimation, so that when the needle is in this position the magnetic axis is inclined $\theta+\lambda$ to the horizontal.

Let $p$ be the perpendicular from the centre of inertia on the plane on which the axis rolls, then $p$ will be a function of $\theta$, whatever be the shape of the rolling surfaces. If both the rolling sections of the ends of the axis are circular we have an equation of the form,

$$p = c - a\sin(\theta+\alpha), \tag{1}$$

where $a$ is the distance of the centre of inertia from the line joining the centres of the rolling sections, and $\alpha$ is the angle which this line makes with the line of collimation.

If $M$ is the magnetic moment, $m$ the mass of the magnet, and $g$ the force of gravity, $I$ the total magnetic force, and $i$ the dip, then, by the conservation of energy, when there is stable equilibrium

$$MI\cos(\theta+\lambda-i) - mgp \tag{2}$$

must be a maximum with respect to $\theta$, or

$$MI\sin(\theta+\lambda-i) = -mg\frac{dp}{d\theta},$$

$$= mga\cos(\theta+\alpha), \tag{3}$$

if the ends of the axis are cylindrical.

Also, if $T$ be the time of vibration about the position of equilibrium,

$$MI + mga\sin(\theta+a) = \frac{4\pi^2 A}{T^2}, \tag{4}$$

where $A$ is the moment of inertia of the needle about its axis of rotation, and $\theta$ is determined by (3).

In determining the dip a reading is taken with the dip-circle in the magnetic meridian and with the graduation towards the west.

Let $\theta_1$ be this reading, then we have

$$MI\sin(\theta_1+\lambda-i) = mga\cos(\theta_1+a). \tag{5}$$

The instrument is now turned about a vertical axis through 180°, so that the graduation is to the east, and if $\theta_2$ is the new reading,

$$MI\sin(\theta_2+\lambda-\pi+i) = mga\cos(\theta_2+a). \tag{6}$$

Taking (6) from (5), and remembering that $\theta_1$ is nearly equal to $i$, and $\theta_2$ nearly equal to $\pi-i$, and that $\lambda$ is a small angle, such that $mga\lambda$ may be neglected in comparison with $MI$,

$$MI(\theta_1-\theta_2+\pi-2i) = 2mga\cos i\cos a. \tag{7}$$

Now take the magnet from its bearings and place it in the deflexion apparatus, Art. 453, so as to indicate its own magnetic moment by the deflexion of a suspended magnet, then

$$M = \tfrac{1}{2}r^3 HD, \tag{8}$$

where $D$ is the tangent of the deflexion.

Next, reverse the magnetism of the needle and determine its new magnetic moment $M'$, by observing a new deflexion the tangent of which is $D'$, then the distance being the same as before,

$$M' = \tfrac{1}{2}r^3 HD', \tag{9}$$

whence

$$MD' = M'D. \tag{10}$$

Then place it on its bearings and take two readings, $\theta_3$ and $\theta_4$, in which $\theta_3$ is nearly $\pi+i$, and $\theta_4$ nearly $-i$,

$$M'I\sin(\theta_3+\lambda'-\pi-i) = mga\cos(\theta_3+a), \tag{11}$$

$$M'I\sin(\theta_4+\lambda'+i) \quad = mga\cos(\theta_4+a), \tag{12}$$

whence, as before,

$$M'I(\theta_3-\theta_4-\pi-2i) = -2mga\cos i\cos a, \tag{13}$$

and on adding (7),

$$MI(\theta_1-\theta_2+\pi-2i)+M'I(\theta_3-\theta_4-\pi-2i) = 0, \tag{14}$$

or

$$D(\theta_1-\theta_2+\pi-2i)+\ D'(\theta_3-\theta_4-\pi-2i) = 0, \tag{15}$$

whence we find the dip

$$i = \frac{D(\theta_1 - \theta_2 + \pi) + D'(\theta_3 - \theta_4 - \pi)}{2D + 2D'}, \qquad (16)$$

where $D$ and $D'$ are the tangents of the deflexions produced by the needle in its first and second magnetizations respectively.

In taking observations with the dip-circle the vertical axis is carefully adjusted so that the plane bearings upon which the axis of the magnet rests are horizontal in every azimuth. The magnet being magnetized so that the end $A$ dips, is placed with its axis on the plane bearings, and observations are taken with the plane of the circle in the magnetic meridian, and with the graduated side of the circle east. Each end of the magnet is observed by means of reading microscopes carried on an arm which moves concentric with the dip-circle. The cross-wires of the microscope are made to coincide with the image of a mark on the magnet, and the position of the arm is then read off on the dip-circle by means of a vernier.

We thus obtain an observation of the end $A$ and another of the end $B$ when the graduations are east. It is necessary to observe both ends in order to eliminate any error arising from the axle of the magnet not being concentric with the dip-circle.

The graduated side is then turned west, and two more observations are made.

The magnet is then turned round so that the ends of the axle are reversed, and four more observations are made looking at the other side of the magnet.

The magnetization of the magnet is then reversed so that the end $B$ dips, the magnetic moment is ascertained, and eight observations are taken in this state, and the sixteen observations combined to determine the true dip.

462.] It is found that in spite of the utmost care the dip, as thus deduced from observations made with one dip-circle, differs perceptibly from that deduced from observations with another dip-circle at the same place. Mr. Broun has pointed out the effect due to ellipticity of the bearings of the axle, and how to correct it by taking observations with the magnet magnetized to different strengths.

The principle of this method may be stated thus. We shall suppose that the error of any one observation is a small quantity not exceeding a degree. We shall also suppose that some unknown but regular force acts upon the magnet, disturbing it from its true position.

If $L$ is the moment of this force, $\theta_0$ the true dip, and $\theta$ the observed dip, then

$$L = MI \sin(\theta - \theta_0), \tag{17}$$

$$= MI(\theta - \theta_0), \tag{18}$$

since $\theta - \theta_0$ is small.

It is evident that the greater $M$ becomes the nearer does the needle approach its proper position. Now let the operation of taking the dip be performed twice, first with the magnetization equal to $M_1$, the greatest that the needle is capable of, and next with the magnetization equal to $M_2$, a much smaller value but sufficient to make the readings distinct and the error still moderate. Let $\theta_1$ and $\theta_2$ be the dips deduced from these two sets of observations, and let $L$ be the mean value of the unknown disturbing force for the eight positions of each determination, which we shall suppose the same for both determinations. Then

$$L = M_1 I(\theta_1 - \theta_0) = M_2 I(\theta_2 - \theta_0). \tag{19}$$

Hence $$\theta_0 = \frac{M_1\theta_1 - M_2\theta_2}{M_1 - M_2}, \qquad L = M_1 M_2 I \frac{\theta_1 - \theta_2}{M_2 - M_1}. \tag{20}$$

If we find that several experiments give nearly equal values for $L$, then we may consider that $\theta_0$ must be very nearly the true value of the dip.

463.] Dr. Joule has recently constructed a new dip-circle, in which the axis of the needle, instead of rolling on horizontal agate planes, is slung on two filaments of silk or spider's thread, the ends of the filaments being attached to the arms of a delicate balance. The axis of the needle thus rolls on two loops of silk fibre, and Dr. Joule finds that its freedom of motion is much greater than when it rolls on agate planes.

In Fig. 18, $NS$ is the needle, $CC'$ is its axis, consisting of a straight cylindrical wire, and $PCQ$, $P'C'Q'$ are the filaments on which the axis rolls. $POQ$ is the balance, consisting of a double bent lever supported by a wire, $O'O'$, stretched horizontally between the prongs of a forked piece, and having a counterpoise $R$ which can be screwed up or down, so that the balance is in neutral equilibrium about $O'O'$.

In order that the needle may be in neutral equilibrium as the needle rolls on the filaments the centre of gravity must neither rise nor fall. Hence the distance $OC$ must remain constant as the needle rolls. This condition will be fulfilled if the arms of the balance $OP$ and $OQ$ are equal, and if the filaments are at right angles to the arms.

Dr. Joule finds that the needle should not be more than five inches long. When it is eight inches long, the bending of the needle tends to diminish the apparent dip by a fraction of a minute. The axis of the needle was originally of steel wire, straightened by being brought to a red heat while stretched by a weight, but Dr. Joule found that with the new suspension it is not necessary to use steel wire, for platinum and even standard gold are hard enough.

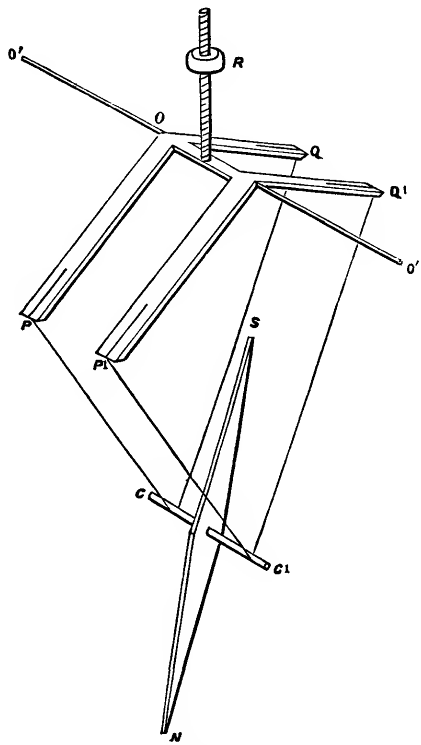

Fig. 18.

The balance is attached to a wire $O'O'$ about a foot long stretched horizontally between the prongs of a fork. This fork is turned round in azimuth by means of a circle at the top of a tripod which supports the whole. Six complete observations of the dip can be obtained in one hour, and the average error of a single observation is a fraction of a minute of arc.

It is proposed that the dip-needle in the Cambridge Physical Laboratory shall be observed by means of a double image instrument, consisting of two totally reflecting prisms placed as in Fig. 19 and mounted on a vertical graduated circle, so that the plane of reflexion may be turned round a horizontal axis nearly coinciding with the prolongation of the axis of the suspended dip-needle. The needle is viewed by means of a telescope placed behind the prisms, and the two ends of the needle are seen together as in Fig. 20. By turning the prisms about the axis of the vertical circle, the images of two lines

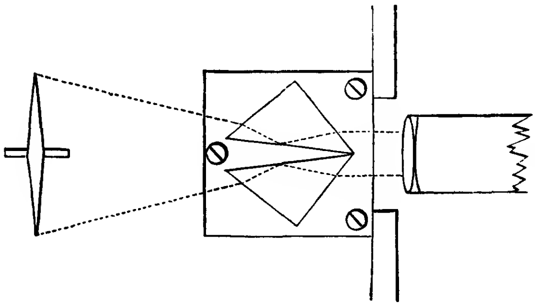

Fig. 19.

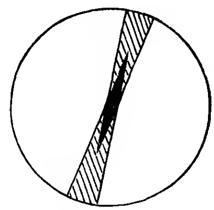

Fig. 20.

drawn on the needle may be made to coincide. The inclination of the needle is thus determined from the reading of the vertical circle.

The total intensity $I$ of the magnetic force in the line of dip may be deduced as follows from the times of vibration $T_1, T_2, T_3, T_4$ in the four positions already described,

$$I = \frac{4\pi^2 A}{2M+2M'}\left\{\frac{1}{T_1^2}+\frac{1}{T_2^2}+\frac{1}{T_3^2}+\frac{1}{T_4^2}\right\}.$$

The values of $M$ and $M'$ must be found by the method of deflexion and vibration formerly described, and $A$ is the moment of inertia of the magnet about its axle.

The observations with a magnet suspended by a fibre are so much more accurate that it is usual to deduce the total force from the horizontal force by means of the equation

$$I = H \sec\theta,$$

where $I$ is the total force, $H$ the horizontal force, and $\theta$ the dip.

464.] The process of determining the dip being a tedious one, is not suitable for determining the continuous variation of the magnetic force. The most convenient instrument for continuous observations is the vertical force magnetometer, which is simply a magnet balanced on knife edges so as to be in stable equilibrium with its magnetic axis nearly horizontal.

If $Z$ is the vertical component of the magnetic force, $M$ the magnetic moment, and $\theta$ the small angle which the magnetic axis makes with the horizon,

$$MZ\cos\theta = mga\cos(\alpha-\theta),$$

where $m$ is the mass of the magnet, $g$ the force of gravity, $a$ the distance of the centre of gravity from the axis of suspension, and $\alpha$ the angle which the plane through the axis and the centre of gravity makes with the magnetic axis.

Hence, for the small variation of vertical force $\delta Z$, there will be since $\theta$ is very small a variation of the angular position of the magnet $\delta\theta$ such that

$$M\delta Z = mga\sin(\alpha-\theta)\,\delta\theta.$$

In practice this instrument is not used to determine the absolute value of the vertical force, but only to register its small variations.

For this purpose it is sufficient to know the absolute value of $Z$ when $\theta = 0$, and the value of $\dfrac{dZ}{d\theta}$.

The value of $Z$, when the horizontal force and the dip are known, is found from the equation $Z = H\tan\theta_0$, where $\theta_0$ is the dip and $H$ the horizontal force.

To find the deflexion due to a given variation of $Z$, take a magnet and place it with its axis east and west, and with its centre at a known distance $r_1$ east or west from the declinometer, as in experiments on deflexion, and let the tangent of deflexion be $D_1$.

Then place it with its axis vertical and with its centre at a distance $r_2$ above or below the centre of the vertical force magnetometer, and let the tangent of the deflexion produced in the magnetometer be $D_2$. Then, if the moment of the deflecting magnet is $M'$,

$$2M = Hr_1^3 D_1 = \frac{dZ}{d\theta} r_2^3 D_2.$$

Hence $$\frac{dZ}{d\theta} = H\frac{r_1^3}{r_2^3}\frac{D_1}{D_2}.$$

The actual value of the vertical force at any instant is

$$Z = Z_0 + \theta\frac{dZ}{d\theta},$$

where $Z_0$ is the value of $Z$ when $\theta = 0$.

For continuous observations of the variations of magnetic force at a fixed observatory the Unifilar Declinometer, the Bifilar Horizontal Force Magnetometer, and the Balance Vertical Force Magnetometer are the most convenient instruments.

At several observatories photographic traces are now produced on prepared paper moved by clock-work, so that a continuous record of the indications of the three instruments at every instant is formed. These traces indicate the variation of the three rectangular components of the force from their standard values. The declinometer gives the force towards mean magnetic west, the bifilar magnetometer gives the variation of the force towards magnetic north, and the balance magnetometer gives the variation of the vertical force. The standard values of these forces, or their values when these instruments indicate their several zeros, are deduced by frequent observations of the absolute declination, horizontal force, and dip.

# CHAPTER VIII.

## ON TERRESTRIAL MAGNETISM.

465.] OUR knowledge of Terrestrial Magnetism is derived from the study of the distribution of magnetic force on the earth's surface at any one time, and of the changes in that distribution at different times.

The magnetic force at any one place and time is known when its three coordinates are known. These coordinates may be given in the form of the declination or azimuth of the force, the dip or inclination to the horizon, and the total intensity.

The most convenient method, however, for investigating the general distribution of magnetic force on the earth's surface is to consider the magnitudes of the three components of the force,

$$\left.\begin{aligned} X &= H\cos\delta, \text{ directed due north,} \\ Y &= H\sin\delta, \text{ directed due west,} \\ Z &= H\tan\theta, \text{ directed vertically downwards,} \end{aligned}\right\} \qquad (1)$$

where $H$ denotes the horizontal force, $\delta$ the declination, and $\theta$ the dip.

If $V$ is the magnetic potential at the earth's surface, and if we consider the earth a sphere of radius $a$, then

$$X = -\frac{1}{a}\frac{dV}{dl}, \qquad Y = -\frac{1}{a\cos l}\frac{dV}{d\lambda}, \qquad Z = \frac{dV}{dr}, \qquad (2)$$

where $l$ is the latitude, $\lambda$ the longitude, and $r$ the distance from the centre of the earth.

A knowledge of $V$ over the surface of the earth may be obtained from the observations of horizontal force alone as follows.

Let $V_0$ be the value of $V$ at the true north pole, then, taking the line-integral along any meridian, we find,

$$V = -a\int_{\frac{\pi}{2}}^{l} X\,dl + V_0, \tag{3}$$

for the value of the potential on that meridian at latitude $l$.

Thus the potential may be found for any point on the earth's surface provided we know the value of $X$, the northerly component at every point, and $V_0$, the value of $V$ at the pole.

Since the forces depend not on the absolute value of $V$ but on its derivatives, it is not necessary to fix any particular value for $V_0$.

The value of $V$ at any point may be ascertained if we know the value of $X$ along any given meridian, and also that of $Y$ over the whole surface.

Let
$$V_2 = -a\int_{\frac{\pi}{2}}^{l} X\,dl + V_0, \tag{4}$$

where the integration is performed along the given meridian from the pole to the parallel $l$, then

$$V = V_l - a\int_{\lambda_0}^{\lambda} Y\cos l\,d\lambda, \tag{5}$$

where the integration is performed along the parallel $l$ from the given meridian $\lambda_0$ to the required point.

These methods imply that a complete magnetic survey of the earth's surface has been made, so that the values of $X$ or of $Y$ or of both are known for every point of the earth's surface at a given epoch. What we actually know are the magnetic components at a certain number of stations. In the civilized parts of the earth these stations are comparatively numerous; in other places there are large tracts of the earth's surface about which we have no data.

## *Magnetic Survey.**

466.] Let us suppose that in a country of moderate size, whose greatest dimensions are a few hundred miles, observations of the declination and the horizontal force have been taken at a considerable number of stations distributed fairly over the country

Within this district we may suppose the value of $V$ to be represented with sufficient accuracy by the formula

$$V = \text{const.} - a(A_1 l + A_2\lambda + \tfrac{1}{2}B_1 l^2 + B_2 l\lambda + \tfrac{1}{2}B_3\lambda^2 + \&\text{c.}), \tag{6}$$

* {The reader should consult Rücker and Thorpe's paper 'A Magnetic Survey of the British Isles,' *Phil. Trans.*, 1890, A, pp. 53–328.}

whence
$$X = A_1 + B_1 l + B_2 \lambda, \tag{7}$$
$$Y \cos l = A_2 + B_2 l + B_3 \lambda. \tag{8}$$

Let there be $n$ stations whose latitudes are $l_1, l_2, \ldots$ &c. and longtitudes $\lambda_1, \lambda_2$, &c., and let $X$ and $Y$ be found for each station.

Let
$$l_0 = \frac{1}{n} \Sigma (l), \quad \text{and } \lambda_0 = \frac{1}{n} \Sigma (\lambda), \tag{9}$$

$l_0$ and $\lambda_0$ may be called the latitude and longitude of the central station. Let

$$X_0 = \frac{1}{n} \Sigma (X), \quad \text{and} \quad Y_0 \cos l_0 = \frac{1}{n} \Sigma (Y \cos l), \tag{10}$$

then $X_0$ and $Y_0$ are the values of $X$ and $Y$ at the imaginary central station, then

$$X = X_0 + B_1 (l - l_0) + B_2 (\lambda - \lambda_0), \tag{11}$$
$$Y \cos l = Y_0 \cos l_0 + B_2 (l - l_0) + B_3 (\lambda - \lambda_0). \tag{12}$$

We have $n$ equations of the form (11) and $n$ of the form (12). If we denote the probable error in the determination of $X$ by $\xi$, and in that of $Y \cos l$ by $\eta$, then we may calculate $\xi$ and $\eta$ on the supposition that they arise from errors of observation of $H$ and $\delta$.

Let the probable error of $H$ be $h$, and that of $\delta$, $\Delta$, then since

$$dX = \cos \delta \,.\, dH - H \sin \delta \,.\, d\delta,$$
$$\xi^2 = h^2 \cos^2 \delta + \Delta^2 H^2 \sin^2 \delta.$$

Similarly
$$\eta^2 = h^2 \sin^2 \delta + \Delta^2 H^2 \cos^2 \delta.$$

If the variations of $X$ and $Y$ from their values as given by equations of the form (11) and (12) considerably exceed the probable errors of observation, we may conclude that they are due to local attractions, and then we have no reason to give the ratio of $\xi$ to $\eta$ any other value than unity.

According to the method of least squares we multiply the equations of the form (11) by $\eta$, and those of the form (12) by $\xi$ to make their probable error the same. We then multiply each equation by the coefficient of one of the unknown quantities $B_1$, $B_2$, or $B_3$ and add the results, thus obtaining three equations from which to find $B_1, B_2, B_3$, viz.

$$P_1 = B_1 b_1 + B_2 b_2,$$
$$\eta^2 P_2 + \xi^2 Q_1 = B_1 \eta^2 b_2 + B_2 (\xi^2 b_1 + \eta^2 b_3) + B_3 \xi^2 b_2,$$
$$Q_2 = \qquad B_2 b_2 \qquad + B_3 b_3;$$

in which we write for conciseness,

$$b_1 = \Sigma(l^2) - nl_0^2, \quad b_2 = \Sigma(l\lambda) - nl_0\lambda_0, \quad b_3 = \Sigma(\lambda^2) - n\lambda_0^2,$$
$$P_1 = \Sigma(lX) - nl_0 X_0, \qquad Q_1 = \Sigma(lY\cos l) - nl_0 Y_0 \cos l_0,$$
$$P_2 = \Sigma(\lambda X) - n\lambda_0 X_0, \qquad Q_2 = \Sigma(\lambda Y\cos l) - n\lambda_0 Y_0 \cos l_0.$$

By calculating $B_1$, $B_2$, and $B_3$, and substituting in equations (11) and (12), we can obtain the values of $X$ and $Y$ at any point within the limits of the survey free from the local disturbances which are found to exist where the rock near the station is magnetic, as most igneous rocks are.

Surveys of this kind can be made only in countries where magnetic instruments can be carried about and set up in a great many stations. For other parts of the world we must be content to find the distribution of the magnetic elements by interpolation between their values at a few stations at great distances from each other.

467.] Let us now suppose that by processes of this kind, or by the equivalent graphical process of constructing charts of the lines of equal values of the magnetic elements, the values of $X$ and $Y$, and thence of the potential $V$, are known over the whole surface of the globe. The next step is to expand $V$ in the form of a series of spherical surface harmonics.

If the earth were magnetized uniformly and in the same direction throughout its interior, $V$ would be a harmonic of the first degree, the magnetic meridians would be great circles passing through two magnetic poles diametrically opposite, the magnetic equator would be a great circle, the horizontal force would be equal at all points of the magnetic equator, and if $H_0$ is this constant value, the value at any other point would be $H = H_0 \cos l'$, where $l'$ is the magnetic latitude. The vertical force at any point would be $Z = 2H_0 \sin l'$, and if $\theta$ is the dip, $\tan\theta$ would be $= 2\tan l'$.

In the case of the earth, the magnetic equator is defined to be the line of no dip. It is not a great circle of the sphere.

The magnetic poles are defined to be the points where there is no horizontal force, or where the dip is 90°. There are two such points, one in the northern and one in the southern regions, but they are not diametrically opposite. and the line joining them is not parallel to the magnetic axis of the earth.

468.] The magnetic poles are the points where the value of $V$

on the surface of the earth is a maximum or minimum, or is stationary.

At any point where the potential is a minimum the north end of the dip-needle points vertically downwards, and if a compass-needle be placed anywhere near such a point, the north end will point towards that point.

At points where the potential is a maximum the south end of the dip-needle points downwards, and in the neighbourhood the south end of the compass-needle points towards the point.

If there are $p$ minima of $V$ on the earth's surface there must be $p-1$ other points, where the north end of the dip-needle points downwards, but where the compass-needle, when carried in a circle round the point, instead of revolving so that its north end points constantly to the centre, revolves in the opposite direction, so as to turn sometimes its north end and sometimes its south end towards the point.

If we call the points where the potential is a minimum true north poles, then these other points may be called false north poles, because the compass-needle is not true to them. If there are $p$ true north poles, there must be $p-1$ false north poles, and in like manner, if there are $q$ true south poles, there must be $q-1$ false south poles. The number of poles of the same name must be odd, so that the opinion at one time prevalent, that there are two north poles and two south poles, is erroneous. According to Gauss there is in fact only one true north pole and one true south pole on the earth's surface, and therefore there are no false poles. The line joining these poles is not a diameter of the earth, and it is not parallel to the earth's magnetic axis.

469.] Most of the early investigators into the nature of the earth's magnetism endeavoured to express it as the result of the action of one or more bar magnets, the positions of the poles of which were to be determined. Gauss was the first to express the distribution of the earth's magnetism in a perfectly general way by expanding its potential in a series of solid harmonics, the coefficients of which he determined for the first four degrees. These coefficients are 24 in number, 3 for the first degree, 5 for the second, 7 for the third, and 9 for the fourth. All these terms are found necessary in order to give a tolerably accurate representation of the actual state of the earth's magnetism.

*To find what Part of the Observed Magnetic Force is due to External and what to Internal Causes.*

470.] Let us now suppose that we have obtained an expansion of the magnetic potential of the earth in spherical harmonics, consistent with the actual direction and magnitude of the horizontal force at every point on the earth's surface, then Gauss has shewn how to determine, from the observed vertical force, whether the magnetic forces are due to causes, such as magnetization or electric currents, within the earth's surface, or whether any part is directly due to causes exterior to the earth's surface.

Let $V$ be the actual potential expanded in a double series of spherical harmonics.

$$V = A_1 \frac{r}{a} + \&c. + A_i \left(\frac{r}{a}\right)^i + \dots\dots\dots$$
$$+ B_1 \left(\frac{r}{a}\right)^{-2} + \&c. + B_i \left(\frac{r}{a}\right)^{-(i+1)} + \dots\dots\dots$$

The first series represents the part of the potential due to causes exterior to the earth, and the second series represents the part due to causes within the earth.

The observations of horizontal force give us the sum of these series when $r = a$, the radius of the earth. The term of the order $i$ is
$$V_i = A_i + B_i.$$

The observations of vertical force give us
$$Z = \frac{dV}{dr},$$
and the term of the order $i$ in $aZ$ is
$$aZ_i = iA_i - (i+1)B_i.$$

Hence the part due to external causes is
$$A_i = \frac{(i+1)\,V_i + aZ_i}{2i+1},$$
and the part due to causes within the earth is
$$B_i = \frac{iV_i - aZ_i}{2i+1}.$$

The expansion of $V$ has hitherto been calculated only for the mean value of $V$ at or near certain epochs. No appreciable part of this mean value appears to be due to causes external to the earth.

471.] We do not yet know enough of the form of the expansion of the solar and lunar parts of the variations of $V$ to determine *by this method* whether any part of these variations arises from magnetic force acting from without. It is certain, however, as the calculations of MM. Stoney and Chambers have shewn, that the principal part of these variations cannot arise from any direct magnetic action of the sun or moon, supposing these bodies to be magnetic *.

472.] The principal changes in the magnetic force to which attention has been directed are as follows.

### I. *The more Regular Variations.*

(1) The Solar variations, depending on the hour of the day and the time of the year.

(2) The Lunar variations, depending on the moon's hour angle and on her other elements of position.

(3) These variations do not repeat themselves in different years, but seem to be subject to a variation of longer period of about eleven years.

(4) Besides this, there is a secular alteration in the state of the earth's magnetism, which has been going on ever since magnetic observations have been made, and is producing changes of the magnetic elements of far greater magnitude than any of the variations of small period.

### II. *The Disturbances.*

473.] Besides the more regular changes, the magnetic elements are subject to sudden disturbances of greater or less amount. It is found that these disturbances are more powerful and frequent at one time than at another, and that at times of great disturbance the laws of the regular variations are masked, though

* Professor Hornstein of Prague has discovered a periodic change in the magnetic elements, the period of which is 26·33 days, almost exactly equal to that of the synodic revolution of the sun, as deduced from the observation of sun-spots near his equator. This method of discovering the time of rotation of the unseen solid body of the sun by its effects on the magnetic needle is the first instalment of the repayment by Magnetism of its debt to Astronomy. *Anzeiger der k. Akad.*, Wien, June 15, 1871. See *Proc. R. S.*, Nov. 16, 1871.

they are very distinct at times of small disturbance. Hence great attention has been paid to these disturbances, and it has been found that disturbances of a particular kind are more likely to occur at certain times of the day, and at certain seasons and intervals of time, though each individual disturbance appears quite irregular. Besides these more ordinary disturbances, there are occasionally times of excessive disturbance, in which the magnetism is strongly disturbed for a day or two. These are called Magnetic Storms. Individual disturbances have been sometimes observed at the same instant in stations widely distant.

Mr. Airy has found that a large proportion of the disturbances at Greenwich correspond with the electric currents collected by electrodes placed in the earth in the neighbourhood, and are such as would be directly produced in the magnet if the earth-current, retaining its actual direction, were conducted through a wire placed *underneath* the magnet.

It has been found that there is an epoch of maximum disturbance every eleven years, and that this appears to coincide with the epoch of maximum number of spots in the sun.

474.] The field of investigation into which we are introduced by the study of terrestrial magnetism is as profound as it is extensive.

We know that the sun and moon act on the earth's magnetism. It has been proved that this action cannot be explained by supposing these bodies magnets. The action is therefore indirect. In the case of the sun part of it may be thermal action, but in the case of the moon we cannot attribute it to this cause. Is it possible that the attraction of these bodies, by causing strains in the interior of the earth, produces (Art. 447) changes in the magnetism already existing in the earth, and so by a kind of tidal action causes the semidiurnal variations?

But the amount of all these changes is very small compared with the great secular changes of the earth's magnetism.

What cause, whether exterior to the earth or in its inner depths, produces such enormous changes in the earth's magnetism, that its magnetic poles move slowly from one part of the globe to another? When we consider that the intensity of the magnetization of the great globe of the earth is quite comparable with that which we produce with much difficulty in our steel magnets, these immense changes in so large a body force us to conclude that we are not yet acquainted with one of the most powerful agents in nature, the scene of whose activity lies in those inner depths of the earth, to the knowledge of which we have so few means of access*.

* {Balfour Stewart suggested that the diurnal variations are due to electric current induced in the rarified air in the upper regions of the atmosphere as it moves across the earth's lines of force. Schuster, *Phil. Trans.* A, 1889, p. 467, by applying Gauss's method, has lately shewn that the greater part of these disturbances have their origin above the surface of the earth.}

# PART IV.

## ELECTROMAGNETISM.

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## CHAPTER I.

### ELECTROMAGNETIC FORCE.

475.] It had been noticed by many different observers that in certain cases magnetism is produced or destroyed in needles by electric discharges through them or near them, and conjectures of various kinds had been made as to the relation between magnetism and electricity, but the laws of these phenomena, and the form of these relations, remained entirely unknown till Hans Christian Örsted*, at a private lecture to a few advanced students at Copenhagen, observed that a wire connecting the ends of a voltaic battery affected a magnet in its vicinity. This discovery he published in a tract entitled *Experimenta circa effectum Conflictûs Electrici in Acum Magneticam*, dated July 21, 1820.

Experiments on the relation of the magnet to bodies charged with electricity had been tried without any result till Örsted endeavoured to ascertain the effect of a wire *heated* by an electric current. He discovered, however, that the current itself, and not the heat of the wire, was the cause of the action, and that the 'electric conflict acts in a revolving manner,' that is, that a magnet placed near a wire transmitting an electric current tends to set itself perpendicular to the wire, and with the

* See another account of Örsted's discovery in a letter from Professor Hansteen in the *Life of Faraday* by Dr. Bence Jones, vol. ii. p. 395.

same end always pointing forwards as the magnet is moved round the wire.

476.] It appears therefore that in the space surrounding a wire transmitting an electric current a magnet is acted on by forces dependent on the position of the wire and on the strength of the current. The space in which these forces act may therefore be considered as a magnetic field, and we may study it in the same way as we have already studied the field in the neighbourhood of ordinary magnets, by tracing the course of the lines of magnetic force, and measuring the intensity of the force at every point.

477.] Let us begin with the case of an indefinitely long straight wire carrying an electric current. If a man were to place himself in imagination in the position of the wire, so that the current should flow from his head to his feet, then a magnet suspended freely before him would set itself so that the end which points north would, under the action of the current, point to his right hand.

The lines of magnetic force are everywhere at right angles to planes drawn through the wire, and are therefore circles each in a plane perpendicular to the wire, which passes through its centre. The pole of a magnet which points north, if carried round one of these circles from left to right, would experience a force acting always in the direction of its motion. The other pole of the same magnet would experience a force in the opposite direction.

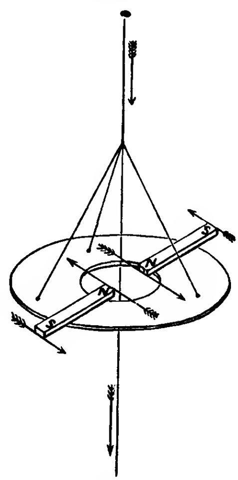

Fig. 21.

478.] To compare these forces let the wire be supposed vertical, and the current a descending one, and let a magnet be placed on an apparatus which is free to rotate about a vertical axis coinciding with the wire. It is found that under these circumstances the current has no effect in causing the rotation of the apparatus as a whole about itself as an axis. Hence the action of the vertical current on the two poles of the magnet is such that the statical moments of the two forces about the current as an axis are equal and opposite. Let $m_1$

and $m_2$ be the strengths of the two poles, $r_1$ and $r_2$ their distances from the axis of the wire, $T_1$ and $T_2$ the intensities of the magnetic force due to the current at the two poles respectively, then the force on $m_1$ is $m_1T_1$, and since it is at right angles to the axis its moment is $m_1T_1r_1$. Similarly that of the force on the other pole is $m_2T_2r_2$, and since there is no motion observed,

$$m_1T_1r_1+m_2T_2r_2 = 0.$$

But we know that in all magnets

$$m_1+m_2 = 0.$$

Hence
$$T_1r_1 = T_2r_2,$$

or the electromagnetic force due to a straight current of infinite length is perpendicular to the current, and varies inversely as the distance from it.

479.] Since the product $Tr$ depends on the strength of the current it may be employed as a measure of the current. This method of measurement is different from that founded upon electrostatic phenomena, and as it depends on the magnetic phenomena produced by electric currents it is called the Electromagnetic system of measurement. In the electromagnetic system if $i$ is the current, $Tr = 2i$.

480.] If the wire be taken for the axis of $z$, then the rectangular components of $T$ are

$$X = -2i\frac{y}{r^2}, \qquad Y = 2i\frac{x}{r^2}, \qquad Z = 0.$$

Here $Xdx+Ydy+Zdz$ is a complete differential, being that of

$$2i\tan^{-1}\frac{y}{x}+C.$$

Hence the magnetic force in the field can be deduced from a potential function, as in several former instances, but the potential is in this case a function having an infinite series of values whose common difference is $4\pi i$. The differential coefficients of the potential with respect to the coordinates have, however, definite and single values at every point.

The existence of a potential function in the field near an electric current is not a self-evident result of the principle of the conservation of energy, for in all actual currents there is a continual expenditure of the electric energy of the battery in overcoming the resistance of the wire, so that unless the amount of this expenditure were accurately known, it might be suspected that part of the energy of the battery was employed in causing work to be done on a magnet moving in a cycle. In fact, if a magnetic pole, $m$, moves round a closed curve which embraces the wire, work is actually done to the amount of $4\pi mi$. It is only for closed paths which do not embrace the wire that the line-integral of the force vanishes. We must therefore for the present consider the law of force and the existence of a potential as resting on the evidence of the experiment already described.

481.] If we consider the space surrounding an infinite straight line we shall see that it is a cyclic space, because it returns into itself. If we now conceive a plane, or any other surface, commencing at the straight line and extending on one side of it to infinity, this surface may be regarded as a diaphragm which reduces the cyclic space to an acyclic one. If from any fixed point lines be drawn to any other point without cutting the diaphragm, and the potential be defined as the line-integral of the force taken along one of these lines, the potential at any point will then have a single definite value.

The magnetic field is now identical in all respects with that due to a magnetic shell coinciding with this surface, the strength of the shell being $i$. This shell is bounded on one edge by the infinite straight line. The other parts of its boundary are at an infinite distance from the part of the field under consideration.

482.] In all actual experiments the current forms a closed circuit of finite dimensions. We shall therefore compare the magnetic action of a finite circuit with that of a magnetic shell of which the circuit is the bounding edge.

It has been shewn by numerous experiments, of which the earliest are those of Ampère, and the most accurate those of Weber, that the magnetic action of a small plane circuit at distances which are great compared with the dimensions of the circuit is the same as that of a magnet whose axis is normal to the plane of the circuit, and whose magnetic moment is equal to the area of the circuit multiplied by the strength of the current*.

* {Ampère, *Théorie des phénomènes électrodynamiques*, 1826; Weber, *Elektrodynamische Maasbestimmungen* (*Abhandlungen der königlich Sächs. Gesellschaft zu Leipzig*, 1850–1852.)}

If the circuit be supposed to be filled up by a surface bounded by the circuit and thus forming a diaphragm, and if a magnetic shell of strength $i$ coinciding with this surface be substituted for the electric current, then the magnetic action of the shell on all distant points will be identical with that of the current.

483.] Hitherto we have supposed the dimensions of the circuit to be small compared with the distance of any part of it from the part of the field examined. We shall now suppose the circuit to be of any form and size whatever, and examine its action at any point $P$ not in the conducting wire itself. The following method, which has important geometrical applications, was introduced by Ampère for this purpose.

Conceive any surface $S$ bounded by the circuit and not passing through the point $P$. On this surface draw two series of lines crossing each other so as to divide it into elementary portions, the dimensions of which are small compared with their distance from $P$, and with the radii of curvature of the surface.

Round each of these elements conceive a current of strength $i$ to flow, the direction of circulation being the same in all the elements as it is in the original circuit.

Along every line forming the division between two contiguous elements two equal currents of strength $i$ flow in opposite directions.

The effect of two equal and opposite currents in the same place is absolutely zero, in whatever aspect we consider the currents. Hence their magnetic effect is zero. The only portions of the elementary circuits which are not neutralized in this way are those which coincide with the original circuit. The total effect of the elementary circuits is therefore equivalent to that of the original circuit.

484.] Now since each of the elementary circuits may be considered as a small plane circuit whose distance from $P$ is great compared with its dimensions, we may substitute for it an elementary magnetic shell of strength $i$ whose bounding edge coincides with the elementary circuit. The magnetic effect of the elementary shell on $P$ is equivalent to that of the elementary circuit. The whole of the elementary shells constitute a magnetic shell of strength $i$, coinciding with the surface $S$ and bounded by the original circuit, and the magnetic action of the whole shell on $P$ is equivalent to that of the circuit.

It is manifest that the action of the circuit is independent of the form of the surface $S$, which was drawn in a perfectly arbitrary manner so as to fill it up. We see from this that the action of a magnetic shell depends only on the form of its edge and not on the form of the shell itself. This result we obtained before, in Art. 410, but it is instructive to see how it may be deduced from electromagnetic considerations.

The magnetic force due to the circuit at any point is therefore identical in magnitude and direction with that due to a magnetic shell bounded by the circuit and not passing through the point, the strength of the shell being numerically equal to that of the current. The direction of the current in the circuit is related to the direction of magnetization of the shell, so that if a man were to stand with his feet on that side of the shell which we call the positive side, and which tends to point to the north, the current in front of him would be from right to left.

485.] The magnetic potential of the circuit, however, differs from that of the magnetic shell for those points which are in the substance of the magnetic shell.

If $\omega$ is the solid angle subtended at the point $P$ by the magnetic shell, reckoned positive when the positive or austral side of the shell is next to $P$, then the magnetic potential at any point not in the shell itself is $\omega\phi$, where $\phi$ is the strength of the shell. At any point in the substance of the shell itself we may suppose the shell divided into two parts whose strengths are $\phi_1$ and $\phi_2$, where $\phi_1+\phi_2 = \phi$, such that the point is on the positive side of $\phi_1$ and on the negative side of $\phi_2$. The potential at this point is

$$\omega(\phi_1+\phi_2)-4\pi\phi_2.$$

On the negative side of the shell the potential becomes $\phi(\omega-4\pi)$. In this case therefore the potential is continuous, and at every point has a single determinate value. In the case of the electric circuit, on the other hand, the magnetic potential at every point not in the conducting wire itself is equal to $i\omega$, where $i$ is the strength of the current, and $\omega$ is the solid angle subtended by a circuit at the point, and is reckoned positive when the current, as seen from $P$, circulates in the direction opposite to that of the hands of a watch.

The quantity $i\omega$ is a function having an infinite series of values whose common difference is $4\pi i$. The differential coefficients of

$i\omega$ with respect to the coordinates have, however, single and determinate values for every point of space.

486.] If a long thin flexible solenoidal magnet were placed in the neighbourhood of an electric circuit, the north and south ends of the solenoid would tend to move in opposite directions round the wire, and if they were free to obey the magnetic force the magnet would finally become wound round the wire in a closed coil. If it were possible to obtain a magnet having only one pole, or poles of unequal strength, such a magnet would be moved round and round the wire continually in one direction, but since the poles of every magnet are equal and opposite, this result can never occur. Faraday, however, has shewn how to produce the continuous rotation of one pole of a magnet round an electric current by making it possible for one pole to go round and round the current while the other pole does not. That this process may be repeated indefinitely, the body of the magnet must be transferred from one side of the current to the other once in each revolution. To do this without interrupting the flow of electricity, the current is split into two branches, so that when one branch is opened to let the magnet pass the current continues to flow through the other. Faraday used for this purpose a circular trough of mercury, as shewn in Fig. 23, Art. 491. The current enters the trough through the wire $AB$, it is divided at $B$, and after flowing through the arcs $BQP$ and $BRP$ it unites at $P$, and leaves the trough through the wire $PO$, the cup of mercury $O$, and a vertical wire beneath $O$, down which the current flows.

The magnet (not shewn in the figure) is mounted so as to be capable of revolving about a vertical axis through $O$, and the wire $OP$ revolves with it. The body of the magnet passes through the aperture of the trough, one pole, say the north pole, being beneath the plane of the trough, and the other above it. As the magnet and the wire $OP$ revolve about the vertical axis, the current is gradually transferred from the branch of the trough which lies in front of the magnet to that which lies behind it, so that in every complete revolution the magnet passes from one side of the current to the other. The north pole of the magnet revolves about the descending current in the direction N.E.S.W., and if $\omega$, $\omega'$ are the solid angles (irrespective of sign) subtended by the circular trough at the two poles, the work done by the electromagnetic force in a complete revolution is

$$mi\,(4\pi-\omega-\omega'),$$

where $m$ is the strength of either pole, and $i$ the strength of the current *.

487.] Let us now endeavour to form a notion of the state of the magnetic field near a linear electric circuit.

Let the value of $\omega$, the solid angle subtended by the circuit, be found for every point of space, and let the surfaces for which $\omega$ is constant be described. These surfaces will be the equipotential surfaces. Each of these surfaces will be bounded by the circuit, and any two surfaces, $\omega_1$ and $\omega_2$, will meet in the circuit at an angle $\frac{1}{2}(\omega_1-\omega_2)$ †.

* [This problem may be discussed as follows: Referring to Fig. 23, Art. 491, let us take $OP$ in any position and introduce imaginary balancing currents $i$ along $BO$ and $x$, $y$ along $OB$. As the magnet attached to $OP$ is carried through a complete revolution no work is done on the south pole by the current $i$, supposed to pass along $ABOZ$, that pole describing a closed curve which does not embrace the current. The north pole however describes a closed curve which does embrace the current, and the work done upon it is $4\pi mi$. We have now to estimate the effects of the currents $x$ in the circuit $BPOB$ and $y$ in the circuit $BRPOB$. The potential of the north pole which is below the planes of those circuits will be

$$-mx\omega_\theta+my\,(\omega-\omega_\theta)\ \text{ and, of the south, }\ -mx\omega'_\theta-my\,(-\omega'+\omega'_\theta),$$

where $\omega_\theta$ and $\omega'_\theta$ denote the solid angles subtended at the two poles by $BOP$, and $\omega$, $\omega'$ those subtended by the circular trough. The resultant potential is

$$my\,(\omega+\omega')-mi\,(\omega_\theta+\omega'_\theta).$$

Hence as $OP$ revolves from $OP$ in the direction $NESW$ back to $OP$ again the potential will change by $-mi\,(\omega+\omega')$. The work done by the currents is therefore that given in the text.]

{The following is a slightly different way of obtaining this result:—The currents through the wires and the mercury trough are equivalent to a circular current $i-x$ round the trough, a current $i$ round the circuit $POB$ and a current $i$ through $AB$, $BO$, and the vertical wire $OZ$. The circular current will evidently not produce any force tending to make either pole travel round a circle co-axial with the circuit of the current. The North pole threads the circuit $AB$, $BO$, and the vertical $OZ$, once in each revolution, the work done on it is therefore $4\pi im$. If $\Omega$ and $\Omega'$ are the numerical values of the solid angle subtended by the circuit $POB$ at the north and south poles of the magnet respectively, then the potential energy of the magnet and circuit is $-mi\,(\Omega+\Omega')$. Hence if $\theta$ is the angle $POB$, the work done on the magnet in a complete revolution is

$$-\int_0^{2\pi} mi\frac{d}{d\theta}(\Omega+\Omega')\,d\theta=-mi\,(\omega+\omega').$$

Hence the whole work done on the magnet is

$$mi\,\{4\pi-(\omega+\omega')\}\}.$$

† {This can be deduced as follows:—Consider a point $P$ on the surface $\omega_1$ near the line of intersection of the two equipotential surfaces, let $O$ be a point on the line of intersection near $P$, then describe a sphere of unit radius with centre $O$. The solid angle subtended at $P$ by the circuit will be measured by the area cut off the unit sphere by the tangent plane at $O$ to the surface $\omega_1$, and by an irregularly shaped cone determined by the shape of the circuit at some distance from $O$. Now consider a point $Q$ on the second surface $\omega_2$ near to $O$, the solid angle subtended by the circuit at this point will be measured by the area cut off the unit sphere with centre $O$ by the

Figure XVIII, at the end of this volume, represents a section of the equipotential surfaces due to a circular current. The small circle represents a section of the conducting wire, and the horizontal line at the bottom of the figure is the perpendicular to the plane of the circular current through its centre. The equipotential surfaces, 24 of which are drawn corresponding to a series of values of $\omega$ differing by $\frac{\pi}{6}$, are surfaces of revolution, having this line for their common axis. They are evidently oblate figures, being flattened in the direction of the axis. They meet each other in the line of the circuit at angles of 15°.

The force acting on a magnetic pole placed at any point of an equipotential surface is perpendicular to this surface, and varies inversely as the distance between consecutive equipotential surfaces. The closed curves surrounding the section of the wire in Fig. XVIII are the lines of force. They are copied from Sir W. Thomson's Paper on 'Vortex Motion *.' See also Art. 702.

*Action of an Electric Circuit on any Magnetic System.*

488.] We are now able to deduce the action of an electric circuit on any magnetic system in its neighbourhood from the theory of magnetic shells. For if we construct a magnetic shell, whose strength is numerically equal to the strength of the current, and whose edge coincides in position with the circuit, while the shell itself does not pass through any part of the magnetic system, the action of the shell on the magnetic system will be identical with that of the electric current.

*Reaction of the Magnetic System on the Electric Circuit.*

489.] From this, applying the principle that action and reaction are equal and opposite, we conclude that the mechanical action of the magnetic system on the electric circuit is identical with its action on a magnetic shell having the circuit for its edge.

The potential energy of a magnetic shell of strength $\phi$ placed

tangent plane to $\omega_2$ at $O$ and by an irregularly shaped cone which, if $P$ and $Q$ are very close together, will be the same as before. Thus the difference between the solid angles is the area of the lune between the tangent planes, and this area is twice the angle between the tangent planes, that is twice the angle at which $\omega_1$ and $\omega_2$ intersect, thus the angle between the surfaces is $\frac{1}{2}(\omega_1-\omega_2)_f$.

* *Trans. R. S. Edin.*, vol. xxv. p. 217, (1869).

in a field of magnetic force of which the potential is $V$, is, by Art. 410,

$$= \phi \iint \left(l\frac{dV}{dx} + m\frac{dV}{dy} + n\frac{dV}{dz}\right) dS,$$

where $l$, $m$, $n$ are the direction-cosines of the normal drawn from the positive side of the element $dS$ of the shell, and the integration is extended over the surface of the shell.

Now the surface-integral

$$N = \iint (la + mb + nc)\, dS,$$

where $a$, $b$, $c$ are the components of the magnetic induction, represents the quantity of magnetic induction through the shell, or, in the language of Faraday, the *number* of lines of magnetic induction, reckoned algebraically, which pass through the shell from the negative to the positive side, lines which pass through the shell in the opposite direction being reckoned negative.

Remembering that the shell does not belong to the magnetic system to which the potential $V$ is due, and that the magnetic force is therefore equal to the magnetic induction, we have

$$a = -\frac{dV}{dx}, \qquad b = -\frac{dV}{dy}, \qquad c = -\frac{dV}{dz},$$

and we may write the value of $M$,

$$M = -\phi N.$$

If $\delta x_1$ represents any displacement of the shell, and $X_1$ the force acting on the shell so as to aid the displacement, then by the principle of conservation of energy,

$$X_1 \delta x_1 + \delta M = 0,$$

or

$$X_1 = \phi \frac{dN}{dx_1}.$$

We have now determined the nature of the force which corresponds to any given displacement of the shell. It aids or resists that displacement accordingly as the displacement increases or diminishes $N$, the number of lines of induction which pass through the shell.

The same is true of the equivalent electric circuit. Any displacement of the circuit will be aided or resisted according as it increases or diminishes the number of lines of induction which pass through the circuit in the positive direction.

We must remember that the positive direction of a line of magnetic induction is the direction in which the pole of a magnet which points north tends to move along the line, and that a line of induction passes through the circuit in the positive direction, when the direction of the line of induction is related to the direction of the current of vitreous electricity in the circuit as the longitudinal to the rotational motion of a right-handed screw. See Art. 23.

490.] It is manifest that the force corresponding to any displacement of the circuit as a whole may be deduced at once from the theory of the magnetic shell. But this is not all. If a portion of the circuit is flexible, so that it may be displaced independently of the rest, we may make the edge of the shell capable of the same kind of displacement by cutting up the surface of the shell into a sufficient number of portions connected by flexible joints. Hence we conclude that if by the displacement of any portion of the circuit in a given direction the number of lines of induction which pass through the circuit can be increased, this displacement will be aided by the electromagnetic force acting on the circuit.

Every portion of the circuit therefore is acted on by a force urging it across the lines of magnetic induction so as to include a greater number of these lines within the embrace of the circuit, and the work done by the force during this displacement is numerically equal to the number of the additional lines of induction multiplied by the strength of the current.

Let the element $ds$ of a circuit, in which a current of strength $i$ is flowing, be moved parallel to itself through a space $\delta x$, it will sweep out an area in the form of a parallelogram whose sides are parallel and equal to $ds$ and $\delta x$ respectively.

If the magnetic induction is denoted by $\mathfrak{B}$, and if its direction makes an angle $\epsilon$ with the normal to the parallelogram, the value of the increment of $N$ corresponding to the displacement is found by multiplying the area of the parallelogram by $\mathfrak{B} \cos \epsilon$. The result of this operation is represented geometrically by the volume of a parallelopiped whose edges represent in magnitude and direction $\delta x$, $ds$, and $\mathfrak{B}$, and it is to be reckoned positive if when we point in these three directions in the order here given the pointer moves round the diagonal of the parallelopiped in the direction of the hands of a watch *. The volume of this parallelopiped is equal to $X\,\delta x$.

If $\theta$ is the angle between $ds$ and $\mathfrak{B}$, the area of the parallelogram whose sides are $ds$ and $\mathfrak{B}$ is $ds.\mathfrak{B}\sin\theta$, and if $\eta$ is the angle which the displacement $\delta x$ makes with the normal to this parallelogram, the volume of the parallelopiped is

$$ds\,.\,\mathfrak{B}\sin\theta\,.\,\delta x\cos\eta = \delta N.$$

Now $$X\,\delta x = i\,\delta N = i\,ds\,.\,\mathfrak{B}\sin\theta\,\delta x\cos\eta,$$

and $$X = i\,ds\,.\,\mathfrak{B}\sin\theta\cos\eta$$

is the force which urges $ds$, resolved in the direction $\delta x$.

The direction of this force is therefore perpendicular to the parallelogram, and its magnitude is equal to $i\,.\,ds\,.\,\mathfrak{B}\sin\theta$.

This is the area of a parallelogram whose sides represent in magnitude and direction $i\,ds$ and $\mathfrak{B}$. The force acting on $ds$ is therefore represented in magnitude by the area of this parallel ogram, and in direction by a normal to its plane drawn in the direction of the longitudinal motion of a right-handed screw, the handle of which is turned from the direction of the current $i\,ds$ to that of the magnetic induction $\mathfrak{B}$.

We may express in the language of Quaternions, both the direction and the magnitude of this force by saying that it is the vector part of the result of multiplying the vector $i\,ds$, the element of the current, by the vector $\mathfrak{B}$, the magnetic induction.

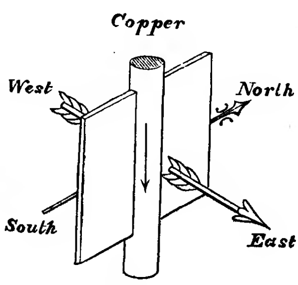

Fig. 22.

491.] We have thus completely determined the force which acts on any portion of an electric circuit placed in a magnetic field. If the circuit is moved in any way so that, after assuming various forms and positions, it returns to its original place, the strength of the current remaining constant during the motion, the whole amount of work done by the electromagnetic forces will be zero. Since this is true of any cycle of motions of the circuit, it follows that it is impossible to maintain by electromagnetic forces a motion of continuous rotation in any part of a linear circuit of constant strength against the resistance of friction, &c.

* {In this rule $ds$ is drawn in the direction of $i$ and the observer is supposed to be at that corner of the parallelopiped from which $dx$, $ds$ and $\mathfrak{B}$ are drawn.}

It is possible, however, to produce continuous rotation provided that at some part of the course of the electric current the current passes from one conductor which slides or glides over another.

When in a circuit there is sliding contact of a conductor over the surface of a smooth solid or a fluid, the circuit can no longer be considered as a single linear circuit of constant strength, but must be regarded as a system of two or of some greater number of circuits of variable strength, the current being so distributed among them that those for which $N$ is increasing have currents in the positive direction, while those for which $N$ is diminishing have currents in the negative direction.

Thus, in the apparatus represented in Fig. 23, $OP$ is a moveable conductor, one end of which rests in a cup of mercury $O$, while the other dips into a circular trough of mercury concentric with $O$.

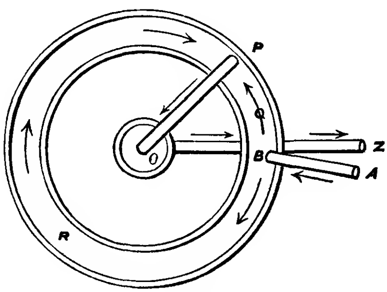

Fig. 23.

The current $i$ enters along $AB$, and divides in the circular trough into two parts, one of which, $x$, flows along the arc $BQP$, while the other, $y$, flows along $BRP$. These currents, uniting at $P$, flow along the moveable conductor $PO$ and the electrode $OZ$ to the zinc end of the battery. The strength of the current along $PO$ and $OZ$ is $x+y$ or $i$.

Here we have two circuits, $ABQPOZ$, the strength of the current in which is $x$, flowing in the positive direction, and $ABRPOZ$, the strength of the current in which is $y$, flowing in the negative direction.

Let $\mathfrak{B}$ be the magnetic induction, and let it be in an upward direction, normal to the plane of the circle.

While $OP$ moves through an angle $\theta$ in the direction opposite to that of the hands of a watch, the area of the first circuit increases by $\frac{1}{2}OP^2.\theta$, and that of the second diminishes by the same quantity. Since the strength of the current in the first circuit is $x$, the work done by it is $\frac{1}{2}x.OP^2.\theta.\mathfrak{B}$, and since the strength of the second is $-y$, the work done by it is $\frac{1}{2}y.OP^2.\theta.\mathfrak{B}$. The whole work done is therefore

$$\tfrac{1}{2}(x+y)\,OP^2.\theta\mathfrak{B} \quad \text{or} \quad \tfrac{1}{2}i.OP^2.\theta\mathfrak{B},$$

depending only on the strength of the current in $PO$. Hence, if $i$ is maintained constant, the arm $OP$ will be carried round and round the circle with a uniform force whose moment is $\frac{1}{2}i.OP^2.\mathfrak{B}$. If, as in northern latitudes, $\mathfrak{B}$ acts downwards, and if the current is inwards, the rotation will be in the negative direction, that is, in the direction $PQBR$.

492.] We are now able to pass from the mutual action of magnets and currents to the action of one circuit on another. For we know that the magnetic properties of an electric circuit $C_1$, with respect to any magnetic system $M_2$, are identical with those of a magnetic shell $S_1$, whose edge coincides with the circuit, and whose strength is numerically equal to that of the electric current. Let the magnetic system $M_2$ be a magnetic shell $S_2$, then the mutual action between $S_1$ and $S_2$ is identical with that between $S_1$ and a circuit $C_2$, coinciding with the edge of $S_2$ and equal in numerical strength, and this latter action is identical with that between $C_1$ and $C_2$.

Hence the mutual action between two circuits $C_1$ and $C_2$ is identical with that between the corresponding magnetic shells $S_1$ and $S_2$.

We have already investigated, in Art. 423, the mutual action of two magnetic shells whose edges are the closed curves $s_1$ and $s_2$.

If we make
$$M = \int_0^{s_2}\int_0^{s_1} \frac{\cos\epsilon}{r}\, ds_1\, ds_2,$$
where $\epsilon$ is the angle between the directions of the elements $ds_1$ and $ds_2$, and $r$ is the distance between them, the integrations being extended one round $s_2$ and one round $s_1$, and if we call $M$ the potential of the two closed curves $s_1$ and $s_2$, then the potential energy due to the mutual action of two magnetic shells whose strengths are $i_1$ and $i_2$ bounded by the two circuits is
$$-i_1 i_2 M,$$
and the force $X$, which aids any displacement $\delta x$, is
$$i_1 i_2 \frac{dM}{dx}.$$

The whole theory of the force acting on any portion of an electric circuit due to the action of another electric circuit may be deduced from this result.

493.] The method which we have followed in this chapter is that of Faraday. Instead of beginning, as we shall do, following

Ampère, in the next chapter, with the direct action of a portion of one circuit on a portion of another, we shew, first, that a circuit produces the same effect on a magnet as a magnetic shell, or, in other words, we determine the nature of the magnetic field due to the circuit. We shew, secondly, that a circuit when placed in any magnetic field experiences the same force as a magnetic shell. We thus determine the force acting on the circuit placed in any magnetic field. Lastly, by supposing the magnetic field to be due to a second electric circuit we determine the action of one circuit on the whole or any portion of the other.

494.] Let us apply this method to the case of a straight current of infinite length acting on a portion of a parallel straight conductor.

Let us suppose that a current $i$ in the first conductor is flowing vertically downwards. In this case the end of a magnet which points north will point to the right-hand of a man (with his feet downwards) looking at it from the axis of the current.

The lines of magnetic induction are therefore horizontal circles, having their centres in the axis of the current, and their positive direction is north, east, south, west.

Let another descending vertical current be placed due west of the first. The lines of magnetic induction due to the first current are here directed towards the north. The direction of the force acting on the second circuit is to be determined by turning the handle of a right-handed screw from the nadir, the direction of the current, to the north, the direction of the magnetic induction. The screw will then move towards the east, that is, the force acting on the second circuit is directed towards the first current, or, in general, since the phenomenon depends only on the relative position of the currents, two parallel circuits conveying currents in the same direction attract each other.

In the same way we may shew that two parallel circuits conveying currents in opposite directions repel one another.

495.] The intensity of the magnetic induction at a distance $r$ from a straight current of strength $i$ is, as we have shewn in Art. 479,

$$2\frac{i}{r}.$$

Hence, a portion of a second conductor parallel to the first, and carrying a current $i'$ in the same direction, will be attracted towards the first with a force

$$F = 2\,ii'\frac{a}{r},$$

where $a$ is the length of the portion considered, and $r$ is its distance from the first conductor.

Since the ratio of $a$ to $r$ is a numerical quantity independent of the absolute value of either of these lines, the product of two currents measured in the electromagnetic system must be of the dimensions of a force, hence the dimensions of the unit current are

$$[i] = [F^{\frac{1}{2}}] = [M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-1}].$$

496.] Another method of determining the direction of the force which acts on a circuit is to consider the relation of the magnetic action of the current to that of other currents and magnets.

If on one side of the wire which carries the current the magnetic action due to the current is in the same or nearly the same direction as that due to other currents, then, on the other side of the wire, these forces will be in opposite or nearly opposite directions, and the force acting on the wire will be from the side on which the forces strengthen each other to the side on which they oppose each other.

Thus, if a descending current is placed in a field of magnetic force directed towards the north, its magnetic action will be to the north on the west side, and to the south on the east side. Hence the forces strengthen each other on the west side and oppose each other on the east side, and the circuit will therefore be acted on by a force from west to east. See Fig. 22, p. 149.

In Fig. XVII at the end of this volume the small circle represents a section of the wire carrying a descending current, and placed in a uniform field of magnetic force acting towards the left-hand of the figure. The magnetic force is greater below the wire than above it. It will therefore be urged from the bottom towards the top of the figure.

497.] If two currents are in the same plane but not parallel, we may apply this principle. Let one of the conductors be an infinite straight wire in the plane of the paper, supposed horizontal. On the right side of the current* the magnetic force acts

* {The right side of the current is the right of an observer with his back against the paper placed so that the current enters at his head and leaves at his feet.}

downwards and on the left side it acts upwards. The same is true of the magnetic force due to any short portion of a second current in the same plane. If the second current is on the right side of the first, the magnetic forces will strengthen each other on its right side and oppose each other on its left side. Hence the circuit conveying the second current will be acted on by a force urging it from its right side to its left side. The magnitude of this force depends only on the position of the second current and not on its direction. If the second circuit is on the left side of the first it will be urged from left to right.

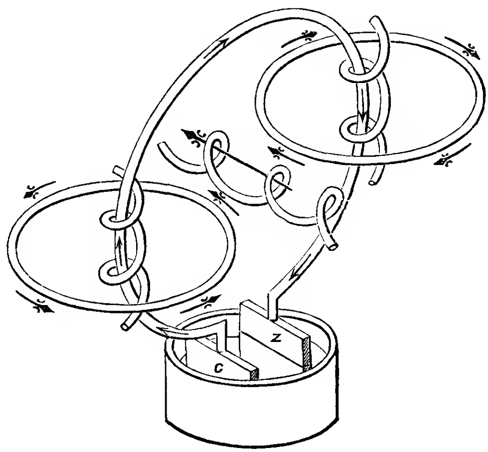


Fig. 24.

Relation between the electric current and the lines of magnetic induction indicated by a right-handed screw.

Hence, if the second current is in the same direction as the first its circuit is attracted; if in the opposite direction it is repelled; if it flows at right angles to the first and away from it, it is urged in the direction of the first current; and if it flows towards the first current, it is urged in the direction opposite to that in which the first current flows.

In considering the mutual action of two currents it is not necessary to bear in mind the relations between electricity and magnetism which we have endeavoured to illustrate by means of a right-handed screw. Even if we have forgotten these relations we shall arrive at correct results, provided we adhere consistently to one of the two possible forms of the relation.

498.] Let us now bring together the magnetic phenomena of the electric circuit so far as we have investigated them.

We may conceive the electric circuit to consist of a voltaic battery, and a wire connecting its extremities, or of a thermo-electric arrangement, or of a charged Leyden jar with a wire connecting its positive and negative coatings, or of any other arrangement for producing an electric current along a definite path.

The current produces magnetic phenomena in its neighbourhood.

If any closed curve be drawn, and the line-integral of the magnetic force taken completely round it, then, if the closed curve is not linked with the circuit, the line-integral is zero, but if it is linked with the circuit, so that the current $i$ flows through the closed curve, the line-integral is $4\pi i$, and is positive if the direction of integration round the closed curve would coincide with that of the hands of a watch as seen by a person passing through it in the direction in which the electric current flows. To a person moving along the closed curve in the direction of integration, and passing through the electric circuit, the direction of the current would appear to be that of the hands of a watch. We may express this in another way by saying that the relation between the directions of the two closed curves may be expressed by describing a right-handed screw round the electric circuit and a right-handed screw round the closed curve. If the direction of rotation of the thread of either, as we pass along it, coincides with the positive direction in the other, then the line-integral will be positive, and in the opposite case it will be negative.

499.] *Note.*—The line-integral $4\pi i$ depends solely on the quantity of the current, and not on any other thing whatever. It does not depend on the nature of the conductor through which the current is passing, as, for instance, whether it be a metal or an electrolyte, or an imperfect conductor. We have reason for believing that even when there is no proper conduction, but merely a variation of electric displacement, as in the glass of a Leyden jar during charge or discharge, the magnetic effect of the electric movement is precisely the same.

Again, the value of the line-integral $4\pi i$ does not depend on the nature of the medium in which the closed curve is drawn. It is the same whether the closed curve is drawn entirely through air, or passes through a magnet, or soft iron, or any other substance, whether paramagnetic or diamagnetic.

500.] When a circuit is placed in a magnetic field the mutual action between the current and the other constituents of the field depends on the surface-integral of the magnetic induction through any surface bounded by that circuit. If by any given motion of the circuit, or of part of it, this surface-integral can be *increased*, there will be a mechanical force tending to move the conductor or the portion of the conductor in the given manner.

The kind of motion of the conductor which increases the surface-integral is motion of the conductor perpendicular to the direction of the current and across the lines of induction.

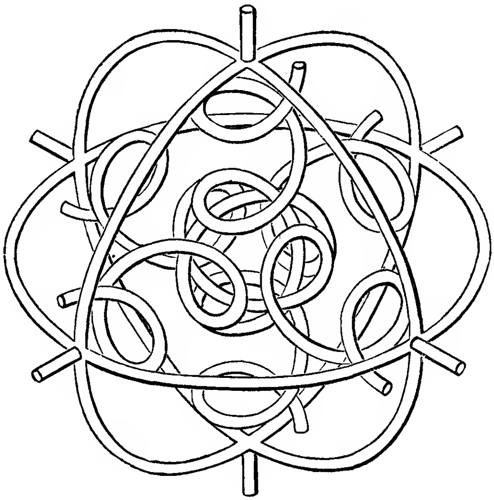

Fig. 25.

Relations between the positive directions of motion and of rotation indicated by three right-handed screws.

If a parallelogram be drawn, whose sides are parallel and proportional to the strength of the current at any point, and to the magnetic induction at the same point, then the force on unit of length of the conductor is numerically equal to the area of this parallelogram, and is perpendicular to its plane, and acts in the direction in which the motion of turning the handle of a right-handed screw from the direction of the current to the direction of the magnetic induction would cause the screw to move.

Hence we have a new electromagnetic definition of a line of magnetic induction. It is that line to which the force on the conductor is always perpendicular.

It may also be defined as a line along which, if an electric current be transmitted, the conductor carrying it will experience no force.

501.] It must be carefully remembered, that the mechanical force which urges a conductor carrying a current across the lines of magnetic force, acts, not on the electric current, but on the conductor which carries it. If the conductor be a rotating disk or a fluid it will move in obedience to this force, and this motion may or may not be accompanied by a change of position of the electric current which it carries. [But if the current itself be free to choose any path through a fixed solid conductor or a network of wires, then, when a constant magnetic force is made to act on the system, the path of the current through the conductors is not permanently altered, but after certain transient phenomena, called induction currents, have subsided, the distribution of the current will be found to be the same as if no magnetic force were in action.]*

The only force which acts on electric currents is electromotive force, which must be distinguished from the mechanical force which is the subject of this chapter.

* {Mr. Hall has discovered (*Phil. Mag.* ix. p. 225, x. p. 301, 1880) that a steady magnetic field does slightly alter the distribution of currents in most conductors, so that the statement in brackets must be regarded as only approximately true.}

# CHAPTER II.

## AMPÈRE'S INVESTIGATION OF THE MUTUAL ACTION OF ELECTRIC CURRENTS.

502.] We have considered in the last chapter the nature of the magnetic field produced by an electric current, and the mechanical action on a conductor carrying an electric current placed in a magnetic field. From this we went on to consider the action of one electric circuit upon another, by determining the action on the first due to the magnetic field produced by the second. But the action of one circuit upon another was originally investigated in a direct manner by Ampère almost immediately after the publication of Örsted's discovery. We shall therefore give an outline of Ampère's method, resuming the method of this treatise in the next chapter.

The ideas which guided Ampère belong to the system which admits direct action at a distance, and we shall find that a remarkable course of speculation and investigation founded on those ideas has been carried on by Gauss, Weber, F. E. Neumann, Riemann, Betti, C. Neumann, Lorenz, and others, with very remarkable results both in the discovery of new facts and in the formation of a theory of electricity. See Arts. 846–866.

The ideas which I have attempted to follow out are those of action through a medium from one portion to the contiguous portion. These ideas were much employed by Faraday, and the development of them in a mathematical form, and the comparison of the results with known facts, have been my aim in several published papers. The comparison, from a philosophical point of view, of the results of two methods so completely opposed in their first principles must lead to valuable data for the study of the conditions of scientific speculation.

503.] Ampère's theory of the mutual action of electric currents is founded on four experimental facts and one assumption.

Ampère's fundamental experiments are all of them examples of what has been called the null method of comparing forces. See Art. 214. Instead of measuring the force by the dynamical effect of communicating motion to a body, or the statical method of placing it in equilibrium with the weight of a body or the elasticity of a fibre, in the null method two forces, due to the same source, are made to act simultaneously on a body already in equilibrium, and no effect is produced, which shews that these forces are themselves in equilibrium. This method is peculiarly valuable for comparing the effects of the electric current when it passes through circuits of different forms. By connecting all the conductors in one continuous series, we ensure that the strength of the current is the same at every point of its course, and since the current begins everywhere throughout its course almost at the same instant, we may prove that the forces due to its action on a suspended body are in equilibrium by observing that the body is not at all affected by the starting or the stopping of the current.

504.] Ampère's balance consists of a light frame capable of revolving about a vertical axis, and carrying a wire which forms two circuits of equal area, in the same plane or in parallel planes, in which the current flows in opposite directions. The object of this arrangement is to get rid of the effects of terrestrial magnetism on the conducting wire. When an electric circuit is free to move it tends to place itself so as to embrace the largest possible number of the lines of induction. If these lines are due to terrestrial magnetism, this position, for a circuit in a vertical plane, will be when the plane of the circuit is magnetic east and west, and when the direction of the current is opposed to the apparent course of the sun.

By rigidly connecting two circuits of equal area in parallel planes, in which equal currents run in opposite directions, a combination is formed which is unaffected by terrestrial magnetism, and is therefore called an Astatic Combination, see Fig. 26. It is acted on, however, by forces arising from currents or magnets which are so near it that they act differently on the two circuits.

505.] Ampère's first experiment is on the effect of two equal currents close together in opposite directions. A wire covered with insulating material is doubled on itself, and placed near one of the circuits of the astatic balance. When a current is made to pass through the wire and the balance, the equilibrium of the balance remains undisturbed, shewing that two equal currents close together in opposite directions neutralize each other. If, instead of two wires side by side, a wire be insulated in the middle of a metal tube, and if the current pass through the wire and back by the tube, the action outside the tube is not only approximately but accurately null. This principle is of great importance in the construction of electric apparatus, as it affords the means of conveying the current to and from any galvano-

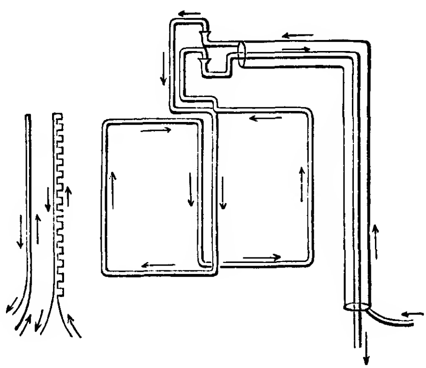

Fig. 26.

meter or other instrument in such a way that no electromagnetic effect is produced by the current on its passage to and from the instrument. In practice it is generally sufficient to bind the wires together, care being taken that they are kept perfectly insulated from each other, but where they must pass near any sensitive part of the apparatus it is better to make one of the conductors a tube and the other a wire inside it. See Art. 683.

506.] In Ampère's second experiment one of the wires is bent and crooked with a number of small sinuosities, but so that in every part of its course it remains very near the straight wire. A current, flowing through the crooked wire and back again through the straight wire, is found to be without influence on the astatic balance. This proves that the effect of the current running through any crooked part of the wire is equivalent to the same current running in the straight line joining its extremities, provided the crooked line is in no part of its course far from the straight one. Hence any small element of a circuit is equivalent to two or more component elements, the relation between the component elements and the resultant element being the same as that between component and resultant displacements or velocities.

507.] In the third experiment a conductor capable of moving only in the direction of its length is substituted for the astatic balance. The current enters the conductor and leaves it at fixed points of space, and it is found that no closed circuit placed in the neighbourhood is able to move the conductor.

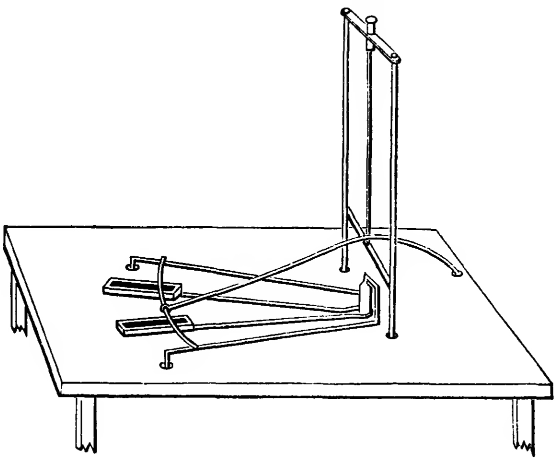

Fig. 27.

The conductor in this experiment is a wire in the form of a circular arc suspended on a frame which is capable of rotation about a vertical axis. The circular arc is horizontal, and its centre coincides with the vertical axis. Two small troughs are filled with mercury till the convex surface of the mercury rises above the level of the troughs. The troughs are placed under the circular arc and adjusted till the mercury touches the wire, which is of copper well amalgamated. The current is made to enter one of these troughs, to traverse the part of the circular arc between the troughs, and to escape by the other trough. Thus part of the circular arc is traversed by the current, and the arc is at the same time capable of moving with considerable freedom in the direction of its length. Any closed currents or magnets may now be made to approach the moveable conductor without producing the slightest tendency to move it in the direction of its length.

508.] In the fourth experiment with the astatic balance two circuits are employed, each similar to one of those in the balance, but one of them, $C$, having dimensions $n$ times greater, and the other, $A$, $n$ times less. These are placed on opposite sides of the circuit of the balance, which we shall call $B$, so that they are similarly placed with respect to it, the distance of $C$ from $B$ being $n$ times greater than the distance of $B$ from $A$.

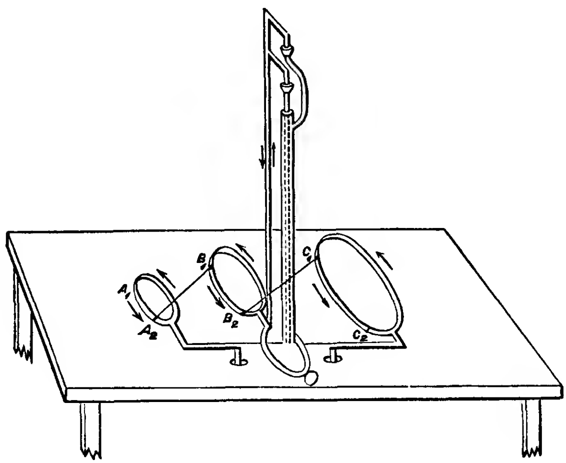


Fig. 28.

The direction and strength of the current is the same in $A$ and $C$. Its direction in $B$ may be the same or opposite. Under these circumstances it is found that $B$ is in equilibrium under the action of $A$ and $C$, whatever be the forms and distances of the three circuits, provided they have the relations given above.

Since the actions between the complete circuits may be considered to be due to actions between the elements of the circuits, we may use the following method of determining the law of these actions.

Let $A_1, B_1, C_1$, Fig. 28, be corresponding elements of the three circuits, and let $A_2, B_2, C_2$ be also corresponding elements in antoher part of the circuits. Then the situation of $B_1$ with respect to $A_2$ is similar to the situation of $C_1$ with respect to $B_2$, but the distance and dimensions of $C_1$ and $B_2$ are $n$ times the distance and dimensions of $B_1$ and $A_2$, respectively. If the law of electromagnetic action is a function of the distance, then the action, whatever be its form or quality, between $B_1$ and $A_2$, may be written

$$F = B_1 \,.\, A_2 f(\overline{B_1 A_2})\, ab,$$

and that between $C_1$ and $B_2$

$$F' = C_1 \,.\, B_2 f(\overline{C_1 B_2})\, bc,$$

where $a$, $b$, $c$ are the strengths of the currents in $A$, $B$, $C$. But $nB_1 = C_1$, $nA_2 = B_2$, $n\overline{B_1 A_2} = \overline{C_1 B_2}$, and $a = c$. Hence

$$F' = n^2 B_1 \,.\, A_2 f(n\overline{B_1 A_2})\, ab,$$

and this is equal to $F$ by experiment, so that we have

$$n^2 f(n\,\overline{A_2 B_1}) = f(\overline{A^2 B_1});$$

or, *the force varies inversely as the square of the distance* *.

509.] It may be observed with reference to these experiments that every electric current forms a closed circuit. The currents used by Ampère, being produced by the voltaic battery, were of course in closed circuits. It might be supposed that in the case of the current of discharge of a conductor by a spark we might have a current forming an open finite line, but according to the views of this book even this case is that of a closed circuit. No experiments on the mutual action of unclosed currents have been made. Hence no statement about the mutual action of two elements of circuits can be said to rest on purely experimental grounds. It is true we may render a portion of a circuit moveable, so as to ascertain the action of the other currents upon it, but these currents, together with that in the moveable portion, necessarily form closed circuits, so that the ultimate result of the experiment is the action of one or more closed currents upon the whole or a part of a closed current.

510.] In the analysis of the phenomena, however, we may regard the action of a closed circuit on an element of itself or of another circuit as the resultant of a number of separate forces, depending on the separate parts into which the first circuit may be conceived, for mathematical purposes, to be divided.

* {Another proof that this experiment leads to the law of the inverse square is given in Art. 523, and the reader will probably find it simpler and more convincing than the preceding.}

This is a merely mathematical analysis of the action, and is therefore perfectly legitimate, whether these forces can really act separately or not.

511.] We shall begin by considering the purely geometrical relations between two lines in space representing the circuits, and between elementary portions of these lines.

Let there be two curves in space in each of which a fixed point is taken, from which the arcs are measured in a defined direction along the curves. Let $A$, $A'$ be these points. Let $PQ$ and $P'Q'$ be elements of the two curves.

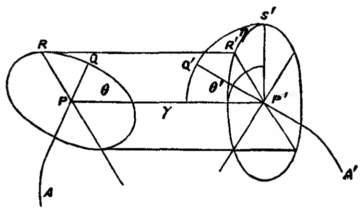


Fig. 29.

Let
$$\left.\begin{aligned} AP &= s, & A'P' &= s', \\ PQ &= ds, & P'Q' &= ds', \end{aligned}\right\} \quad (1)$$

and let the distance $PP'$ be denoted by $r$. Let the angle $P'PQ$ be denoted by $\theta$, and $PP'Q'$ by $\theta'$, and let the angle between the planes of these angles be denoted by $\eta$.

The relative position of the two elements is sufficiently defined by their distance $r$ and the three angles $\theta$, $\theta'$, and $\eta$, for if these be given their relative position is as completely determined as if they formed part of the same rigid body.

512.] If we use rectangular coordinates and make $x, y, z$ the coordinates of $P$, and $x', y', z'$ those of $P'$, and if we denote by $l, m, n$ and by $l', m', n'$ the direction-cosines of $PQ$, and of $P'Q'$ respectively, then

$$\left.\begin{aligned} \frac{dx}{ds} &= l, & \frac{dy}{ds} &= m, & \frac{dz}{ds} &= n, \\ \frac{dx'}{ds'} &= l', & \frac{dy'}{ds'} &= m', & \frac{dz'}{ds'} &= n', \end{aligned}\right\} \quad (2)$$

and
$$\left.\begin{aligned} l(x'-x)+m(y'-y)+n(z'-z) &= r\cos\theta, \\ l'(x'-x)+m'(y'-y)+n'(z'-z) &= -r\cos\theta', \\ ll'+mm'+nn' &= \cos\epsilon, \end{aligned}\right\} \quad (3)$$

where $\epsilon$ is the angle between the directions of the elements themselves, and

$$\cos\epsilon = -\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos\eta. \quad (4)$$

Again,
$$r^2 = (x'-x)^2 + (y'-y)^2 + (z'-z)^2, \quad (5)$$

$$\left.\begin{aligned}\text{whence}\quad r\frac{dr}{ds} &= -(x'-x)\frac{dx}{ds}-(y'-y)\frac{dy}{ds}-(z'-z)\frac{dz}{ds},\\ &= -r\cos\theta.\\ \text{Similarly}\quad r\frac{dr}{ds'} &= \quad (x'-x)\frac{dx'}{ds'}+(y'-y)\frac{dy'}{ds'}+(z'-z)\frac{dz'}{ds'},\\ &= -r\cos\theta';\end{aligned}\right\} \qquad (6)$$

and differentiating $r\dfrac{dr}{ds}$ with respect to $s'$,

$$\left.\begin{aligned} r\frac{d^2r}{ds\,ds'}+\frac{dr}{ds}\frac{dr}{ds'} &= -\frac{dx}{ds}\frac{dx'}{ds'}-\frac{dy}{ds}\frac{dy'}{ds'}-\frac{dz}{ds}\frac{dz'}{ds'},\\ &= -(ll'+mm'+nn'),\\ &= -\cos\epsilon.\end{aligned}\right\} \qquad (7)$$

We can therefore express the three angles $\theta$, $\theta'$, and $\eta$, and the auxiliary angle $\epsilon$ in terms of the differential coefficients of $r$ with respect to $s$ and $s'$ as follows,

$$\left.\begin{aligned}\cos\theta &= -\frac{dr}{ds},\\ \cos\theta' &= -\frac{dr}{ds'},\\ \cos\epsilon &= -r\frac{d^2r}{ds\,ds'}-\frac{dr}{ds}\frac{dr}{ds'},\\ \sin\theta\sin\theta'\cos\eta &= -r\frac{d^2r}{ds\,ds'}.\end{aligned}\right\} \qquad (8)$$

513.] We shall next consider in what way it is mathematically conceivable that the elements $PQ$ and $P'Q'$ might act on each other, and in doing so we shall not at first assume that their mutual action is necessarily in the line joining them.

We have seen that we may suppose each element resolved into other elements, provided that these components, when combined according to the rule of addition of vectors, produce the original element as their resultant.

We shall therefore consider $ds$ as resolved into $\cos\theta\,ds = \alpha$ in the direction of $r$, and $\sin\theta\,ds = \beta$ in a direction perpendicular to $r$ in the plane $P'PQ$.


Fig. 30.

We shall also consider $ds'$ as resolved into $\cos\theta'\,ds' = \alpha'$ in the direction of $r$ reversed, $\sin\theta'\cos\eta\,ds' = \beta'$ in a direction parallel to that in which $\beta$ was measured, and $\sin\theta' \sin\eta\, ds' = \gamma'$ in a direction perpendicular to $\alpha'$ and $\beta'$.

Let us consider the action between the components $\alpha$ and $\beta$ on the one hand, and $\alpha'$, $\beta'$, $\gamma'$ on the other.

(1) $\alpha$ and $\alpha'$ are in the same straight line. The force between them must therefore be in this line. We shall suppose it to be an attraction $$= A\alpha\alpha' i i',$$
where $A$ is a function of $r$, and $i$, $i'$ are the intensities of the currents in $ds$ and $ds'$ respectively. This expression satisfies the condition of changing sign with $i$ and with $i'$.

(2) $\beta$ and $\beta'$ are parallel to each other and perpendicular to the line joining them. The action between them may be written
$$B\beta\beta' i i'.$$

This force is evidently in the line joining $\beta$ and $\beta'$, for it must be in the plane in which they both lie, and if we were to measure $\beta$ and $\beta'$ in the reversed direction, the value of this expression would remain the same, which shews that, if it represents a force, that force has no component in the direction of $\beta$, and must therefore be directed along $r$. Let us assume that this expression, when positive, represents an attraction.

(3) $\beta$ and $\gamma'$ are perpendicular to each other and to the line joining them. The only action possible between elements so related is a couple whose axis is parallel to $r$. We are at present engaged with forces, so we shall leave this out of account *.

(4) The action of $\alpha$ and $\beta'$, if they act on each other, must be expressed by $$C\alpha\beta' i i'.$$

The sign of this expression is reversed if we reverse the direction in which we measure $\beta'$. It must therefore represent either a force in the direction of $\beta'$, or a couple in the plane of $\alpha$ and $\beta'$. As we are not investigating couples, we shall take it as a force acting on $\alpha$ in the direction of $\beta'$.

There is of course an equal force acting on $\beta'$ in the opposite direction.

* {It might be objected that we have no right to assume there is no force in this case, inasmuch as such a rule as that there was a force on $\beta$ at right angles to both $\beta$ and $\gamma'$, and in the direction to which $\gamma'$ would be brought by a right-handed screw through 90° round $\beta$, would indicate a force which would satisfy the condition of reversing if either of the components were reversed but not if both. The reason for assuming that such a force does not exist, is that the direction of the force would be determined merely by the direction of the currents, and not by their relative position. Thus for example, it would change from a repulsive to an attractive force between the elements, if in Fig. 30 $P'$ were to the left instead of the right of $P$.}

We have for the same reason a force

$$C\alpha\gamma' ii'$$

acting on $\alpha$ in the direction of $\gamma'$, and a force

$$C\beta\alpha' ii'$$

acting on $\beta$ in the direction opposite to that in which $\beta$ is measured.

514.] Collecting our results, we find that the action on $ds$ is compounded of the following forces,

$$\left.\begin{aligned} X &= (A\alpha\alpha' + B\beta\beta')ii' \text{ in the direction of } r,\\ Y &= C(\alpha\beta' - \alpha'\beta)ii' \text{ in the direction of } \beta,\\ \text{and}\quad Z &= C\alpha\gamma' ii' \text{ in the direction of } \gamma'. \end{aligned}\right\} \tag{9}$$

Let us suppose that this action on $ds$ is the resultant of three forces, $Rii'dsds'$ acting in the direction of $r$, $Sii'dsds'$ acting in the direction of $ds$, and $S'ii'dsds'$ acting in the direction of $ds'$; then in terms of $\theta$, $\theta'$, and $\eta$,

$$\left.\begin{aligned} R &= \quad A + 2C\cos\theta\cos\theta' + B\sin\theta\sin\theta'\cos\eta,\\ S &= -C\cos\theta', \qquad\qquad S' = C\cos\theta. \end{aligned}\right\} \tag{10}$$

In terms of the differential coefficients of $r$

$$\left.\begin{aligned} R &= A + 2C\frac{dr}{ds}\frac{dr}{ds'} - Br\frac{d^2r}{ds\,ds'},\\ S &= C\frac{dr}{ds'}, \qquad\qquad S' = -C\frac{dr}{ds}. \end{aligned}\right\} \tag{11}$$

In terms of $l$, $m$, $n$, and $l'$, $m'$, $n'$,

$$\left.\begin{aligned} R &= -(A+2C+B)\frac{1}{r^2}(l\xi+m\eta+n\zeta)(l'\xi+m'\eta+n'\zeta) + B(ll'+mm'+nn'),\\ S &= C\frac{1}{r}(l'\xi+m'\eta+n'\zeta), \qquad S' = C\frac{1}{r}(l\xi+m\eta+n\zeta), \end{aligned}\right\} \tag{12}$$

where $\xi$, $\eta$, $\zeta$ are written for $x'-x$, $y'-y$, and $z'-z$ respectively.

515.] We have next to calculate the force with which the finite current $s'$ acts on the finite current $s$. The current $s$ extends from $A$, where $s = 0$, to $P$, where it has the value $s$. The current $s'$ extends from $A'$, where $s' = 0$, to $P'$, where it has the value $s'$. The coordinates of points on either current are functions of $s$ or of $s'$.

If $F$ is any function of the position of a point, then we shall use the subscript $_{(s,\,0)}$ to denote the excess of its value at $P$ over that at $A$, thus

$$F_{(s,\,0)} = F_P - F_A.$$

Such functions necessarily disappear when the circuit is closed.

Let the components of the total force with which $A'P'$ acts on $AP$ be $ii'X$, $ii'Y$, and $ii'Z$. Then the component parallel to $X$ of the force with which $ds'$ acts on $ds$ will be $ii'\frac{d^2X}{ds\,ds'}ds\,ds'$.

Hence
$$\frac{d^2X}{ds\,ds'} = R\frac{\xi}{r} + Sl + S'l'. \tag{13}$$

Substituting the values of $R$, $S$, and $S'$ from (12), remembering that
$$l'\xi + m'\eta + n'\zeta = r\frac{dr}{ds'}, \tag{14}$$

and arranging the terms with respect to $l$, $m$, $n$, we find
$$\begin{aligned}\frac{d^2X}{ds\,ds'} = l&\left\{-(A+2C+B)\frac{1}{r^2}\frac{dr}{ds'}\xi^2 + C\frac{dr}{ds'} + (B+C)\frac{l'\xi}{r}\right\}\\ +m&\left\{-(A+2C+B)\frac{1}{r^2}\frac{dr}{ds'}\xi\eta + C\frac{l'\eta}{r} + B\frac{m'\xi}{r}\right\}\\ +n&\left\{-(A+2C+B)\frac{1}{r^2}\frac{dr}{ds'}\xi\zeta + C\frac{l'\zeta}{r} + B\frac{n'\xi}{r}\right\}.\end{aligned} \tag{15}$$

Since $A$, $B$, and $C$ are functions of $r$, we may write
$$P = \int_r^\infty (A+2C+B)\frac{1}{r^2}dr, \qquad Q = \int_r^\infty C\,dr, \tag{16}$$

the integration being taken between $r$ and $\infty$ because $A$, $B$, $C$ vanish when $r = \infty$.

Hence
$$(A+B)\frac{1}{r^2} = -\frac{dP}{dr}, \quad \text{and} \quad C = -\frac{dQ}{dr}. \tag{17}$$

516.] Now we know, by Ampère's third case of equilibrium, that when $s'$ is a closed circuit, the force acting on $ds$ is perpendicular to the direction of $ds$, or, in other words, the component of the force in the direction of $ds$ itself is zero. Let us therefore assume the direction of the axis of $x$ so as to be parallel to $ds$ by making $l = 1$, $m = 0$, $n = 0$. Equation (15) then becomes
$$\frac{d^2X}{ds\,ds'} = \frac{dP}{ds'}\xi^2 - \frac{dQ}{ds'} + (B+C)\frac{l'\xi}{r}. \tag{18}$$

To find $\frac{dX}{ds}$, the force on $ds$ referred to unit of length, we must integrate this expression with respect to $s'$. Integrating the first term by parts, we find
$$\frac{dX}{ds} = (P\xi^2 - Q)_{(s',0)} - \int_0^{s'}(2Pr - B - C)\frac{l'\xi}{r}ds'. \tag{19}$$

When $s'$ is a closed circuit this expression must be zero. The first term will disappear of itself. The second term, however, will not in general disappear in the case of a closed circuit unless the quantity under the sign of integration is always zero. Hence, to satisfy Ampère's condition, we must put

$$P = \frac{1}{2r}(B+C). \tag{20}$$

517.] We can now eliminate $P$, and find the general value of $\frac{dX}{ds}$,

$$\frac{dX}{ds} = \left\{\frac{B+C}{2}\frac{\xi}{r}(l\xi+m\eta+n\zeta)+Q\right\}_{(s',\,0)}$$
$$+m\int_0^{s'}\frac{B-C}{2}\frac{m'\xi-l'\eta}{r}ds' - n\int_0^{s'}\frac{B-C}{2}\frac{l'\zeta-n'\xi}{r}ds'. \tag{21}$$

When $s'$ is a closed circuit the first term of this expression vanishes, and if we make

$$\left.\begin{aligned} \alpha' &= \int_0^{s'}\frac{B-C}{2}\frac{n'\eta-m'\zeta}{r}ds', \\ \beta' &= \int_0^{s'}\frac{B-C}{2}\frac{l'\zeta-n'\xi}{r}ds', \\ \gamma' &= \int_0^{s'}\frac{B-C}{2}\frac{m'\xi-l'\eta}{r}ds', \end{aligned}\right\} \tag{22}$$

where the integration is extended round the closed circuit $s'$, we may write

$$\left.\begin{aligned} \frac{dX}{ds} &= m\gamma'-n\beta'. \\ \text{Similarly}\qquad \frac{dY}{ds} &= n\alpha'-l\gamma', \\ \frac{dZ}{ds} &= l\beta'-m\alpha'. \end{aligned}\right\} \tag{23}$$

The quantities $\alpha'$, $\beta'$, $\gamma'$ are sometimes called the determinants of the circuit $s'$ referred to the point $P$. Their resultant is called by Ampère the directrix of the electrodynamic action.

It is evident from the equation, that the force whose components are $\frac{dX}{ds}ds$, $\frac{dY}{ds}ds$, and $\frac{dZ}{ds}ds$ is perpendicular both to $ds$ and to this directrix, and is represented numerically by the area of the parallelogram whose sides are $ds$ and the directrix.

In the language of quaternions, the resultant force on $ds$ is the vector part of the product of the directrix multiplied by $ds$.

Since we already know that the directrix is the same thing as the magnetic force due to a unit current in the circuit $s'$, we shall henceforth speak of the directrix as the magnetic force due to the circuit.

518.] We shall now complete the calculation of the components of the force acting between two finite currents, whether closed or open.

Let $\rho$ be a new function of $r$, such that

$$\rho = \tfrac{1}{2}\int_r^\infty (B-C)\,dr, \tag{24}$$

then by (17) and (20)

$$A+B = r\frac{d^2}{dr^2}(Q+\rho) - \frac{d}{dr}(Q+\rho), \tag{25}$$

and equations (11) become

$$\left.\begin{aligned} R &= -\frac{d\rho}{dr}\cos\epsilon + r\frac{d^2}{ds\,ds'}(Q+\rho), \\ S &= -\frac{dQ}{ds'}, \qquad S' = \frac{dQ}{ds}. \end{aligned}\right\} \tag{26}$$

With these values of the component forces, equation (13) becomes

$$\begin{aligned} \frac{d^2X}{ds\,ds'} &= -\cos\epsilon\frac{d\rho}{dr}\frac{\xi}{r} + \xi\frac{d^2}{ds\,ds'}(Q+\rho) - l\frac{dQ}{ds'} + l'\frac{dQ}{ds}, \\ &= \cos\epsilon\frac{d\rho}{dx} + \frac{d^2\{(Q+\rho)\xi\}}{ds\,ds'} + l\frac{d\rho}{ds'} - l'\frac{d\rho}{ds}. \end{aligned} \tag{27}$$

519.] Let

$$F = \int_0^s l\rho\,ds, \qquad G = \int_0^s m\rho\,ds, \qquad H = \int_0^s n\rho\,ds, \tag{28}$$

$$F' = \int_0^{s'} l'\rho\,ds', \qquad G' = \int_0^{s'} m'\rho\,ds', \qquad H' = \int_0^{s'} n'\rho\,ds'. \tag{29}$$

These quantities have definite values for any given point of space. When the circuits are closed, they correspond to the components of the vector-potentials of the circuits.

Let $L$ be a new function of $r$, such that

$$L = \int_0^r r(Q+\rho)\,dr, \tag{30}$$

and let $M$ be the double integral

$$\int_0^{s'}\int_0^s \rho\cos\epsilon\,ds\,ds', \tag{31}$$

which, when the circuits are closed, becomes their mutual potential, then (27) may be written

$$\frac{d^2 X}{ds\,ds'} = \frac{d^2}{ds\,ds'}\left\{\frac{dM}{dx} - \frac{dL}{dx} + F - F'\right\}. \tag{32}$$

520.] Integrating, with respect to $s$ and $s'$, between the given limits, we find

$$X = \frac{dM}{dx} - \frac{d}{dx}(L_{PP'} - L_{AP'} - L_{A'P} + I_{AA'}),$$
$$+ F_{P'} - F_{A'} - F'_{P} + F'_{A}, \tag{33}$$

where the subscripts of $L$ indicate the distance, $r$, of which the quantity $L$ is a function, and the subscripts of $F$ and $F'$ indicate the points at which their values are to be taken.

The expressions for $Y$ and $Z$ may be written down from this. Multiplying the three components by $dx$, $dy$, and $dz$ respectively, we obtain

$$\begin{aligned} X\,dx + Y\,dy + Z\,dz = DM - D(L_{PP'} - L_{AP'} - L_{A'P} + L_{AA'}) \\ - (F'dx + G'dy + H'dz)_{(P-A)} \\ + (F\,dx + G\,dy + H\,dz)_{(P'-A')}, \end{aligned} \tag{34}$$

where $D$ is the symbol of a complete differential.

Since $F\,dx + G\,dy + H\,dz$ is not in general a complete differential of a function of $x$, $y$, $z$, $X\,dx + Y\,dy + Z\,dz$ is not in general a complete differential for currents either of which is not closed.

521.] If, however, both currents are closed, the terms in $L$, $F$, $G$, $H$, $F'$, $G'$, $H'$ disappear, and

$$X\,dx + Y\,dy + Z\,dz = DM, \tag{35}$$

where $M$ is the mutual potential of two closed circuits carrying unit currents. The quantity $M$ expresses the work done by the electromagnetic forces on either conducting circuit when it is moved parallel to itself from an infinite distance to its actual position. Any alteration of its position, by which $M$ is *increased*, will be *assisted* by the electromagnetic forces.

It may be shewn, as in Arts. 490, 596, that when the motion of the circuit is not parallel to itself the forces acting on it are still determined by the variation of $M$, the potential of the one circuit on the other.

522.] The only experimental fact which we have made use of in this investigation is the fact established by Ampère that the action of a closed circuit on any portion of another circuit is perpendicular to the direction of the latter. Every other part of the investigation depends on purely mathematical considerations depending on the properties of lines in space. The reasoning therefore may be presented in a much more condensed and appropriate form by the use of the ideas and language of the mathematical method specially adapted to the expression of such geometrical relations—the *Quaternions* of Hamilton.

This has been done by Professor Tait in the *Quarterly Journal of Mathematics*, 1866, and in his treatise on *Quaternions*, § 399, for Ampère's original investigation, and the student can easily adapt the same method to the somewhat more general investigation given here.

523.] Hitherto we have made no assumption with respect to the quantities $A$, $B$, $C$, except that they are functions of $r$, the distance between the elements. We have next to ascertain the form of these functions, and for this purpose we make use of Ampère's fourth case of equilibrium, Art. 508, in which it is shewn that if all the linear dimensions and distances of a system of two circuits be altered in the same proportion, the currents remaining the same, the force between the two circuits will remain the same.

Now the force between the circuits for unit currents is $\dfrac{dM}{dx}$, and since this is independent of the dimensions of the system, it must be a numerical quantity. Hence $M$ itself, the coefficient of the mutual potential of the circuits, must be a quantity of the dimensions of a line. It follows, from equation (31), that $\rho$ must be the reciprocal of a line, and therefore by (24), $B-C$ must be the inverse square of a line. But since $B$ and $C$ are both functions of $r$, $B-C$ must be the inverse square of $r$ or some numerical multiple of it.

524.] The multiple we adopt depends on our system of measurement. If we adopt the electromagnetic system, so called because it agrees with the system already established for magnetic measurements, the value of $M$ ought to coincide with that of the potential of two magnetic shells of strength unity whose boundaries are the two circuits respectively. The value of $M$ in that case is, by Art. 423,

$$M = \iint \frac{\cos\epsilon}{r}\, ds\, ds', \tag{36}$$

the integration being performed round both circuits in the positive direction. Adopting this as the numerical value of $M$, and comparing with (31), we find

$$\rho = \frac{1}{r}, \quad \text{and} \quad B - C = \frac{2}{r^2}. \tag{37}$$

525.] We may now express the components of the force on $ds$ arising from the action of $ds'$ in the most general form consistent with experimental facts.

The force on $ds$ is compounded of an attraction

$$\left.\begin{aligned} &Rii'ds\,ds' = \frac{1}{r^2}\Big(\frac{dr}{ds}\frac{dr}{ds'} - 2r\frac{d^2r}{ds\,ds'}\Big)ii'ds\,ds' + r\frac{d^2Q}{ds\,ds'}ii'ds\,ds' \\ &\text{in the direction of } r, \\ &Sii'ds\,ds' = -\frac{dQ}{ds'}ii'ds\,ds' \text{ in the direction of } ds, \\ &\text{and } S'ii'ds\,ds' = \frac{dQ}{ds}ii'ds\,ds' \text{ in the direction of } ds', \end{aligned}\right\} \tag{38}$$

where $Q = \int_r^\infty C\,dr$, and since $C$ is an unknown function of $r$, we know only that $Q$ is some function of $r$.

526.] The quantity $Q$ cannot be determined, without assumptions of some kind, from experiments in which the active current forms a closed circuit. If we suppose with Ampère that the action between the elements $ds$ and $ds'$ is in the line joining them, then $S$ and $S'$ must disappear, and $Q$ must be constant, or zero. The force is then reduced to an attraction whose value is

$$Rii'ds\,ds' = \frac{1}{r^2}\Big(\frac{dr}{ds}\frac{dr}{ds'} - 2r\frac{d^2r}{ds\,ds'}\Big)ii'ds\,ds'. \tag{39}$$

Ampère, who made this investigation long before the magnetic system of units had been established, uses a formula having a numerical value half of this, namely

$$'jj'd\,ds' = \frac{1}{r^2}\Big(\frac{1}{2}\frac{dr}{ds}\frac{dr}{ds'} - r\frac{d^2r}{ds\,ds'}\Big)jj'ds\,ds'. \tag{40}$$

Here the strength of a current is measured in what is called electrodynamic measure. If $i$, $i'$ are the strengths of the currents in electromagnetic measure, and $j$, $j'$ the same in electrodynamic measure, then it is plain that

$$jj' = 2ii', \quad \text{or} \quad j = \sqrt{2}\,i. \tag{41}$$

Hence the unit current adopted in electromagnetic measure is greater than that adopted in electrodynamic measure in the ratio of $\sqrt{2}$ to 1.

The only title of the electrodynamic unit to consideration is that it was originally adopted by Ampère, the discoverer of the law of action between currents. The continual recurrence of $\sqrt{2}$ in calculations founded on it is inconvenient, and the electromagnetic system has the great advantage of coinciding numerically with all our magnetic formulae. As it is difficult for the student to bear in mind whether he is to multiply or to divide by $\sqrt{2}$, we shall henceforth use only the electromagnetic system, as adopted by Weber and most other writers.

Since the form and value of $Q$ have no effect on any of the experiments hitherto made, in which the active current at least is always a closed one, we may, if we please, adopt any value of $Q$ which appears to us to simplify the formulae.

Thus Ampère assumes that the force between two elements is in the line joining them. This gives $Q = 0$,

$$Rii'ds\,ds' = \frac{1}{r^2}\left(\frac{dr}{ds}\frac{dr}{ds'} - 2r\frac{d^2r}{ds\,ds'}\right)ii'ds\,ds', \quad S = 0, \quad S' = 0. \quad (42)$$

Grassmann* assumes that two elements in the same straight line have no mutual action. This gives

$$Q = -\frac{1}{2r}, \quad R = -\frac{3}{2r}\frac{d^2r}{ds\,ds'}, \quad S = -\frac{1}{2r^2}\frac{dr}{ds'}, \quad S' = \frac{1}{2r^2}\frac{dr}{ds}. \quad (43)$$

We might, if we pleased, assume that the attraction between two elements at a given distance is proportional to the cosine of the angle between them. In this case

$$Q = -\frac{1}{r}, \quad R = \frac{1}{r^2}\cos\epsilon, \quad S = -\frac{1}{r^2}\frac{dr}{ds'}, \quad S' = \frac{1}{r^2}\frac{dr}{ds}. \quad 44)$$

Finally, we might assume that the attraction and the oblique forces depend only on the angles which the elements make with the line joining them, and then we should have

$$Q = -\frac{2}{r}, \quad R = -3\frac{1}{r^2}\frac{dr}{ds}\frac{dr}{ds'}, \quad S = -\frac{2}{r^2}\frac{dr}{ds'}, \quad S' = \frac{2}{r^2}\frac{dr}{ds}. \quad (45)$$

527.] Of these four different assumptions that of Ampère is undoubtedly the best, since it is the only one which makes the forces on the two elements not only equal and opposite but in the straight line which joins them.

* Pogg., *Ann.* 64, p. 1 (1845).

## CHAPTER III.

### ON THE INDUCTION OF ELECTRIC CURRENTS.

528.] The discovery by Örsted of the magnetic action of an electric current led by a direct process of reasoning to that of magnetization by electric currents, and of the mechanical action between electric currents. It was not, however, till 1831 that Faraday, who had been for some time endeavouring to produce electric currents by magnetic or electric action, discovered the conditions of magneto-electric induction. The method which Faraday employed in his researches consisted in a constant appeal to experiment as a means of testing the truth of his ideas, and a constant cultivation of ideas under the direct influence of experiment. In his published researches we find these ideas expressed in language which is all the better fitted for a nascent science, because it is somewhat alien from the style of physicists who have been accustomed to establish mathematical forms of thought.

The experimental investigation by which Ampère established the laws of the mechanical action between electric currents is one of the most brilliant achievements in science.

The whole, theory and experiment, seems as if it had leaped, full grown and full armed, from the brain of the 'Newton of electricity.' It is perfect in form, and unassailable in accuracy, and it is summed up in a formula from which all the phenomena may be deduced, and which must always remain the cardinal formula of electro-dynamics.

The method of Ampère, however, though cast into an inductive form, does not allow us to trace the formation of the ideas which guided it. We can scarcely believe that Ampère really discovered the law of action by means of the experiments which he describes. We are led to suspect, what, indeed, he tells us himself*, that he discovered the law by some process which he has not shewn us, and that when he had afterwards built up a perfect demonstration he removed all traces of the scaffolding by which he had raised it.

Faraday, on the other hand, shews us his unsuccessful as well as his successful experiments, and his crude ideas as well as his developed ones, and the reader, however inferior to him in inductive power, feels sympathy even more than admiration, and is tempted to believe that, if he had the opportunity, he too would be a discoverer. Every student should therefore read Ampère's research as a splendid example of scientific style in the statement of a discovery, but he should also study Faraday for the cultivation of a scientific spirit, by means of the action and reaction which will take place between the newly discovered facts as introduced to him by Faraday and the nascent ideas in his own mind.

It was perhaps for the advantage of science that Faraday, though thoroughly conscious of the fundamental forms of space, time, and force, was not a professed mathematician. He was not tempted to enter into the many interesting researches in pure mathematics which his discoveries would have suggested if they had been exhibited in a mathematical form, and he did not feel called upon either to force his results into a shape acceptable to the mathematical taste of the time, or to express them in a form which mathematicians might attack. He was thus left at leisure to do his proper work, to coordinate his ideas with his facts, and to express them in natural, untechnical language.

It is mainly with the hope of making these ideas the basis of a mathematical method that I have undertaken this treatise.

529.] We are accustomed to consider the universe as made up of parts, and mathematicians usually begin by considering a single particle, and then conceiving its relation to another particle, and so on. This has generally been supposed the most natural method. To conceive of a particle, however, requires a process of abstraction, since all our perceptions are related to extended bodies, so that the idea of the *all* that is in our consciousness at a given instant is perhaps as primitive an idea as

* *Théorie des phénomènes Électrodynamiques*, p. 9.

that of any individual thing. Hence there may be a mathematical method in which we proceed from the whole to the parts instead of from the parts to the whole. For example, Euclid, in his first book, conceives a line as traced out by a point, a surface as swept out by a line, and a solid as generated by a surface. But he also defines a surface as the boundary of a solid, a line as the edge of a surface, and a point as the extremity of a line.

In like manner we may conceive the potential of a material system as a function found by a certain process of integration with respect to the masses of the bodies in the field, or we may suppose these masses themselves to have no other mathematical meaning than the volume-integrals of $\frac{1}{4\pi}\nabla^2\Psi$, where $\Psi$ is the potential.

In electrical investigations we may use formulae in which the quantities involved are the distances of certain bodies, and the electrifications or currents in these bodies, or we may use formulae which involve other quantities, each of which is continuous through all space.

The mathematical process employed in the first method is integration along lines, over surfaces, and throughout finite spaces, those employed in the second method are partial differential equations and integrations throughout all space.

The method of Faraday seems to be intimately related to the second of these modes of treatment. He never considers bodies as existing with nothing between them but their distance, and acting on one another according to some function of that distance. He conceives all space as a field of force, the lines of force being in general curved, and those due to any body extending from it on all sides, their directions being modified by the presence of other bodies. He even speaks * of the lines of force belonging to a body as in some sense part of itself, so that in its action on distant bodies it cannot be said to act where it is not. This, however, is not a dominant idea with Faraday. I think he would rather have said that the field of space is full of lines of force, whose arrangement depends on that of the bodies in the field, and that the mechanical and electrical action on each body is determined by the lines which abut on it.

* *Exp. Res.*, vol. ii. p. 293; vol. iii. p. 447.

## PHENOMENA OF MAGNETO-ELECTRIC INDUCTION*.

530.] 1. *Induction by Variation of the Primary Current.*

Let there be two conducting circuits, the Primary and the Secondary circuit. The primary circuit is connected with a voltaic battery by which the primary current may be produced, maintained, stopped, or reversed. The secondary circuit includes a galvanometer to indicate any currents which may be formed in it. This galvanometer is placed at such a distance from all parts of the primary circuit that the primary current has no sensible direct influence on its indications.

Let part of the primary circuit consist of a straight wire, and part of the secondary circuit of a straight wire near and parallel to the first, the other parts of the circuits being at a greater distance from each other.

It is found that at the instant of sending a current through the straight wire of the primary circuit the galvanometer of the secondary circuit indicates a current in the secondary straight wire in the *opposite* direction. This is called the induced current. If the primary current is maintained constant, the induced current soon disappears, and the primary current appears to produce no effect on the secondary circuit. If now the primary current is stopped, a secondary current is observed, which is in the *same* direction as the primary current. Every variation of the primary current produces electromotive force in the secondary circuit. When the primary current increases, the electromotive force is in the opposite direction to the current. When it diminishes, the electromotive force is in the same direction as the current. When the primary current is constant, there is no electromotive force.

These effects of induction are increased by bringing the two wires nearer together. They are also increased by forming them into two circular or spiral coils placed close together, and still more by placing an iron rod or a bundle of iron wires inside the coils.

2. *Induction by Motion of the Primary Circuit.*

We have seen that when the primary current is maintained constant and at rest the secondary current rapidly disappears.

* Read Faraday's *Experimental Researches*, Series i and ii.

Now let the primary current be maintained constant, but let the primary straight wire be made to approach the secondary straight wire. During the approach there will be a secondary current in the *opposite* direction to the primary.

If the primary circuit be moved away from the secondary, there will be a secondary current in the *same* direction as the primary.

3. *Induction by Motion of the Secondary Circuit.*

If the secondary circuit be moved, the secondary current is opposite to the primary when the secondary wire is approaching the primary wire, and in the same direction when it is receding from it.

In all cases the direction of the secondary current is such that the mechanical action between the two conductors is opposite to the direction of motion, being a repulsion when the wires are approaching, and an attraction when they are receding. This very important fact was established by Lenz *.

4. *Induction by the Relative Motion of a Magnet and the Secondary Circuit.*

If we substitute for the primary circuit a magnetic shell, whose edge coincides with the circuit, whose strength is numerically equal to that of the current in the circuit, and whose austral face corresponds to the positive face of the circuit, then the phenomena produced by the relative motion of this shell and the secondary circuit are the same as those observed in the case of the primary circuit.

531.] The whole of these phenomena may be summed up in one law. When the number of lines of magnetic induction which pass through the secondary circuit in the positive direction is altered, an electromotive force acts round the circuit, which is measured by the rate of decrease of the magnetic induction through the circuit.

532.] For instance, let the rails of a railway be insulated from the earth, but connected at one terminus through a galvanometer, and let the circuit be completed by the wheels and axle of a railway carriage at a distance $x$ from the terminus. Neglecting the height of the axle above the level of the rails,

* Pogg., *Ann.* xxxi. p. 483 (1834).

the induction through the secondary circuit is due to the vertical component of the earth's magnetic force, which in northern latitudes is directed downwards. Hence, if $b$ is the gauge of the railway, the horizontal area of the circuit is $bx$, and the surface-integral of the magnetic induction through it is $Zbx$, where $Z$ is the vertical component of the magnetic force of the earth. Since $Z$ is downwards, the lower face of the circuit is to be reckoned positive, and the positive direction of the circuit itself is north, east, south, west, that is, in the direction of the sun's apparent diurnal course.

Now let the carriage be set in motion, then $x$ will vary, and there will be an electromotive force in the circuit whose value is $-Zb\frac{dx}{dt}$.

If $x$ is increasing, that is, if the carriage is moving away from the terminus, this electromotive force is in the negative direction, or north, west, south, east. Hence the direction of this force through the axle is from right to left. If $x$ were diminishing, the absolute direction of the force would be reversed, but since the direction of the motion of the carriage is also reversed, the electromotive force on the axle is still from right to left, the observer in the carriage being always supposed to move face forwards. In southern latitudes, where the south end of the needle dips, the electromotive force on a moving body is from left to right.

Hence we have the following rule for determining the electromotive force on a wire moving through a field of magnetic force. Place, in imagination, your head and feet in the positions occupied by the ends of a compass-needle which point north and south respectively; turn your face in the forward direction of motion, then the electromotive force due to the motion will be from left to right.

533.] As these directional relations are important, let us take another illustration. Suppose a metal girdle laid round the earth at the equator, and a metal wire laid along the meridian of Greenwich from the equator to the north pole.

Let a great quadrantal arch of metal be constructed, of which one extremity is pivoted on the north pole, while the other is carried round the equator, sliding on the great girdle of the earth, and following the sun in his daily course. There will then be an electromotive force along the moving quadrant, acting from the pole towards the equator.

The electromotive force will be the same whether we suppose the earth at rest and the quadrant moved from east to west, or whether we suppose the quadrant at rest and the earth turned from west to east. If we suppose the earth to rotate, the electromotive force will be the same whatever be the form of the part of the circuit fixed in space of which one end touches one of the poles and the other the equator. The current in this part of the circuit is from the pole to the equator.

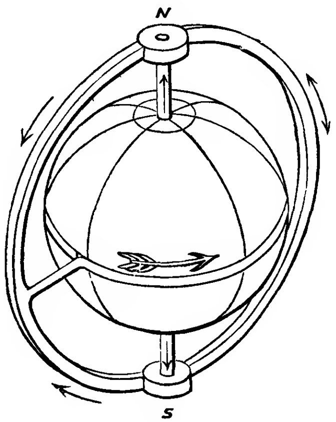

Fig. 31.

The other part of the circuit, which is fixed with respect to the earth, may also be of any form, and either within or without the earth. In this part the current is from the equator to either pole.

534.] The intensity of the electromotive force of magneto-electric induction is entirely independent of the nature of the substance of the conductor in which it acts, and also of the nature of the conductor which carries the inducing current.

To shew this, Faraday* made a conductor of two wires of different metals insulated from one another by a silk covering, but twisted together, and soldered together at one end. The other ends of the wires were connected with a galvanometer. In this way the wires were similarly situated with respect to the primary circuit, but if the electromotive force were stronger in the one wire than in the other it would produce a current which would be indicated by the galvanometer. He found, however, that such a combination may be exposed to the most powerful electromotive forces due to induction without the galvanometer being affected. He also found that whether the two branches of the compound conductor consisted of two metals, or of a metal and an electrolyte, the galvanometer was not affected†.

* *Exp. Res.*, 195. † Ib., 200.

Hence the electromotive force on any conductor depends only on the form and the motion of that conductor, together with the strength, form, and motion of the electric currents in the field.

535.] Another negative property of electromotive force is that it has of itself no tendency to cause the mechanical motion of any body, but only to cause a current of electricity within it.

If it actually produces a current in the body, there will be mechanical action due to that current, but if we prevent the current from being formed, there will be no mechanical action on the body itself. If the body is electrified, however, the electromotive force will move the body, as we have described in Electrostatics.

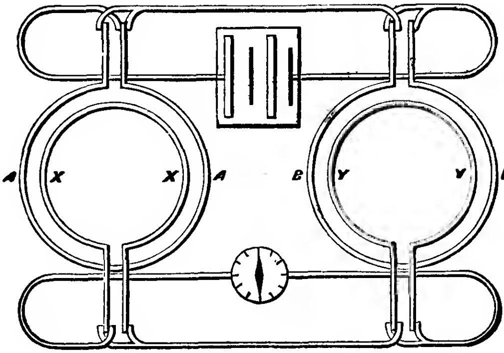


Fig. 32.

536.] The experimental investigation of the laws of the induction of electric currents in fixed circuits may be conducted with considerable accuracy by methods in which the electromotive force, and therefore the current, in the galvanometer circuit is rendered zero.

For instance, if we wish to shew that the induction of the coil $A$ on the coil $X$ is equal to that of $B$ upon $Y$, we place the first pair of coils $A$ and $X$ at a sufficient distance from the second pair $B$ and $Y$. We then connect $A$ and $B$ with a voltaic battery, so that we can make the same primary current flow through $A$ in the positive direction and then through $B$ in the negative direction. We also connect $X$ and $Y$ with a galvanometer, so that the secondary current, if it exists, shall flow in the same direction through $X$ and $Y$ in series.

Then, if the induction of $A$ on $X$ is equal to that of $B$ on $Y$, the galvanometer will indicate no induction current when the battery circuit is closed or broken.

The accuracy of this method increases with the strength of the primary current and the sensitiveness of the galvanometer to instantaneous currents, and the experiments are much more easily performed than those relating to electromagnetic attractions, where the conductor itself has to be delicately suspended.

A very instructive series of well-devised experiments of this kind is described by Professor Felici of Pisa *.

I shall only indicate briefly some of the laws which may be proved in this way.

(1) The electromotive force of the induction of one circuit on another is independent of the area of the section of the conductors and of the material of which they are made †.

For we can exchange any one of the circuits in the experiment for another of a different section and material, but of the same form, without altering the result.

(2) The induction of the circuit $A$ on the circuit $X$ is equal to that of $X$ upon $A$.

For if we put $A$ in the galvanometer circuit, and $X$ in the battery circuit, the equilibrium of electromotive force is not disturbed.

(3) The induction is proportional to the inducing current.

For if we have ascertained that the induction of $A$ on $X$ is equal to that of $B$ on $Y$, and also to that of $C$ on $Z$, we may make the battery current first flow through $A$, and then divide itself in any proportion between $B$ and $C$. Then if we connect $X$ reversed, $Y$ and $Z$ direct, all in series, with the galvanometer, the electromotive force in $X$ will balance the sum of the electromotive forces in $Y$ and $Z$.

(4) In pairs of circuits forming systems geometrically similar the induction is proportional to their linear dimensions.

For if the three pairs of circuits above mentioned are all similar, but if the linear dimension of the first pair is the sum of the corresponding linear dimensions of the second and third pairs, then, if $A$, $B$, and $C$ are connected in series with the

* *Annales de Chimie*, xxxiv. p. 64 (1852), and *Nuovo Cimento*, ix. p. 345 (1859).

† {This statement is not necessarily strictly true if one or more of the materials is magnetic, for in this case the distribution of the lines of magnetic force are disturbed by the magnetism induced in the wires.}

battery, and if $X$ reversed, $Y$ and $Z$ are in series with the galvanometer, there will be equilibrium.

(5) The electromotive force produced in a coil of $n$ windings by a current in a coil of $m$ windings is proportional to the product $m\,n$.

537.] For experiments of the kind we have been considering the galvanometer should be as sensitive as possible, and its needle as light as possible, so as to give a sensible indication of a very small transient current. The experiments on induction due to motion require the needle to have a somewhat longer period of vibration, so that there may be time to effect certain motions of the conductors while the needle is not far from its position of equilibrium. In the former experiments, the electromotive forces in the galvanometer circuit were in equilibrium during the whole time, so that no current passed through the galvanometer coil. In those now to be described, the electromotive forces act first in one direction and then in the other, so as to produce in succession two currents in opposite directions through the galvanometer, and we have to show that the impulses on the galvanometer needle due to these successive currents are in certain cases equal and opposite.

The theory of the application of the galvanometer to the measurement of transient currents will be considered more at length in Art. 748. At present it is sufficient for our purpose to observe that as long as the galvanometer needle is near its position of equilibrium the deflecting force of the current is proportional to the current itself, and if the whole time of action of the current is small compared with the period of vibration of the needle, the final velocity of the magnet will be proportional to the total quantity of electricity in the current. Hence, if two currents pass in rapid succession, conveying equal quantities of electricity in opposite directions, the needle will be left without any final velocity.

Thus, to shew that the induction currents in the secondary circuit, due to the closing and the breaking of the primary circuit, are equal in total quantity but opposite in direction, we may arrange the primary circuit in connexion with the battery, so that by touching a key the current may be sent through the primary circuit, or by removing the finger the contact may be broken at pleasure. If the key is pressed down for some time, the galvanometer in the secondary circuit indicates, at the time of making contact, a transient current in the opposite direction to the primary current. If contact be maintained, the induction current simply passes and disappears. If we now break contact, another transient current passes in the opposite direction through the secondary circuit, and the galvanometer needle receives an impulse in the opposite direction.

But if we make contact only for an instant, and then break contact, the two induced currents pass through the galvanometer in such rapid succession that the needle, when acted on by the first current, has not time to move a sensible distance from its position of equilibrium before it is stopped by the second, and, on account of the exact equality between the quantities of these transient currents, the needle is stopped dead.

If the needle is watched carefully, it appears to be jerked suddenly from one position of rest to another position of rest very near the first.

In this way we prove that the quantity of electricity in the induction current, when contact is broken, is exactly equal and opposite to that in the induction current when contact is made.

538.] Another application of this method is the following, which is given by Felici in the second series of his *Researches*.

It is always possible to find many different positions of the secondary coil $B$, such that the making or the breaking of contact in the primary coil $A$ produces no induction current in $B$. The positions of the two coils are in such cases said to be *conjugate* to each other.

Let $B_1$ and $B_2$ be two of these positions. If the coil $B$ be suddenly moved from the position $B_1$ to the position $B_2$, the algebraical sum of the transient currents in the coil $B$ is exactly zero, so that the galvanometer needle is left at rest when the motion of $B$ is completed.

This is true in whatever way the coil $B$ is moved from $B_1$ to $B_2$, and also whether the current in the primary coil $A$ be continued constant, or made to vary during the motion.

Again, let $B'$ be any other position of $B$ not conjugate to $A$, so that the making or breaking of contact in $A$ produces an induction current when $B$ is in the position $B'$.

Let the contact be made when $B$ is in the conjugate position $B_1$, there will be no induction current. Move $B$ to $B'$, there will be an induction current due to the motion, but if $B$ is moved rapidly to $B'$, and the primary contact then broken, the induction current due to breaking contact will exactly annul the effect of that due to the motion, so that the galvanometer needle will be left at rest. Hence the current due to the motion from a conjugate position to any other position is equal and opposite to the current due to breaking contact in the latter position.

Since the effect of making contact is equal and opposite to that of breaking it, it follows that the effect of making contact when the coil $B$ is in any position $B'$ is equal to that of bringing the coil from any conjugate position $B_1$ to $B'$ while the current is flowing through $A$.

If the change of the relative position of the coils is made by moving the primary circuit instead of the secondary, the result is found to be the same.

539.] It follows from these experiments that the total induction current in $B$ during the simultaneous motion of $A$ from $A_1$ to $A_2$, and of $B$ from $B_1$ to $B_2$, while the current in $A$ changes from $\gamma_1$ to $\gamma_2$, depends only on the initial state $A_1$, $B_1$, $\gamma_1$, and the final state $A_2$, $B_2$, $\gamma_2$, and not at all on the nature of the intermediate states through which the system may pass.

Hence the value of the total induction current must be of the form

$$F(A_2, B_2, \gamma_2) - F(A_1, B_1, \gamma_1),$$

where $F$ is a function of $A$, $B$, and $\gamma$.

With respect to the form of this function, we know, by Art. 536, that when there is no motion, and therefore $A_1 = A_2$ and $B_1 = B_2$, the induction current is proportional to the primary current. Hence $\gamma$ enters simply as a factor, the other factor being a function of the form and position of the circuits $A$ and $B$.

We also know that the value of this function depends on the relative and not on the absolute positions of $A$ and $B$, so that it must be capable of being expressed as a function of the distances of the different elements of which the circuits are composed, and of the angles which these elements make with each other.

Let $M$ be this function, then the total induction current may be written

$$C\{M_1\gamma_1 - M_2\gamma_2\},$$

where $C$ is the conductivity of the secondary circuit, and $M_1$, $\gamma_1$ are the original, and $M_2$, $\gamma_2$ the final values of $M$ and $\gamma$.

These experiments, therefore, shew that the total current of induction depends on the change which takes place in a certain quantity, $M\gamma$, and that this change may arise either from variation of the primary current $\gamma$, or from any motion of the primary or secondary circuit which alters $M$.

540.] The conception of such a quantity, on the changes of which, and not on its absolute magnitude, the induction current depends, occurred to Faraday at an early stage of his *Researches* *. He observed that the secondary circuit, when at rest in an electromagnetic field which remains of constant intensity, does not shew any electrical effect, whereas, if the same state of the field had been suddenly produced, there would have been a current. Again, if the primary circuit is removed from the field, or the magnetic forces abolished, there is a current of the opposite kind. He therefore recognised in the secondary circuit, when in the electromagnetic field, a 'peculiar electrical condition of matter,' to which he gave the name of the Electrotonic State. He afterwards found that he could dispense with this idea by means of considerations founded on the lines of magnetic force †, but even in his latest *Researches* ‡, he says, 'Again and again the idea of an *electrotonic* state § has been forced on my mind.'

The whole history of this idea in the mind of Faraday, as shewn in his published *Researches*, is well worthy of study. By a course of experiments, guided by intense application of thought, but without the aid of mathematical calculations, he was led to recognise the existence of something which we now know to be a mathematical quantity, and which may even be called the fundamental quantity in the theory of electromagnetism. But as he was led up to this conception by a purely experimental path, he ascribed to it a physical existence, and supposed it to be a peculiar condition of matter, though he was ready to abandon this theory as soon as he could explain the phenomena by any more familiar forms of thought.

Other investigators were long afterwards led up to the same idea by a purely mathematical path, but, so far as I know, none of them recognised, in the refined mathematical idea of the potential of two circuits, Faraday's bold hypothesis of an electrotonic state. Those, therefore, who have approached this subject

* *Exp. Res.*, series i. 60.
† Ib., series ii. 242.
‡ Ib., 3269.
§ Ib., 60, 1114, 1661, 1729, 1733.

in the way pointed out by those eminent investigators who first reduced its laws to a mathematical form, have sometimes found it difficult to appreciate the scientific accuracy of the statements of laws which Faraday, in the first two series of his *Researches*, has given with such wonderful completeness.

The scientific value of Faraday's conception of an electrotonic state consists in its directing the mind to lay hold of a certain quantity, on the changes of which the actual phenomena depend. Without a much greater degree of development than Faraday gave it, this conception does not easily lend itself to the explanation of the phenomena. We shall return to this subject again in Art. 584.

541.] A method which, in Faraday's hands, was far more powerful is that in which he makes use of those lines of magnetic force which were always in his mind's eye when contemplating his magnets or electric currents, and the delineation of which by means of iron filings he rightly regarded * as a most valuable aid to the experimentalist.

Faraday looked on these lines as expressing, not only by their direction that of the magnetic force, but by their number and concentration the intensity of that force, and in his later *Researches* † he shews how to conceive of unit lines of force. I have explained in various parts of this treatise the relation between the properties which Faraday recognised in the lines of force and the mathematical conditions of electric and magnetic forces, and how Faraday's notion of unit lines and of the number of lines within certain limits may be made mathematically precise. See Arts. 82, 404, 490.

In the first series of his *Researches* ‡ he shews clearly how the direction of the current in a conducting circuit, part of which is moveable, depends on the mode in which the moving part cuts through the lines of magnetic force.

In the second series § he shews how the phenomena produced by variation of the strength of a current or a magnet may be explained, by supposing the system of lines of force to expand from or contract towards the wire or magnet as its power rises or falls.

I am not certain with what degree of clearness he then held the doctrine afterwards so distinctly laid down by him ‖, that

* *Exp. Res.*, 3234. † Ib., 3122. ‡ Ib., 114.
§ Ib., 238. ‖ Ib., 3082, 3087, 3113.

the moving conductor, as it cuts the lines of force, sums up the action due to an area or section of the lines of force. This, however, appears no new view of the case after the investigations of the second series * have been taken into account.

The conception which Faraday had of the continuity of the lines of force precludes the possibility of their suddenly starting into existence in a place where there were none before. If, therefore, the number of lines which pass through a conducting circuit is made to vary, it can only be by the circuit moving across the lines of force, or else by the lines of force moving across the circuit. In either case a current is generated in the circuit.

The number of the lines of force which at any instant pass through the circuit is mathematically equivalent to Faraday's earlier conception of the electrotonic state of that circuit, and it is represented by the quantity $M\gamma$.

It is only since the definitions of electromotive force, Arts. 69, 274, and its measurement have been made more precise, that we can enunciate completely the true law of magneto-electric induction in the following terms:—

The total electromotive force acting round a circuit at any instant is measured by the rate of decrease of the number of lines of magnetic force which pass through it.

When integrated with respect to the time this statement becomes:—

The time-integral of the total electromotive force acting round any circuit, together with the number of lines of magnetic force which pass through the circuit, is a constant quantity.

Instead of speaking of the number of lines of magnetic force, we may speak of the magnetic induction through the circuit, or the surface-integral of magnetic induction extended over any surface bounded by the circuit.

We shall return again to this method of Faraday. In the meantime we must enumerate the theories of induction which are founded on other considerations.

## *Lenz's Law.*

542.] In 1834, Lenz † enunciated the following remarkable relation between the phenomena of the mechanical action of

* *Exp. Res.*, 217, &c.

† Pogg., *Ann.* xxxi. p. 483 (1834).

electric currents, as defined by Ampère's formula, and the induction of electric currents by the relative motion of conductors. An earlier attempt at a statement of such a relation was given by Ritchie in the *Philosophical Magazine* for January of the same year, but the direction of the induced current was in every case stated wrongly. Lenz's law is as follows:—

*If a constant current flows in the primary circuit A, and if, by the motion of A, or of the secondary circuit B, a current is induced in B, the direction of this induced current will be such that, by its electromagnetic action on A, it tends to oppose the relative motion of the circuits.*

On this law F. E. Neumann * founded his mathematical theory of induction, in which he established the mathematical laws of the induced currents due to the motion of the primary or secondary conductor. He shewed that the quantity $M$, which we have called the potential of the one circuit on the other, is the same as the electromagnetic potential of the one circuit on the other, which we have already investigated in connection with Ampère's formula. We may regard F. E. Neumann, therefore, as having completed for the induction of currents the mathematical treatment which Ampère had applied to their mechanical action.

543.] A step of still greater scientific importance was soon after made by Helmholtz in his *Essay on the Conservation of Force* †, and by Sir W. Thomson ‡, working somewhat later, but independently of Helmholtz. They shewed that the induction of electric currents discovered by Faraday could be mathematically deduced from the electromagnetic actions discovered by Örsted and Ampère by the application of the principle of the Conservation of Energy.

Helmholtz takes the case of a conducting circuit of resistance $R$, in which an electromotive force $A$, arising from a voltaic or thermoelectric arrangement, acts. The current in the circuit at any instant is $I$. He supposes that a magnet is in motion in the neighbourhood of the circuit, and that its potential with respect to the conductor is $V$, so that, during any small interval of time

* *Berlin Akad.*, 1845 and 1847.

† Read before the Physical Society of Berlin, July 23, 1847. Translated in Taylor's 'Scientific Memoirs,' part ii. p. 114.

‡ *Trans. Brit. Ass.*, 1848, and *Phil. Mag.*, Dec. 1851. See also his paper on 'Transient Electric Currents,' *Phil. Mag.*, June 1853.

$dt$, the energy communicated to the magnet by the electromagnetic action is $I\frac{dV}{dt}dt$.

The work done in generating heat in the circuit is, by Joule's law, Art. 242, $I^2 R\,dt$, and the work spent by the electromotive force $A$, in maintaining the current $I$ during the time $dt$, is $A I dt$. Hence, since the total work done must be equal to the work spent,

$$A I dt = I^2 R dt + I\frac{dV}{dt}dt,$$

whence we find the intensity of the current

$$I = \frac{A - \frac{dV}{dt}}{R}.$$

Now the value of $A$ may be what we please. Let, therefore, $A = 0$, and then

$$I = -\frac{1}{R}\frac{dV}{dt},$$

or, there will be a current due to the motion of the magnet, equal to that due to an electromotive force $-\frac{dV}{dt}$.

The whole induced current during the motion of the magnet from a place where its potential is $V_1$ to a place where its potential is $V_2$, is

$$\int I\,dt = -\frac{1}{R}\int\frac{dV}{dt}dt = \frac{1}{R}(V_1 - V_2),$$

and therefore the total current is independent of the velocity or the path of the magnet, and depends only on its initial and final positions.

Helmholtz in his original investigation adopted a system of units founded on the measurement of the heat generated in the conductor by the current. Considering the unit of current as arbitrary, the unit of resistance is that of a conductor in which this unit current generates unit of heat in unit of time. The unit of electromotive force in this system is that required to produce the unit of current in the conductor of unit resistance. The adoption of this system of units necessitates the introduction into the equations of a quantity $a$, which is the mechanical equivalent of the unit of heat. As we invariably adopt either the electrostatic or the electromagnetic system of units, this factor does not occur in the equations here given.

544.] Helmholtz also deduces the current of induction when a conducting circuit and a circuit carrying a constant current are made to move relatively to one another*.

Let $R_1$, $R_2$ be the resistances, $I_1$, $I_2$ the currents, $A_1$, $A_2$ the external electromotive forces, and $V$ the potential of the one

* {The proofs given in Arts. 543 and 544 are not satisfactory, as they neglect any variations which may occur in the currents and also any change which may occur in the Kinetic Energy due to the motion of the circuits. It is in fact as impossible to deduce the equations of induction of two circuits from the principle of the Conservation of Energy alone as it would be to deduce the equations of motion of a system with two degrees of freedom without using any principle beyond that of the Conservation of Energy.

If we apply the principle of the Conservation of Energy to the case of two currents, we get one equation, which we may deduce as follows:—Let $L$, $M$, $N$ be the coefficient of self-induction of the first circuit, the coefficient of mutual induction of the two circuits and the self-induction of the second circuit respectively (Art. 578). Let $T_e$ be the Kinetic Energy due to the currents round the circuits, and let the rest of the notation be the same as in Art. 544. Then (Art. 578)

$$T_e = \tfrac{1}{2}LI_1^2 + MI_1I_2 + \tfrac{1}{2}NI_2^2,$$

$$\delta T_e = \frac{dT_e}{dI_1}\delta I_1 + \frac{dT_e}{dI_2}\delta I_2 + \Sigma\frac{dT_e}{dx}\delta x, \tag{1}$$

where $x$ is a coordinate of any type helping to fix the position of the circuit.

Since $T_e$ is a homogeneous quadratic function of $I_1$, $I_2$,

$$2T_e = I_1\frac{dT_e}{dI_1} + I_2\frac{dT_e}{dI_2},$$

hence

$$2\delta T_e = \delta I_1\frac{dT_e}{dI_1} + I_1\delta\frac{dT_e}{dI_1} + \delta I_2\cdot\frac{dT_e}{dI_2} + I_2\delta\frac{dT_e}{dI_2}. \tag{2}$$

Subtracting (1) from (2), we get

$$\delta T_e = I_1\delta\frac{dT_e}{dI_1} + I_2\delta\frac{dT_e}{dI_2} - \Sigma\frac{dT_e}{dx}\delta x. \tag{3}$$

But $\frac{dT_e}{dx}$ is the force of type $x$ acting on the system, hence, since we suppose no external force acts on the system, $\Sigma\frac{dT_e}{dx}\delta x$ will be the increase in Kinetic Energy $T_m$ due to the motion of the system, hence (3) gives,

$$\delta(T_e + T_m) = I_1\delta\frac{dT_e}{dI_1} + I_2\delta\frac{dT_e}{dI_2}. \tag{4}$$

The work done by the batteries in a time $\delta t$ is

$$A_1I_1\delta t + A_2I_2\delta t.$$

The heat produced in the same time is by Joule's Law,

$$(R_1I_1^2 + R_2I_2^2)\delta t.$$

By the Conservation of Energy the work done by the batteries must equal the heat produced in the circuit plus the increase in the energy of the system, hence

$$A_1I_1\delta t + A_2I_2\delta t = (R_1I_1^2 + R_2I_2^2)\delta t + \delta(T_e + T_m).$$

Substituting for $\delta(T_e + T_m)$ from (4) we get

$$I_1\left\{A_1 - R_1I_1 - \frac{d}{dt}\frac{dT_e}{dI_1}\right\} + I_2\left\{A_2 - R_2I_2 - \frac{d}{dt}\frac{dT_e}{dI_2}\right\} = 0,$$

or $$I_1\left\{A_1 - R_1I_1 - \frac{d}{dt}(LI_1 + MI_2)\right\} + I_2\left\{A_2 - R_2I_2 - \frac{d}{dt}(MI_1 + NI_2)\right\} = 0. \quad (5$$

The equations of induction are the two quantities inside the brackets equated to zero, the principle of the Conservation of Energy however only shows that the left-hand side of (5) is zero, not that each bracket is separately zero. A rigid proof of the equations of induced currents is given in Art. 581.}

circuit on the other due to unit current in each, then we have, as before,

$$A_1 I_1 + A_2 I_2 = I_1^2 R_1 + I_2^2 R_2 + I_1 I_2 \frac{dV}{dt}.$$

If we suppose $I_1$ to be the primary current, and $I_2$ so much less than $I_1$, that it does not by its induction produce any sensible alteration in $I_1$, so that we may put $I_1 = \frac{A_1}{R_1}$, then

$$I_2 = \frac{A_2 - I_1 \frac{dV}{dt}}{R_2},$$

a result which may be interpreted exactly as in the case of the magnet.

If we suppose $I_2$ to be the primary current, and $I_1$ to be very much smaller than $I_2$, we get for $I_1$,

$$I_1 = \frac{A_1 - I_2 \frac{dV}{dt}}{R_1}.$$

This shews that for equal currents the electromotive force of the first circuit on the second is equal to that of the second on the first, whatever be the forms of the circuits.

Helmholtz does not in this memoir discuss the case of induction due to the strengthening or weakening of the primary current, or the induction of a current on itself. Thomson * applied the same principle to the determination of the mechanical value of a current, and pointed out that when work is done by the mutual action of two constant currents, their mechanical action is *increased* by the same amount, so that the battery has to supply *double* that amount of work, in addition to that required to maintain the currents against the resistance of the circuits †.

545.] The introduction, by W. Weber, of a system of absolute units for the measurement of electrical quantities is one of the most important steps in the progress of the science. Having already, in conjunction with Gauss, placed the measurement of magnetic quantities in the first rank of methods of precision, Weber proceeded in his *Electrodynamic Measurements* not only to lay down sound principles for fixing the units to be employed,

* Mechanical Theory of Electrolysis, *Phil. Mag.*, Dec. 1851.

† Nichol's *Cyclopaedia of Physical Science*, ed. 1860, Article 'Magnetism, Dynamical Relations of,' and *Reprint*, § 571.

but to make determinations of particular electrical quantities in terms of these units, with a degree of accuracy previously unattempted. Both the electromagnetic and the electrostatic systems of units owe their development and practical application to these researches.

Weber has also formed a general theory of electric action from which he deduces both electrostatic and electromagnetic force, and also the induction of electric currents. We shall consider this theory, with some of its more recent developments, in a separate chapter. See Art. 846.

## CHAPTER IV.

### ON THE INDUCTION OF A CURRENT ON ITSELF.

546.] FARADAY has devoted the ninth series of his *Researches* to the investigation of a class of phenomena exhibited by the current in a wire which forms the coil of an electromagnet.

Mr. Jenkin has observed that, although it is impossible to produce a sensible shock by the direct action of a voltaic system consisting of only one pair of plates, yet, if the current is made to pass through the coil of an electromagnet, and if contact is then broken between the extremities of two wires held one in each hand, a smart shock will be felt. No such shock is felt on making contact.

Faraday shewed that this and other phenomena, which he describes, are due to the same inductive action which he had already observed the current to exert on neighbouring conductors. In this case, however, the inductive action is exerted on the same conductor which carries the current, and it is so much the more powerful as the wire itself is nearer to the different elements of the current than any other wire can be.

547.] He observes, however*, that 'the first thought that arises in the mind is that the electricity circulates with something like momentum or inertia in the wire.' Indeed, when we consider one particular wire only, the phenomena are exactly analogous to those of a pipe full of water flowing in a continued stream. If while the stream is flowing we suddenly close the end of the pipe, the momentum of the water produces a sudden pressure, which is much greater than that due to the head of water, and may be sufficient to burst the pipe.

If the water has the means of escaping through a narrow jet when the principal aperture is closed, it will be projected with a

* *Exp. Res.*, 1077.

velocity much greater than that due to the head of water, and if it can escape through a valve into a chamber, it will do so, even when the pressure in the chamber is greater than that due to the head of water.

It is on this principle that the hydraulic ram is constructed, by which a small quantity of water may be raised to a great height by means of a large quantity flowing down from a much lower level.

548.] These effects of the inertia of the fluid in the tube depend solely on the quantity of fluid running through the tube, on its length, and on its section in different parts of its length. They do not depend on anything outside the tube, nor on the form into which the tube may be bent, provided its length remains the same.

With a wire conveying a current this is not the case, for if a long wire is doubled on itself the effect is very small, if the two parts are separated from each other it is greater, if it is coiled up into a helix it is still greater, and greatest of all if, when so coiled, a piece of soft iron is placed inside the coil.

Again, if a second wire is coiled up with the first, but insulated from it, then, if the second wire does not form a closed circuit, the phenomena are as before, but if the second wire forms a closed circuit, an induction current is formed in the second wire, and the effects of self-induction in the first wire are retarded.

549.] These results shew clearly that, if the phenomena are due to momentum, the momentum is certainly not that of the electricity in the wire, because the same wire, conveying the same current, exhibits effects which differ according to its form; and even when its form remains the same, the presence of other bodies, such as a piece of iron or a closed metallic circuit, affects the result.

550.] It is difficult, however, for the mind which has once recognised the analogy between the phenomena of self-induction and those of the motion of material bodies, to abandon altogether the help of this analogy, or to admit that it is entirely superficial and misleading. The fundamental dynamical idea of matter, as capable by its motion of becoming the recipient of momentum and of energy, is so interwoven with our forms of thought that, whenever we catch a glimpse of it in any part of nature, we feel that a path is before us leading, sooner or later, to the complete understanding of the subject.

551.] In the case of the electric current, we find that, when the electromotive force begins to act, it does not at once produce the full current, but that the current rises gradually. What is the electromotive force doing during the time that the opposing resistance is not able to balance it? It is increasing the electric current.

Now an ordinary force, acting on a body in the direction of its motion, increases its momentum, and communicates to it kinetic energy, or the power of doing work on account of its motion.

In like manner the unresisted part of the electromotive force has been employed in increasing the electric current. Has the electric current, when thus produced, either momentum or kinetic energy?

We have already shewn that it has something very like momentum, that it resists being suddenly stopped, and that it can exert, for a short time, a great electromotive force.

But a conducting circuit in which a current has been set up has the power of doing work in virtue of this current, and this power cannot be said to be something very like energy, for it is really and truly energy.

Thus, if the current be left to itself, it will continue to circulate till it is stopped by the resistance of the circuit. Before it is stopped, however, it will have generated a certain quantity of heat, and the amount of this heat in dynamical measure is equal to the energy originally existing in the current.

Again, when the current is left to itself, it may be made to do mechanical work by moving magnets, and the inductive effect of these motions will, by Lenz's law, stop the current sooner than the resistance of the circuit alone would have stopped it. In this way part of the energy of the current may be transformed into mechanical work instead of heat.

552.] It appears, therefore, that a system containing an electric current is a seat of energy of some kind; and since we can form no conception of an electric current except as a kinetic phenomenon*, its energy must be kinetic energy, that is to say, the energy which a moving body has in virtue of its motion.

We have already shewn that the electricity in the wire cannot

* Faraday, *Exp. Res.* 283.

be considered as the moving body in which we are to find this energy, for the energy of a moving body does not depend on anything external to itself, whereas the presence of other bodies near the current alters its energy.

We are therefore led to enquire whether there may not be some motion going on in the space outside the wire, which is not occupied by the electric current, but in which the electromagnetic effects of the current are manifested.

I shall not at present enter on the reasons for looking in one place rather than another for such motions, or for regarding these motions as of one kind rather than another.

What I propose now to do is to examine the consequences of the assumption that the phenomena of the electric current are those of a moving system, the motion being communicated from one part of the system to another by forces, the nature and laws of which we do not yet even attempt to define, because we can eliminate these forces from the equations of motion by the method given by Lagrange for any connected system.

In the next five chapters of this treatise I propose to deduce the main structure of the theory of electricity from a dynamical hypothesis of this kind, instead of following the path which has led Weber and other investigators to many remarkable discoveries and experiments, and to conceptions, some of which are as beautiful as they are bold. I have chosen this method because I wish to shew that there are other ways of viewing the phenomena which appear to me more satisfactory, and at the same time are more consistent with the methods followed in the preceding parts of this book than those which proceed on the hypothesis of direct action at a distance.

## CHAPTER V.

### ON THE EQUATIONS OF MOTION OF A CONNECTED SYSTEM.

553.] IN the fourth section of the second part of his *Mécanique Analytique,* Lagrange has given a method of reducing the ordinary dynamical equations of the motion of the parts of a connected system to a number equal to that of the degrees of freedom of the system.

The equations of motion of a connected system have been given in a different form by Hamilton, and have led to a great extension of the higher part of pure dynamics*.

As we shall find it necessary, in our endeavours to bring electrical phenomena within the province of dynamics, to have our dynamical ideas in a state fit for direct application to physical questions, we shall devote this chapter to an exposition of these dynamical ideas from a physical point of view.

554.] The aim of Lagrange was to bring dynamics under the power of the calculus. He began by expressing the elementary dynamical relations in terms of the corresponding relations of pure algebraical quantities, and from the equations thus obtained he deduced his final equations by a purely algebraical process. Certain quantities (expressing the reactions between the parts of the system called into play by its physical connexions) appear in the equations of motion of the component parts of the system, and Lagrange's investigation, as seen from a mathematical point of view, is a method of eliminating these quantities from the final equations.

In following the steps of this elimination the mind is exercised in calculation, and should therefore be kept free from the

* See Professor Cayley's 'Report on Theoretical Dynamics,' *British Association,* 1857; and Thomson and Tait's *Natural Philosophy.*

intrusion of dynamical ideas. Our aim, on the other hand, is to cultivate our dynamical ideas. We therefore avail ourselves of the labours of the mathematicians, and retranslate their results from the language of the calculus into the language of dynamics, so that our words may call up the mental image, not of some algebraical process, but of some property of moving bodies.

The language of dynamics has been considerably extended by those who have expounded in popular terms the doctrine of the Conservation of Energy, and it will be seen that much of the following statement is suggested by the investigation in Thomson and Tait's *Natural Philosophy*, especially the method of beginning with the theory of impulsive forces.

I have applied this method so as to avoid the explicit consideration of the motion of any part of the system except the coordinates or variables, on which the motion of the whole depends. It is doubtless important that the student should be able to trace the connexion of the motion of each part of the system with that of the variables, but it is by no means necessary to do this in the process of obtaining the final equations, which are independent of the particular form of these connexions.

### *The Variables.*

555.] The number of degrees of freedom of a system is the number of data which must be given in order completely to determine its position. Different forms may be given to these data, but their number depends on the nature of the system itself, and cannot be altered.

To fix our ideas we may conceive the system connected by means of suitable mechanism with a number of moveable pieces, each capable of motion along a straight line, and of no other kind of motion. The imaginary mechanism which connects each of these pieces with the system must be conceived to be free from friction, destitute of inertia, and incapable of being strained by the action of the applied forces. The use of this mechanism is merely to assist the imagination in ascribing position, velocity, and momentum to what appear, in Lagrange's investigation, as pure algebraical quantities.

Let $q$ denote the position of one of the moveable pieces as defined by its distance from a fixed point in its line of motion.

We shall distinguish the values of $q$ corresponding to the different pieces by the suffixes $_1$, $_2$, &c. When we are dealing with a set of quantities belonging to one piece only we may omit the suffix.

When the values of all the variables ($q$) are given, the position of each of the moveable pieces is known, and, in virtue of the imaginary mechanism, the configuration of the entire system is determined.

*The Velocities.*

556.] During the motion of the system the configuration changes in some definite manner, and since the configuration at each instant is fully defined by the values of the variables ($q$), the velocity of every part of the system, as well as its configuration, will be completely defined if we know the values of the variables ($q$), together with their velocities

$$\left(\frac{dq}{dt}, \text{ or, according to Newton's notation, } \dot{q}\right).$$

*The Forces.*

557.] By a proper regulation of the motion of the variables, any motion of the system, consistent with the nature of the connexions, may be produced. In order to produce this motion by moving the variable pieces, forces must be applied to these pieces.

We shall denote the force which must be applied to any variable $q_r$ by $F_r$. The system of forces ($F$) is mechanically equivalent (in virtue of the connexions of the system) to the system of forces, whatever it may be, which really produces the motion.

*The Momenta.*

558.] When a body moves in such a way that its configuration, with respect to the force which acts on it, remains always the same, (as, for instance, in the case of a force acting on a single particle in the line of its motion,) the moving force is measured by the rate of increase of the momentum. If $F$ is the moving force, and $p$ the momentum,

$$F = \frac{dp}{dt},$$

whence $$p = \int F dt.$$

The time-integral of a force is called the Impulse of the force;

so that we may assert that the momentum is the impulse of the force which would bring the body from a state of rest into the given state of motion.

In the case of a connected system in motion, the configuration is continually changing at a rate depending on the velocities ($\dot{q}$), so that we can no longer assume that the momentum is the time-integral of the force which acts on it.

But the increment $\delta q$ of any variable cannot be greater than $\dot{q}'\,\delta t$, where $\delta t$ is the time during which the increment takes place, and $\dot{q}'$ is the greatest value of the velocity during that time. In the case of a system moving from rest under the action of forces always in the same direction, this is evidently the final velocity.

If the final velocity and configuration of the system are given, we may conceive the velocity to be communicated to the system in a very small time $\delta t$, the original configuration differing from the final configuration by quantities $\delta q_1$, $\delta q_2$, &c., which are less than $\dot{q}_1 \delta t$, $\dot{q}_2 \delta t$, &c., respectively.

The smaller we suppose the increment of time $\delta t$, the greater must be the impressed forces, but the time-integral, or impulse, of each force will remain finite. The limiting value of the impulse, when the time is diminished and ultimately vanishes, is defined as the *instantaneous* impulse, and the momentum $p$, corresponding to any variable $q$, is defined as the impulse corresponding to that variable, when the system is brought instantaneously from a state of rest into the given state of motion.

This conception, that the momenta are capable of being produced by instantaneous impulses on the system at rest, is introduced only as a method of defining the magnitude of the momenta, for the momenta of the system depend only on the instantaneous state of motion of the system, and not on the process by which that state was produced.

In a connected system the momentum corresponding to any variable is in general a linear function of the velocities of all the variables, instead of being, as in the dynamics of a particle, simply proportional to the velocity.

The impulses required to change the velocities of the system suddenly from $\dot{q}_1$, $\dot{q}_2$, &c. to $\dot{q}_1'$, $\dot{q}_2'$, &c. are evidently equal to $p_1'-p_1$, $p_2'-p_2$, the changes of momentum of the several variables.

*Work done by a Small Impulse.*

559.] The work done by the force $F_1$ during the impulse is the space-integral of the force, or

$$W = \int F_1\, dq_1,$$
$$= \int F_1\, \dot{q}_1\, dt.$$

If $\dot{q}_1'$ is the greatest and $\dot{q}_1''$ the least value of the velocity $\dot{q}_1$ during the action of the force, $W$ must be less than

$$\dot{q}_1 \int F\, dt \quad \text{or} \quad \dot{q}_1'\,(p_1' - p_1),$$

and greater than $\dot{q}_1'' \int F\, dt$ or $\dot{q}_1''(p_1' - p_1)$.

If we now suppose the impulse $\int F\, dt$ to be diminished without limit, the values of $\dot{q}_1'$ and $\dot{q}_1''$ will approach and ultimately coincide with that of $\dot{q}_1$, and we may write $p_1' - p_1 = \delta p_1$, so that the work done is ultimately

$$\delta W_1 = \dot{q}_1 \delta p_1,$$

or, *the work done by a very small impulse is ultimately the product of the impulse and the velocity.*

*Increment of the Kinetic Energy.*

560.] When work is done in setting a conservative system in motion, energy is communicated to it, and the system becomes capable of doing an equal amount of work against resistances before it is reduced to rest.

The energy which a system possesses in virtue of its motion is called its Kinetic Energy, and is communicated to it in the form of the work done by the forces which set it in motion.

If $T$ be the kinetic energy of the system, and if it becomes $T + \delta T$, on account of the action of an infinitesimal impulse whose components are $\delta p_1$, $\delta p_2$, &c., the increment $\delta T$ must be the sum of the quantities of work done by the components of the impulse, or in symbols,

$$\delta T = \dot{q}_1 \delta p_1 + \dot{q}_2 \delta p_2 + \text{\&c.},$$
$$= \Sigma\,(\dot{q}\,\delta p). \qquad (1)$$

The instantaneous state of the system is completely defined if the variables and the momenta are given. Hence the kinetic energy, which depends on the instantaneous state of the system, can be expressed in terms of the variables ($q$), and the momenta ($p$). This is the mode of expressing $T$ introduced by Hamilton. When $T$ is expressed in this way we shall distinguish it by the suffix $_p$, thus, $T_p$.

The complete variation of $T_p$ is

$$\delta T_p = \Sigma\left(\frac{dT_p}{dp}\delta p\right) + \Sigma\left(\frac{dT_p}{dq}\delta q\right). \tag{2}$$

The last term may be written

$$\Sigma\left(\frac{dT_p}{dq}\dot{q}\,\delta t\right),$$

which diminishes with $\delta t$, and ultimately vanishes with it when the impulse becomes instantaneous.

Hence, equating the coefficients of $\delta p$ in equations (1) and (2), we obtain

$$\dot{q} = \frac{dT_p}{dp}, \tag{3}$$

or, *the velocity corresponding to the variable $q$ is the differential coefficient of $T_p$ with respect to the corresponding momentum $p$.*

We have arrived at this result by the consideration of impulsive forces. By this method we have avoided the consideration of the change of configuration during the action of the forces. But the instantaneous state of the system is in all respects the same, whether the system was brought from a state of rest to the given state of motion by the transient application of impulsive forces, or whether it arrived at that state in any manner, however gradual.

In other words, the variables, and the corresponding velocities and momenta, depend on the actual state of motion of the system at the given instant, and not on its previous history.

Hence, the equation (3) is equally valid, whether the state of motion of the system is supposed due to impulsive forces, or to forces acting in any manner whatever.

We may now therefore dismiss the consideration of impulsive forces, together with the limitations imposed on their time of action, and on the changes of configuration during their action.

*Hamilton's Equations of Motion.*

561.] We have already shewn that

$$\frac{dT_p}{dp} = \dot{q}. \tag{4}$$

Let the system move in any arbitrary way, subject to the conditions imposed by its connexions, then the variations of $p$ and $q$ are

$$\delta p = \frac{dp}{dt}\delta t, \qquad \delta q = \dot{q}\,\delta t. \tag{5}$$

Hence
$$\frac{dT_p}{dp}\delta p = \frac{dp}{dt}\dot{q}\,\delta t,$$

$$= \frac{dp}{dt}\delta q, \tag{6}$$

and the complete variation of $T_p$ is

$$\delta T_p = \Sigma\left(\frac{dT_p}{dp}\delta p + \frac{dT_p}{dq}\delta q\right),$$

$$= \Sigma\left(\left(\frac{dp}{dt} + \frac{dT_p}{dq}\right)\delta q\right). \tag{7}$$

But the increment of the kinetic energy arises from the work done by the impressed forces, or

$$\delta T_p = \Sigma\,(F\delta q). \tag{8}$$

In these two expressions the variations $\delta q$ are all independent of each other, so that we are entitled to equate the coefficients of each of them in the two expressions (7) and (8). We thus obtain

$$F_r = \frac{dp_r}{dt} + \frac{dT_p}{dq_r}, \tag{9}$$

where the momentum $p_r$ and the force $F_r$ belong to the variable $q_r$*.

There are as many equations of this form as there are variables. These equations were given by Hamilton. They shew that the force corresponding to any variable is the sum of two parts. The first part is the rate of increase of the momentum of that variable with respect to the time. The second part is the rate of increase of the kinetic energy per unit of increment of the variable, the other variables and all the momenta being constant.

* {This proof does not seem conclusive as $\delta q$ is assumed to be equal to $\dot{q}\delta t$, that is to $\frac{dT_p}{dp}\delta t$, so that all we can legitimately deduce from (7) and (8) is

$$\Sigma\left\{\left(\frac{dp_r}{dt} + \frac{dT_p}{dq_r} - P_r\right)\frac{dT_p}{dp_r}\right\} = 0.$$

## *The Kinetic Energy expressed in Terms of the Momenta and Velocities.*

562.] Let $p_1$, $p_2$, &c. be the momenta, and $\dot{q}_1$, $\dot{q}_2$, &c. the velocities at a given instant, and let $\mathrm{p}_1$, $\mathrm{p}_2$, &c., $\dot{\mathrm{q}}_1$, $\dot{\mathrm{q}}_2$, &c. be another system of momenta and velocities, such that

$$\mathrm{p}_1 = n p_1, \qquad \dot{\mathrm{q}}_1 = n\dot{q}_1, \text{ \&c.} \tag{10}$$

It is manifest that the systems $\mathrm{p}$, $\dot{\mathrm{q}}$ will be consistent with each other if the systems $p$, $\dot{q}$ are so.

Now let $n$ vary by $\delta n$. The work done by the force $F_1$ is

$$F_1\,\delta\mathrm{q}_1 = \dot{\mathrm{q}}_1\,\delta\mathrm{p}_1 = \dot{q}_1 p_1 n\,\delta n. \tag{11}$$

Let $n$ increase from 0 to 1, then the system is brought from a state of rest into the state of motion $(\dot{q}, p)$, and the whole work expended in producing this motion is

$$(\dot{q}_1 p_1 + \dot{q}_2 p_2 + \text{\&c.})\int_0^1 n\,dn. \tag{12}$$

But
$$\int_0^1 n\,dn = \tfrac{1}{2},$$

and the work spent in producing the motion is equivalent to the kinetic energy. Hence

$$T_{p\dot{q}} = \tfrac{1}{2}(p_1\dot{q}_1 + p_2\dot{q}_2 + \text{\&c.}), \tag{13}$$

where $T_{p\dot{q}}$ denotes the kinetic energy expressed in terms of the momenta and velocities. The variables $q_1$, $q_2$, &c. do not enter into this expression.

The kinetic energy is therefore half the sum of the products of the momenta into their corresponding velocities.

When the kinetic energy is expressed in this way we shall denote it by the symbol $T_{p\dot{q}}$. It is a function of the momenta and velocities only, and does not involve the variables themselves.

563.] There is a third method of expressing the kinetic energy, which is generally, indeed, regarded as the fundamental one. By solving the equations (3) we may express the momenta in terms of the velocities, and then, introducing these values in (13), we shall have an expression for $T$ involving only the velocities and the variables. When $T$ is expressed in this form we shall indicate it by the symbol $T_{\dot{q}}$. This is the form in which the kinetic energy is expressed in the equations of Lagrange.

564.] It is manifest that, since $T_p$, $T_{\dot q}$, and $T_{p\dot q}$ are three different expressions for the same thing,

$$T_p + T_{\dot q} - 2\,T_{p\dot q} = 0,$$

or

$$T_p + T_{\dot q} - p_1\dot q_1 - p_2\dot q_2 - \&c. = 0. \tag{14}$$

Hence, if all the quantities $p$, $q$, and $\dot q$ vary,

$$\begin{aligned} &\left(\frac{dT_p}{dp_1} - \dot q_1\right)\delta p_1 + \left(\frac{dT_p}{dp_2} - \dot q_2\right)\delta p_1 + \&c. \\ &+ \left(\frac{dT_{\dot q}}{d\dot q_1} - p_1\right)\delta\dot q_1 + \left(\frac{dT_{\dot q}}{d\dot q_2} - p_2\right)\delta\dot q_2 + \&c. \\ &+ \left(\frac{dT_p}{dq_1} + \frac{dT_{\dot q}}{dq_1}\right)\delta q_1 + \left(\frac{dT_p}{dq_2} + \frac{dT_{\dot q}}{dq_2}\right)\delta q_2 + \&c. = 0. \end{aligned} \tag{15}$$

The variations $\delta p$ are not independent of the variations $\delta q$ and $\delta\dot q$, so that we cannot at once assert that the coefficient of each variation in this equation is zero. But we know, from equations (3), that

$$\frac{dT_p}{dp_1} - \dot q_1 = 0,\ \&c., \tag{16}$$

so that the terms involving the variations $\delta p$ vanish of themselves.

The remaining variations $\delta\dot q$ and $\delta q$ are now all independent, so that we find, by equating to zero the coefficients of $\delta\dot q_1$, &c.,

$$p_1 = \frac{dT_{\dot q}}{d\dot q_1}, \qquad p_2 = \frac{dT_{\dot q}}{d\dot q_2},\ \&c.; \tag{17}$$

or, *the components of momentum are the differential coefficients of $T_{\dot q}$ with respect to the corresponding velocities.*

Again, by equating to zero the coefficients of $\delta q_1$, &c.,

$$\frac{dT_p}{dq_1} + \frac{dT_{\dot q}}{dq_1} = 0; \tag{18}$$

or, *the differential coefficient of the kinetic energy with respect to any variable $q_1$ is equal in magnitude but opposite in sign when $T$ is expressed as a function of the velocities instead of as a function of the momenta.*

In virtue of equation (18) we may write the equation of motion (9),

$$F_1 = \frac{dp_1}{dt} - \frac{dT_{\dot q}}{dq_1}, \tag{19}$$

or

$$F_1 = \frac{d}{dt}\frac{dT_{\dot q}}{d\dot q_1} - \frac{dT_{\dot q}}{dq_1}, \tag{20}$$

which is the form in which the equations of motion were given by Lagrange.

565.] In the preceding investigation we have avoided the consideration of the form of the function which expresses the kinetic energy in terms either of the velocities or of the momenta. The only explicit form which we have assigned to it is

$$T_{p\dot{q}} = \tfrac{1}{2}(p_1\dot{q}_1 + p_2\dot{q}_2 + \&c.), \tag{21}$$

in which it is expressed as half the sum of the products of the momenta each into its corresponding velocity.

We may express the velocities in terms of the differential coefficients of $T_p$ with respect to the momenta, as in equation (3),

$$T_p = \tfrac{1}{2}\left(p_1 \frac{dT_p}{dp_1} + p_2 \frac{dT_p}{dp_2} + \&c.\right). \tag{22}$$

This shews that $T_p$ is a homogeneous function of the second degree of the momenta $p_1$, $p_2$, &c.

We may also express the momenta in terms of $T_{\dot{q}}$, and we find

$$T_{\dot{q}} = \tfrac{1}{2}\left(\dot{q}_1 \frac{dT_{\dot{q}}}{d\dot{q}_1} + \dot{q}_2 \frac{dT_{\dot{q}}}{d\dot{q}_2} + \&c.\right), \tag{23}$$

which shews that $T_{\dot{q}}$ is a homogeneous function of the second degree with respect to the velocities $\dot{q}_1$, $\dot{q}_2$, &c.

If we write

$$P_{11} \text{ for } \frac{d^2T_{\dot{q}}}{d\dot{q}_1^{\,2}}, \qquad P_{12} \text{ for } \frac{d^2T_{\dot{q}}}{d\dot{q}_1 d\dot{q}_2}, \ \&c.$$

and

$$Q_{11} \text{ for } \frac{d^2T_p}{dp_1^{\,2}}, \qquad Q_{12} \text{ for } \frac{d^2T_p}{dp_1\, dp_2}, \ \&c.;$$

then, since $T_{\dot{q}}$ and $T_p$ are functions of the second degree of $\dot{q}$ and $p$ respectively, both the $P$'s and the $Q$'s will be functions of the variables $q$ only, and independent of the velocities and the momenta. We thus obtain the expressions for $T$,

$$2T_{\dot{q}} = P_{11}\dot{q}_1^{\,2} + 2P_{12}\dot{q}_1\dot{q}_2 + \&c., \tag{24}$$

$$2T_p = Q_{11}p_1^{\,2} + 2Q_{12}p_1p_2 + \&c. \tag{25}$$

The momenta are expressed in terms of the velocities by the linear equations

$$p_1 = P_{11}\dot{q}_1 + P_{12}\dot{q}_2 + \&c., \tag{26}$$

and the velocities are expressed in terms of the momenta by the linear equations

$$\dot{q}_1 = Q_{11}p_1 + Q_{12}p_2 + \&c. \tag{27}$$

In treatises on the dynamics of a rigid body, the coefficients corresponding to $P_{11}$, in which the suffixes are the same, are called Moments of Inertia, and those corresponding to $P_{12}$, in which the suffixes are different, are called Products of Inertia.

We may extend these names to the more general problem which is now before us, in which these quantities are not, as in the case of a rigid body, absolute constants, but are functions of the variables $q_1$, $q_2$, &c.

In like manner we may call the coefficients of the form $Q_{11}$ Moments of Mobility, and those of the form $Q_{12}$ Products of Mobility. It is not often, however, that we shall have occasion to speak of the coefficients of mobility.

566.] The kinetic energy of the system is a quantity essentially positive or zero. Hence, whether it be expressed in terms of the velocities, or in terms of the momenta, the coefficients must be such that no real values of the variables can make $T$ negative.

There are thus a set of necessary conditions which the values of the coefficients $P$ must satisfy. These conditions are as follows:

The quantities $P_{11}$, $P_{12}$, &c. must all be positive.

The $n-1$ determinants formed in succession from the determinant

$$\begin{vmatrix} P_{11}, & P_{12}, & P_{13}, \ldots\ldots P_{1n} \\ P_{12}, & P_{22}, & P_{23}, \ldots\ldots P_{2n} \\ P_{13}, & P_{23}, & P_{33}, \ldots\ldots P_{3n} \\ \\ P_{1n}, & P_{2n}, & P_{3n}, \ldots\ldots P_{nn} \end{vmatrix}$$

by the omission of terms with suffix 1, then of terms with either 1 or 2 in their suffix, and so on, must all be positive.

The number of conditions for $n$ variables is therefore $2n-1$.

The coefficients $Q$ are subject to conditions of the same kind.

567.] In this outline of the fundamental principles of the dynamics of a connected system, we have kept out of view the mechanism by which the parts of the system are connected. We have not even written down a set of equations to indicate how the motion of any part of the system depends on the variation of the variables. We have confined our attention to the variables, their velocities and momenta, and the forces which act on the pieces representing the variables. Our only assumptions are, that the connexions of the system are such that the time is not explicitly contained in the equations of condition, and that the principle of the conservation of energy is applicable to the system.

Such a description of the methods of pure dynamics is not unnecessary, because Lagrange and most of his followers, to whom we are indebted for these methods, have in general confined themselves to a demonstration of them, and, in order to devote their attention to the symbols before them, they have endeavoured to banish all ideas except those of pure quantity, so as not only to dispense with diagrams, but even to get rid of the ideas of velocity, momentum, and energy, after they have been once for all supplanted by symbols in the original equations. In order to be able to refer to the results of this analysis in ordinary dynamical language, we have endeavoured to retranslate the principal equations of the method into language which may be intelligible without the use of symbols.

As the development of the ideas and methods of pure mathematics has rendered it possible, by forming a mathematical theory of dynamics, to bring to light many truths which could not have been discovered without mathematical training, so, if we are to form dynamical theories of other sciences, we must have our minds imbued with these dynamical truths as well as with mathematical methods.

In forming the ideas and words relating to any science, which, like electricity, deals with forces and their effects, we must keep constantly in mind the ideas appropriate to the fundamental science of dynamics, so that we may, during the first development of the science, avoid inconsistency with what is already established, and also that when our views become clearer, the language we have adopted may be a help to us and not a hindrance.

# CHAPTER VI.

## DYNAMICAL THEORY OF ELECTROMAGNETISM.

568.] We have shewn, in Art. 552, that, when an electric current exists in a conducting circuit, it has a capacity for doing a certain amount of mechanical work, and this independently of any external electromotive force maintaining the current. Now capacity for performing work is nothing else than energy, in whatever way it arises, and all energy is the same in kind, however it may differ in form. The energy of an electric current is either of that form which consists in the actual motion of matter, or of that which consists in the capacity for being set in motion, arising from forces acting between bodies placed in certain positions relative to each other.

The first kind of energy, that of motion, is called Kinetic energy, and when once understood it appears so fundamental a fact of nature that we can hardly conceive the possibility of resolving it into anything else. The second kind of energy, that depending on position, is called Potential energy, and is due to the action of what we call forces, that is to say, tendencies towards change of relative position. With respect to these forces, though we may accept their existence as a demonstrated fact, yet we always feel that every explanation of the mechanism by which bodies are set in motion forms a real addition to our knowledge.

569.] The electric current cannot be conceived except as a kinetic phenomenon. Even Faraday, who constantly endeavoured to emancipate his mind from the influence of those suggestions which the words 'electric current' and 'electric fluid' are too apt to carry with them, speaks of the electric current as 'something progressive, and not a mere arrangement *.'

* *Exp. Res.*, 283.

The effects of the current, such as electrolysis, and the transfer of electrification from one body to another, are all progressive actions which require time for their accomplishment, and are therefore of the nature of motions.

As to the velocity of the current, we have shewn that we know nothing about it, it may be the tenth of an inch in an hour, or a hundred thousand miles in a second*. So far are we from knowing its absolute value in any case, that we do not even know whether what we call the positive direction is the actual direction of the motion or the reverse.

But all that we assume here is that the electric current involves motion of some kind. That which is the cause of electric currents has been called Electromotive Force. This name has long been used with great advantage, and has never led to any inconsistency in the language of science. Electromotive force is always to be understood to act on electricity only, not on the bodies in which the electricity resides. It is never to be confounded with ordinary mechanical force, which acts on bodies only, not on the electricity in them. If we ever come to know the formal relation between electricity and ordinary matter, we shall probably also know the relation between electromotive force and ordinary force.

570.] When ordinary force acts on a body, and when the body yields to the force, the work done by the force is measured by the product of the force into the amount by which the body yields. Thus, in the case of water forced through a pipe, the work done at any section is measured by the fluid pressure at the section multiplied into the quantity of water which crosses the section.

In the same way the work done by an electromotive force is measured by the product of the electromotive force into the quantity of electricity which crosses a section of the conductor under the action of the electromotive force.

The work done by an electromotive force is of exactly the same kind as the work done by an ordinary force, and both are measured by the same standards or units.

Part of the work done by an electromotive force acting on a conducting circuit is spent in overcoming the resistance of the circuit, and this part of the work is thereby converted into heat.

* *Exp. Res.*, 1648.

Another part of the work is spent in producing the electromagnetic phenomena observed by Ampère, in which conductors are made to move by electromagnetic forces. The rest of the work is spent in increasing the kinetic energy of the current, and the effects of this part of the action are shewn in the phenomena of the induction of currents observed by Faraday.

We therefore know enough about electric currents to recognise, in a system of material conductors carrying currents, a dynamical system which is the seat of energy, part of which may be kinetic and part potential.

The nature of the connexions of the parts of this system is unknown to us, but as we have dynamical methods of investigation which do not require a knowledge of the mechanism of the system, we shall apply them to this case.

We shall first examine the consequences of assuming the most general form for the function which expresses the kinetic energy of the system.

571.] Let the system consist of a number of conducting circuits, the form and position of which are determined by the values of a system of variables $x_1$, $x_2$, &c., the number of which is equal to the number of degrees of freedom of the system.

If the whole kinetic energy of the system were that due to the motion of these conductors, it would be expressed in the form

$$T = \tfrac{1}{2}(x_1 x_1)\dot{x}_1^{\,2} + \&c. + (x_1 x_2)\dot{x}_1\dot{x}_2 + \&c.,$$

where the symbols $(x_1 x_1)$, &c. denote the quantities which we have called moments of inertia, and $(x_1 x_2)$, &c. denote the products of inertia.

If $X'$ is the impressed force, tending to increase the coordinate $x$, which is required to produce the actual motion, then, by Lagrange's equation,
$$\frac{d}{dt}\frac{dT}{d\dot{x}} - \frac{dT}{dx} = X'.$$

When $T$ denotes the energy due to the visible motion only, we shall indicate it by the suffix $_m$, thus, $T_m$.

But in a system of conductors carrying electric currents, part of the kinetic energy is due to the existence of these currents. Let the motion of the electricity, and of anything whose motion is governed by that of the electricity, be determined by another set of coordinates $y_1$, $y_2$, &c., then $T$ will be a homogeneous function of squares and products of all the velocities of the two sets of coordinates. We may therefore divide $T$ into three portions, in the first of which, $T_m$, the velocities of the coordinates $x$ only occur, while in the second, $T_e$, the velocities of the coordinates $y$ only occur, and in the third, $T_{me}$, each term contains the product of the velocities of two coordinates of which one is an $x$ and the other a $y$.

We have therefore $$T = T_m + T_e + T_{me},$$

where
$$T_m = \tfrac{1}{2}(x_1 x_1)\dot{x}_1^2 + \&c. + (x_1 x_2)\dot{x}_1\dot{x}_2 + \&c.,$$
$$T_e = \tfrac{1}{2}(y_1 y_1)\dot{y}_1^2 + \&c. + (y_1 y_2)\dot{y}_1\dot{y}_2 + \&c.,$$
$$T_{me} = (x_1 y_1)\dot{x}_1\dot{y}_1 + \&c.$$

572.] In the general dynamical theory, the coefficients of every term may be functions of all the coordinates, both $x$ and $y$. In the case of electric currents, however, it is easy to see that the coordinates of the class $y$ do not enter into the coefficients.

For, if all the electric currents are maintained constant, and the conductors at rest, the whole state of the field will remain constant. But in this case the coordinates $y$ are variable, though the velocities $\dot{y}$ are constant. Hence the coordinates $y$ cannot enter into the expression for $T$, or into any other expression of what actually takes place.

Besides this, in virtue of the equation of continuity, if the conductors are of the nature of linear circuits, only one variable is required to express the strength of the current in each conductor. Let the velocities $\dot{y}_1$, $\dot{y}_2$, &c. represent the strengths of the currents in the several conductors.

All this would be true, if, instead of electric currents, we had currents of an incompressible fluid running in flexible tubes. In this case the velocities of these currents would enter into the expression for $T$, but the coefficients would depend only on the variables $x$, which determine the form and position of the tubes.

In the case of the fluid, the motion of the fluid in one tube does not directly affect that of any other tube, or of the fluid in it. Hence, in the value of $T_e$, only the squares of the velocities $\dot{y}$, and not their products, occur, and in $T_{me}$ any velocity $\dot{y}$ is associated only with those velocities of the form $\dot{x}$ which belong to its own tube.

In the case of electrical currents we know that this restriction does not hold, for the currents in different circuits act on each other. Hence we must admit the existence of terms involving products of the form $\dot{y}_1 \dot{y}_2$, and this involves the existence of something in motion, whose motion depends on the strength of both electric currents $\dot{y}_1$ and $\dot{y}_2$. This moving matter, whatever it is, is not confined to the interior of the conductors carrying the two currents, but probably extends throughout the whole space surrounding them.

573.] Let us next consider the form which Lagrange's equations of motion assume in this case. Let $X'$ be the impressed force corresponding to the coordinate $x$, one of those which determine the form and position of the conducting circuits. This is a force in the ordinary sense, a tendency towards change of position. It is given by the equation

$$X' = \frac{d}{dt}\frac{dT}{d\dot{x}} - \frac{dT}{dx}.$$

We may consider this force as the sum of three parts, corresponding to the three parts into which we divided the kinetic energy of the system, and we may distinguish them by the same suffixes. Thus $\quad X' = X'_m + X'_e + X'_{me}.$

The part $X'_m$ is that which depends on ordinary dynamical considerations, and we need not attend to it.

Since $T_e$ does not contain $\dot{x}$, the first term of the expression for $X'_e$ is zero, and its value is reduced to

$$X'_e = -\frac{dT_e}{dx}.$$

This is the expression for the mechanical force which must be applied to a conductor to balance the electromagnetic force, and it asserts that it is measured by the rate of *diminution* of the purely electrokinetic energy due to the variation of the coordinate $x$. The electromagnetic force, $X_e$, which brings this external mechanical force into play, is equal and opposite to $X'_e$, and is therefore measured by the rate of *increase* of the electrokinetic energy corresponding to an increase of the coordinate $x$. The value of $X_e$, since it depends on squares and products of the currents, remains the same if we reverse the directions of all the currents.

The third part of $X'$ is

$$X'_{me} = \frac{d}{dt}\frac{dT_{me}}{d\dot{x}} - \frac{dT_{me}}{dx}.$$

The quantity $T_{me}$ contains only products of the form $\dot{x}\dot{y}$, so that $\frac{dT_{me}}{d\dot{x}}$ is a linear function of the strengths of the currents $\dot{y}$. The first term, therefore, depends on the rate of variation of the strengths of the currents, and indicates a mechanical force on the conductor, which is zero when the currents are constant, and which is positive or negative according as the currents are increasing or decreasing in strength.

The second term depends, not on the variation of the currents, but on their actual strengths. As it is a linear function with respect to these currents, it changes sign when the currents change sign. Since every term involves a velocity $\dot{x}$, it is zero when the conductors are at rest. There are also terms arising from the time variations of the coefficients of $\dot{y}$ in $\frac{dT_{me}}{d\dot{x}}$: these remarks apply also to them.

We may therefore investigate these terms separately. If the conductors are at rest, we have only the first term to deal with. If the currents are constant, we have only the second.

574.] As it is of great importance to determine whether any part of the kinetic energy is of the form $T_{me}$, consisting of products of ordinary velocities and strengths of electric currents, it is desirable that experiments should be made on this subject with great care.

The determination of the forces acting on bodies in rapid motion is difficult. Let us therefore attend to the first term, which depends on the variation of the strength of the current.

If any part of the kinetic energy depends on the product of an ordinary velocity and the strength of a current, it will probably be most easily observed when the velocity and the current are in the same or in opposite directions. We therefore take a circular coil of a great many windings, and suspend it by a fine vertical wire, so that its windings are horizontal, and the coil is capable of rotating about a vertical axis, either in the same direction as the current in the coil, or in the opposite direction.

We shall suppose the current to be conveyed into the coil by means of the suspending wire, and, after passing round the windings, to complete its circuit by passing downwards through a wire in the same line with the suspending wire and dipping into a cup of mercury.

Since the action of the horizontal component of terrestrial magnetism would tend to turn this coil round a horizontal axis when the current flows through it, we shall suppose that the horizontal component of terrestrial magnetism is exactly neutralized by means of fixed magnets, or that the experiment is made at the magnetic pole. A vertical mirror is attached to the coil to detect any motion in azimuth.

Now let a current be made to pass through the coil in the direction N.E.S.W. If electricity were a fluid like water, flowing along the wire, then, at the moment of starting the current, and as long as its velocity is increasing, a force would require to be supplied to produce the angular momentum of the fluid in passing round the coil, and as this must be supplied by the elasticity of the suspending wire, the coil would at first rotate in the opposite direction or W.S.E.N., and this would be detected by means of the mirror. On stopping the current there would be another movement of the mirror, this time in the same direction as that of the current.

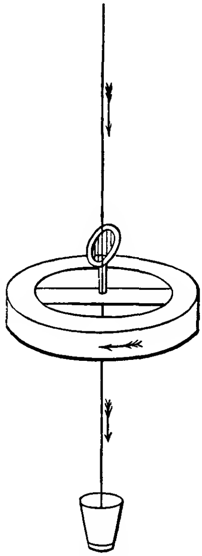

Fig. 33.

No phenomenon of this kind has yet been observed. Such an action, if it existed, might be easily distinguished from the already known actions of the current by the following peculiarities.

(1) It would occur only when the strength of the current varies, as when contact is made or broken, and not when the current is constant.

All the known *mechanical* actions of the current depend on the strength of the currents, and not on the rate of variation. The electromotive action in the case of induced currents cannot be confounded with this electromagnetic action.

(2) The direction of this action would be reversed when that of all the currents in the field is reversed.

All the known mechanical actions of the current remain the same when all the currents are reversed, since they depend on squares and products of these currents.

If any action of this kind were discovered, we should be able to regard one of the so-called kinds of electricity, either the positive or the negative kind, as a real substance, and we should be able to describe the electric current as a true motion of this substance in a particular direction. In fact, if electrical motions were in any way comparable with the motions of ordinary matter, terms of the form $T_{me}$ would exist, and their existence would be manifested by the mechanical force $X_{me}$.

According to Fechner's hypothesis, that an electric current consists of two equal currents of positive and negative electricity, flowing in opposite directions through the same conductor, the terms of the second class $T_{me}$ would vanish, each term belonging to the positive current being accompanied by an equal term of opposite sign belonging to the negative current, and the phenomena depending on these terms would have no existence.

It appears to me, however, that while we derive great advantage from the recognition of the many analogies between the electric current and a current of material fluid, we must carefully avoid making any assumption not warranted by experimental evidence, and that there is, as yet, no experimental evidence to shew whether the electric current is really a current of a material substance, or a double current, or whether its velocity is great or small as measured in feet per second.

A knowledge of these things would amount to at least the beginnings of a complete dynamical theory of electricity, in which we should regard electrical action, not, as in this treatise, as a phenomenon due to an unknown cause, subject only to the general laws of dynamics, but as the result of known motions of known portions of matter, in which not only the total effects and final results, but the whole intermediate mechanism and details of the motion, are taken as the objects of study.

575.] The experimental investigation of the second term of $X_{me}$, namely $\dfrac{dT_{me}}{dx}$, is more difficult, as it involves the observation of the effect of forces on a body in rapid motion.

The apparatus shewn in Fig. 34, which I had constructed in 1861, is intended to test the existence of a force of this kind.

The electromagnet $A$ is capable of rotating about the horizontal axis $BB'$, within a ring which itself revolves about a vertical axis.

Let $A$, $B$, $C$ be the moments of inertia of the electromagnet about the axis of the coil, the horizontal axis $BB'$, and a third axis $CC'$ respectively.

Let $\theta$ be the angle which $CC'$ makes with the vertical, $\phi$ the azimuth of the axis $BB'$, and $\psi$ a variable on which the motion of electricity in the coil depends.

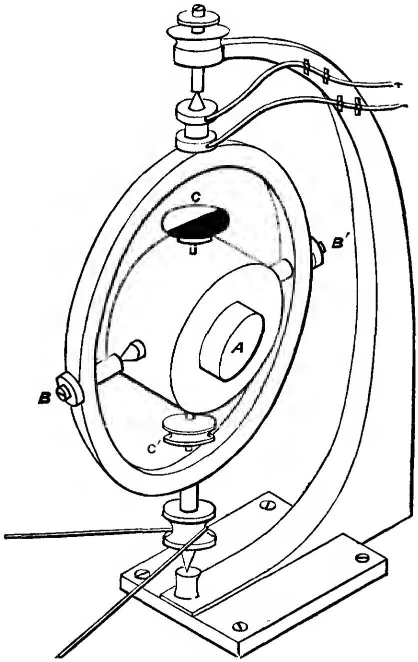


Fig. 34.

Then the kinetic energy $T$ of the electromagnet may be written

$$2\,T = A\dot{\phi}^2 \sin^2\theta + B\dot{\theta}^2 + C\dot{\phi}^2\cos^2\theta + E\,(\dot{\phi}\sin\theta + \dot{\psi})^2,$$

where $E$ is a quantity which may be called the moment of inertia of the electricity in the coil.

If $\Theta$ is the moment of the impressed force tending to increase $\theta$, we have, by the equations of dynamics,

$$\Theta = B\frac{d^2\theta}{dt^2} - \{(A-C)\,\dot{\phi}^2\sin\theta\cos\theta + E\dot{\phi}\cos\theta\,(\dot{\phi}\sin\theta + \dot{\psi})\}.$$

By making $\Psi$, the impressed force tending to increase $\psi$, equal to zero, we obtain

$$\dot{\phi}\sin\theta + \dot{\psi} = \gamma,$$

a constant, which we may consider as representing the strength of the current in the coil.

If $C$ is somewhat greater than $A$; $\Theta$ will be zero, and the equilibrium about the axis $BB'$ will be stable when

$$\sin\theta = \frac{E\gamma}{(C-A)\dot{\phi}}.$$

This value of $\theta$ depends on that of $\gamma$, the electric current, and is positive or negative according to the direction of the current.

The current is passed through the coil by its bearings at $B$ and $B'$, which are connected with the battery by means of springs rubbing on metal rings placed on the vertical axis.

To determine the value of $\theta$, a disk of paper is placed at $C$, divided by a diameter parallel to $BB'$ into two parts, one of which is painted red and the other green.

When the instrument is in motion a red circle is seen at $C$ when $\theta$ is positive, the radius of which indicates roughly the value of $\theta$. When $\theta$ is negative, a green circle is seen at $C$.

By means of nuts working on screws attached to the electromagnet, the axis $CC'$ is adjusted to be a principal axis having its moment of inertia just exceeding that round the axis $A$, so as to make the instrument very sensitive to the action of the force if it exists.

The chief difficulty in the experiments arose from the disturbing action of the earth's magnetic force, which caused the electromagnet to act like a dip-needle. The results obtained were on this account very rough, but no evidence of any change in $\theta$ could be obtained even when an iron core was inserted in the coil, so as to make it a powerful electromagnet.

If, therefore, a magnet contains matter in rapid rotation, the angular momentum of this rotation must be very small compared with any quantities which we can measure, and we have as yet no evidence of the existence of the terms $T_{me}$ derived from their mechanical action.

576.] Let us next consider the forces acting on the currents of electricity, that is, the electromotive forces.

Let $Y$ be the effective electromotive force due to induction, the electromotive force which must act on the circuit from without to balance it is $Y' = -Y$, and, by Lagrange's equation,

$$Y = -Y' = -\frac{d}{dt}\frac{dT}{d\dot{y}} + \frac{dT}{dy}.$$

Since there are no terms in $T$ involving the coordinate $y$, the second term is zero, and $Y$ is reduced to its first term. Hence, electromotive force cannot exist in a system at rest, and with constant currents.

Again, if we divide $Y$ into three parts, $Y_m$, $Y_e$, and $Y_{me}$, corresponding to the three parts of $T$, we find that, since $T_m$ does not contain $\dot{y}$, $Y_m = 0$.

We also find $$Y_e = -\frac{d}{dt}\frac{dT_e}{d\dot{y}}.$$

Here $\frac{dT_e}{d\dot{y}}$ is a linear function of the currents, and this part of the electromotive force is equal to the rate of change of this function. This is the electromotive force of induction discovered by Faraday. We shall consider it more at length afterwards.

577.] From the part of $T$, depending on velocities multiplied by currents, we find $$Y_{me} = -\frac{d}{dt}\frac{dT_{me}}{d\dot{y}}.$$

Now $\frac{dT_{me}}{d\dot{y}}$ is a linear function of the velocities of the conductors. If, therefore, any terms of $T_{me}$ have an actual existence, it would be possible to produce an electromotive force independently of all existing currents by simply altering the velocities of the conductors. For instance, in the case of the suspended coil at Art. 574, if, when the coil is at rest, we suddenly set it in rotation about the vertical axis, an electromotive force would be called into action proportional to the acceleration of this motion. It would vanish when the motion became uniform, and be reversed when the motion was retarded.

Now few scientific observations can be made with greater precision than that which determines the existence or non-existence of a current by means of a galvanometer. The delicacy of this method far exceeds that of most of the arrangements for measuring the mechanical force acting on a body. If, therefore, any currents could be produced in this way they would be detected, even if they were very feeble. They would be distinguished from ordinary currents of induction by the following characteristics.

(1) They would depend entirely on the motions of the conductors, and in no degree on the strength cf currents or magnetic forces already in the field.

(2) They would depend not on the absolute velocities of the conductors, but on their accelerations, and on squares and products of velocities, and they would change when the acceleration becomes a retardation, though the absolute velocity is the same.

Now in all the cases actually observed, the induced currents depend altogether on the strength and the variation of currents in the field, and cannot be excited in a field devoid of magnetic force and of currents. In so far as they depend on the motion of conductors, they depend on the absolute velocity, and not on the change of velocity of these motions.

We have thus three methods of detecting the existence of the terms of the form $T_{me}$, none of which have hitherto led to any positive result. I have pointed them out with the greater care because it appears to me important that we should attain the greatest amount of certitude within our reach on a point bearing so strongly on the true theory of electricity.

Since, however, no evidence has yet been obtained of such terms, I shall now proceed on the assumption that they do not exist, or at least that they produce no sensible effect, an assumption which will considerably simplify our dynamical theory. We shall have occasion, however, in discussing the relation of magnetism to light, to shew that the motion which constitutes light may enter as a factor into terms involving the motion which constitutes magnetism.

# CHAPTER VII.

## THEORY OF ELECTRIC CIRCUITS.

578.] We may now confine our attention to that part of the kinetic energy of the system which depends on squares and products of the strengths of the electric currents. We may call this the Electrokinetic Energy of the system. The part depending on the motion of the conductors belongs to ordinary dynamics, and we have seen that the part depending on products of velocities and currents does not exist.

Let $A_1$, $A_2$, &c. denote the different conducting circuits. Let their form and relative position be expressed in terms of the variables $x_1$, $x_2$, &c. the number of which is equal to the number of degrees of freedom of the mechanical system. We shall call these the Geometrical Variables.

Let $y_1$ denote the quantity of electricity which has crossed a given section of the conductor $A_1$ since the beginning of the time $t$. The strength of the current will be denoted by $\dot{y}_1$, the fluxion of this quantity.

We shall call $\dot{y}_1$ the actual current, and $y_1$ the integral current. There is one variable of this kind for each circuit in the system.

Let $T$ denote the electrokinetic energy of the system. It is a homogeneous function of the second degree with respect to the strengths of the currents, and is of the form

$$T = \tfrac{1}{2} L_1 \dot{y}_1^2 + \tfrac{1}{2} L_2 \dot{y}_2^2 + \&\text{c.} + M_{12} \dot{y}_1 \dot{y}_2 + \&\text{c.}, \qquad (1)$$

where the coefficients $L$, $M$, &c. are functions of the geometrical variables $x_1$, $x_2$, &c. The electrical variables $y_1$, $y_2$ do not enter into the expression.

We may call $L_1$, $L_2$, &c. the electric moments of inertia of the circuits $A_1$, $A_2$, &c., and $M_{12}$ the electric product of inertia of the two circuits $A_1$ and $A_2$. When we wish to avoid the language of the dynamical theory, we shall call $L_1$ the coefficient of self-induction of the circuit $A_1$, and $M_{12}$ the coefficient of mutual induction of the circuits $A_1$ and $A_2$. $M_{12}$ is also called the potential of the circuit $A_1$ with respect to $A_2$. These quantities depend only on the form and relative position of the circuits. We shall find that in the electromagnetic system of measurement they are quantities of the dimension of a line. See Art. 627.

By differentiating $T$ with respect to $\dot{y}_1$ we obtain the quantity $p_1$, which, in the dynamical theory, may be called the momentum corresponding to $y_1$. In the electric theory we shall call $p_1$ the electrokinetic momentum of the circuit $A_1$. Its value is

$$p_1 = L_1\dot{y}_1 + M_{12}\dot{y}_2 + \&c.$$

The electrokinetic momentum of the circuit $A_1$ is therefore made up of the product of its own current into its coefficient of self-induction, together with the sum of the products of the currents in the other circuits, each into the coefficient of mutual induction of $A_1$ and that other circuit.

### *Electromotive Force.*

579.] Let $E$ be the impressed electromotive force in the circuit $A$, arising from some cause, such as a voltaic or thermo-electric battery, which would produce a current independently of magneto-electric induction.

Let $R$ be the resistance of the circuit, then, by Ohm's law, an electromotive force $R\dot{y}$ is required to overcome the resistance, leaving an electromotive force $E-R\dot{y}$ available for changing the momentum of the circuit. Calling this force $Y'$, we have, by the general equations,

$$Y' = \frac{dp}{dt} - \frac{dT}{dy},$$

but since $T$ does not involve $y$, the last term disappears.

Hence, the equation of electromotive force is

$$E - R\dot{y} = Y' = \frac{dp}{dt},$$

or

$$E = R\dot{y} + \frac{dp}{dt}.$$

The impressed electromotive force $E$ is therefore the sum of two parts. The first, $R\dot{y}$, is required to maintain the current $\dot{y}$ against the resistance $R$. The second part is required to increase the electromagnetic momentum $p$. This is the electromotive force which must be supplied from sources independent of magneto-electric induction. The electromotive-force arising from magneto-electric induction alone is evidently $-\frac{dp}{dt}$, or, *the rate of decrease of the electrokinetic momentum of the circuit.*

*Electromagnetic Force.*

580.] Let $X'$ be the impressed mechanical force arising from external causes, and tending to increase the variable $x$. By the general equations

$$X' = \frac{d}{dt}\frac{dT}{d\dot{x}} - \frac{dT}{dx}.$$

Since the expression for the electrokinetic energy does not contain the velocity ($\dot{x}$), the first term of the second member disappears, and we find

$$X' = -\frac{dT}{dx}.$$

Here $X'$ is the external force required to balance the forces arising from electrical causes. It is usual to consider this force as the reaction against the electromagnetic force, which we shall call $X$, and which is equal and opposite to $X'$.

Hence
$$X = \frac{dT}{dx},$$

or, *the electromagnetic force tending to increase any variable is equal to the rate of increase of the electrokinetic energy per unit increase of that variable, the currents being maintained constant.*

If the currents are maintained constant by a battery during a displacement in which a quantity, $W$, of work is done by electromotive force, the electrokinetic energy of the system will be at the same time increased by $W$. Hence the battery will be drawn upon for a double quantity of energy, or 2 $W$, in addition to that which is spent in generating heat in the circuit. This was first pointed out by Sir W. Thomson *. Compare this result with the electrostatic property in Art. 93.

* Nichol's *Cyclopaedia of the Physical Sciences*, ed. 1860, article 'Magnetism, Dynamical Relations of.'

*Case of Two Circuits.*

581.] Let $A_1$ be called the Primary Circuit, and $A_2$ the Secondary Circuit. The electrokinetic energy of the system may be written

$$T = \tfrac{1}{2} L\dot{y}_1^2 + M\dot{y}_1\dot{y}_2 + \tfrac{1}{2} N\dot{y}_2^2,$$

where $L$ and $N$ are the coefficients of self-induction of the primary and secondary circuits respectively, and $M$ is the coefficient of their mutual induction.

Let us suppose that no electromotive force acts on the secondary circuit except that due to the induction of the primary current. We have then

$$E_2 = R_2\dot{y}_2 + \frac{d}{dt}(M\dot{y}_1 + N\dot{y}_2) = 0.$$

Integrating this equation with respect to $t$, we have

$$R_2 y_2 + M\dot{y}_1 + N\dot{y}_2 = C, \text{ a constant},$$

where $y_2$ is the integral current in the secondary circuit.

The method of measuring an integral current of short duration will be described in Art. 748, and it is easy in most cases to ensure that the duration of the secondary current shall be very short.

Let the values of the variable quantities in the equation at the end of the time $t$ be accented, then, if $y_2$ is the integral current, or the whole quantity of electricity which flows through a section of the secondary circuit during the time $t$,

$$R_2 y_2 = M\dot{y}_1 + N\dot{y}_2 - (M'\dot{y}_1' + N'\dot{y}_2').$$

If the secondary current arises entirely from induction, its initial value $\dot{y}_2$ must be zero if the primary current is constant, and the conductors are at rest before the beginning of the time $t$.

If the time $t$ is sufficient to allow the secondary current to die away, $\dot{y}_2'$, its final value, is also zero, so that the equation becomes

$$R_2 y_2 = M\dot{y}_1 - M'\dot{y}_1'.$$

The integral current of the secondary circuit depends in this case on the initial and final values of $M\dot{y}_1$.

*Induced Currents.*

582.] Let us begin by supposing the primary circuit broken, or $\dot{y}_1 = 0$, and let a current $\dot{y}_1'$ be established in it when contact is made.

The equation which determines the secondary integral current is

$$R_2 y_2 = -M' \dot{y}_1'.$$

When the circuits are placed side by side, and in the same direction, $M'$ is a positive quantity. Hence, when contact is made in the primary circuit, a negative current is induced in the secondary circuit.

When the contact is broken in the primary circuit, the primary current ceases, and the induced integral current is $y_2$, where

$$R_2 y_2 = M \dot{y}_1.$$

The secondary current is in this case positive.

If the primary current is maintained constant, and the form or relative position of the circuits altered so that $M$ becomes $M'$, the integral secondary current is $y_2$, where

$$R_2 y_2 = (M - M')\dot{y}_1.$$

In the case of two circuits placed side by side and in the same direction $M$ diminishes as the distance between the circuits increases. Hence, the induced current is positive when this distance is increased and negative when it is diminished.

These are the elementary cases of induced currents described in Art. 530.

*Mechanical Action between the Two Circuits.*

583.] Let $x$ be any one of the geometrical variables on which the form and relative position of the circuits depend, the electromagnetic force tending to increase $x$ is

$$X = \tfrac{1}{2}\dot{y}_1^2 \frac{dL}{dx} + \dot{y}_1 \dot{y}_2 \frac{dM}{dx} + \tfrac{1}{2}\dot{y}_2^2 \frac{dN}{dx}.$$

If the motion of the system corresponding to the variation of $x$ is such that each circuit moves as a rigid body, $L$ and $N$ will be independent of $x$, and the equation will be reduced to the form

$$X = \dot{y}_1 \dot{y}_2 \frac{dM}{dx}.$$

Hence, if the primary and secondary currents are of the same sign, the force $X$, which acts between the circuits, will tend to move them so as to increase $M$.

If the circuits are placed side by side, and the currents flow in the same direction, $M$ will be increased by their being brought nearer together. Hence the force $X$ is in this case an attraction.

584.] The whole of the phenomena of the mutual action of two circuits, whether the induction of currents or the mechanical force between them, depend on the quantity $M$, which we have called the coefficient of mutual induction. The method of calculating this quantity from the geometrical relations of the circuits is given in Art. 524, but in the investigations of the next chapter we shall not assume a knowledge of the mathematical form of this quantity. We shall consider it as deduced from experiments on induction, as, for instance, by observing the integral current when the secondary circuit is suddenly moved from a given position to an infinite distance, or to any position in which we know that $M = 0$.

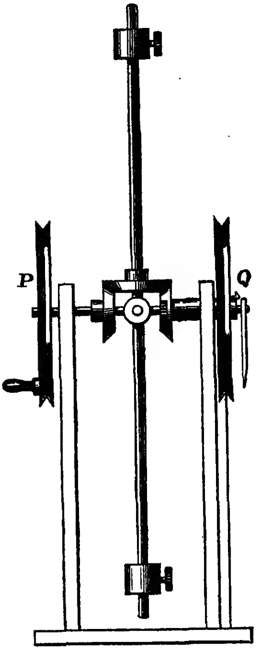

Fig. 34 *a*.

NOTE.—{There is a model in the Cavendish Laboratory designed by Maxwell which illustrates very clearly the laws of the induction of currents.

It is represented in Fig. 34 *a*. $P$ and $Q$ are two disks, the rotation of $P$ represents the primary current, that of $Q$ the secondary. These disks are connected together by a differential gearing. The intermediate wheel carries a fly-wheel the moment of inertia of which can be altered by moving weights inwards or outwards. The resistance of the secondary circuit is represented by the friction of a string passing over $Q$ and kept tight by an elastic band. If the disk $P$ is set in rotation (a current started in the primary) the disk $Q$ will turn in the opposite direction (inverse current when the primary is started). When the velocity of rotation of $P$ becomes uniform, $Q$ is at rest (no current in the secondary when the primary current is constant); if the disk $P$ is stopped, $Q$ commences to rotate in the direction in which $P$ was previously moving (direct current in the secondary on breaking the circuit). The effect of an iron core in increasing the induction can be illustrated by increasing the moment of inertia of the fly-wheel.}

# CHAPTER VIII.

## EXPLORATION OF THE FIELD BY MEANS OF THE SECONDARY CIRCUIT.

585.] WE have proved in Arts. 582, 583, 584 that the electromagnetic action between the primary and the secondary circuit depends on the quantity denoted by $M$, which is a function of the form and relative position of the two circuits.

Although this quantity $M$ is in fact the same as the potential of the two circuits, the mathematical form and properties of which we deduced in Arts. 423, 492, 521, 539 from magnetic and electromagnetic phenomena, we shall here make no reference to these results, but begin again from a new foundation, without any assumptions except those of the dynamical theory as stated in Chapter VII.

The electrokinetic momentum of the secondary circuit consists of two parts (Art. 578), one, $Mi_1$, depending on the primary current $i_1$, while the other, $Ni_2$, depends on the secondary current $i_2$. We are now to investigate the first of these parts, which we shall denote by $p$, where

$$p = Mi_1. \qquad (1)$$

We shall also suppose the primary circuit fixed, and the primary current constant. The quantity $p$, the electrokinetic momentum of the secondary circuit, will in this case depend only on the form and position of the secondary circuit, so that if any closed curve be taken for the secondary circuit, and if the direction along this curve, which is to be reckoned positive, be chosen, the value of $p$ for this closed curve is determinate. If the opposite direction along the curve had been chosen as the positive direction, the sign of the quantity $p$ would have been reversed.

586.] Since the quantity $p$ depends on the form and position of the circuit, we may suppose that each portion of the circuit contributes something to the value of $p$, and that the part contributed by each portion of the circuit depends on the form and position of that portion only, and not on the position of other parts of the circuit.

This assumption is legitimate, because we are not now considering a *current*, the parts of which may, and indeed do, act on one another, but a mere *circuit*, that is, a closed curve along which a current *may* flow, and this is a purely geometrical figure, the parts of which cannot be conceived to have any physical action on each other.

We may therefore assume that the part contributed by the element $ds$ of the circuit is $J\,ds$, where $J$ is a quantity depending on the position and direction of the element $ds$. Hence, the value of $p$ may be expressed as a line-integral

$$p = \int J\,ds, \tag{2}$$

where the integration is to be extended once round the circuit.

587.] We have next to determine the form of the quantity $J$. In the first place, if $ds$ is reversed in direction, $J$ is reversed in sign. Hence, if two circuits $ABCE$ and $AECD$ have the arc $AEC$ common, but reckoned in opposite directions in the two circuits, the sum of the values of $p$ for the two circuits $ABCE$ and $AECD$ will be equal to the value of $p$ for the circuit $ABCD$, which is made up of the two circuits.

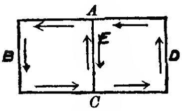

Fig. 35.

For the parts of the line-integral depending on the arc $AEC$ are equal but of opposite sign in the two partial circuits, so that they destroy each other when the sum is taken, leaving only those parts of the line-integral which depend on the external boundary of $ABCD$.

In the same way we may shew that if a surface bounded by a closed curve be divided into any number of parts, and if the boundary of each of these parts be considered as a circuit, the positive direction round every circuit being the same as that round the external closed curve, then the value of $p$ for the closed curve is equal to the sum of the values of $p$ for all the circuits. See Art. 483.

588.] Let us now consider a portion of a surface, the dimensions of which are so small with respect to the principal radii of curvature of the surface that the variation of the direction of the normal within this portion may be neglected. We shall also suppose that if any very small circuit be carried parallel to itself from one part of this portion to another, the value of $p$ for the small circuit is not sensibly altered. This will evidently be the case if the dimensions of the portion of surface are small enough compared with its distance from the primary circuit.

*If any closed curve be drawn on this portion of the surface, the value of $p$ will be proportional to its area.*

For the areas of any two circuits may be divided into small elements all of the same dimensions, and having the same value of $p$. The areas of the two circuits are as the numbers of these elements which they contain, and the values of $p$ for the two circuits are also in the same proportion.

Hence, the value of $p$ for the circuit which bounds any element $dS$ of a surface is of the form

$$IdS,$$

where $I$ is a quantity depending on the position of $dS$ and on the direction of its normal. We have therefore a new expression for $p$,

$$p = \iint IdS, \tag{3}$$

where the double integral is extended over any surface bounded by the circuit.

589.] Let $ABCD$ be a circuit, of which $AC$ is an elementary portion, so small that it may be considered straight. Let $APB$ and $CQB$ be small equal areas in the same plane, then the value of $p$ will be the same for the small circuits $APB$ and $CQB$,

or $$p(APB) = p(CQB).$$

Hence $$p(APBQCD) = p(ABQCD) + p(APB),$$
$$= p(ABQCD) + p(CQB),$$
$$= p(ABCD),$$

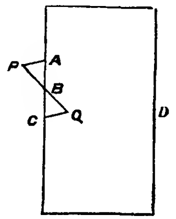

Fig. 36.

or the value of $p$ is not altered by the substitution of the crooked line $APQC$ for the straight line $AC$, provided the area of the circuit is not sensibly altered. This, in fact, is the principle established by Ampère's second experiment (Art. 506), in which a crooked portion of a circuit is shewn to be equivalent to a straight portion provided no part of the crooked portion is at a sensible distance from the straight portion.

If therefore we substitute for the element $ds$ three small elements, $dx$, $dy$, and $dz$, drawn in succession, so as to form a continuous path from the beginning to the end of the element $ds$, and if $F\,dx$, $G\,dy$, and $H\,dz$ denote the elements of the line-integral corresponding to $dx$, $dy$, and $dz$ respectively, then

$$J\,ds = F\,dx + G\,dy + H\,dz. \tag{4}$$

590.] We are now able to determine the mode in which the quantity $J$ depends on the direction of the element $ds$. For, by (4),

$$J = F\frac{dx}{ds} + G\frac{dy}{ds} + H\frac{dz}{ds}. \tag{5}$$

This is the expression for the resolved part, in the direction of $ds$, of a vector, the components of which, resolved in the directions of the axes of $x$, $y$, and $z$, are $F$, $G$, and $H$ respectively.

If this vector be denoted by $\mathfrak{A}$, and the vector from the origin to a point of the circuit by $\rho$, the element of the circuit will be $d\rho$, and the quaternion expression for $J\,ds$ will be

$$-S\,.\,\mathfrak{A}\,d\rho.$$

We may now write equation (2) in the form

$$p = \int\Big(F\frac{dx}{ds} + G\frac{dy}{ds} + H\frac{dz}{ds}\Big)ds, \tag{6}$$

$$\text{or } p = -\int S\,.\,\mathfrak{A}\,d\rho. \tag{7}$$

The vector $\mathfrak{A}$ and its constituents $F$, $G$, $H$ depend on the position of $ds$ in the field, and not on the direction in which it is drawn. They are therefore functions of $x$, $y$, $z$, the co-ordinates of $ds$, and not of $l$, $m$, $n$, its direction-cosines.

The vector $\mathfrak{A}$ represents in direction and magnitude the time-integral of the electromotive intensity which a particle placed at the point $(x, y, z)$ would experience if the primary current were suddenly stopped. We shall therefore call it the Electrokinetic Momentum *at the point* $(x, y, z)$. It is identical with the quantity which we investigated in Art. 405 under the name of the vector-potential of magnetic induction.

The electrokinetic momentum of any finite line or circuit is the line-integral, extended along the line or circuit, of the resolved part of the electrokinetic momentum at each point of the same.

591.] Let us next determine the value of $p$ for the elementary rectangle $ABCD$, of which the sides are $dy$ and $dz$, the positive direction being from the direction of the axis of $y$ to that of $z$.

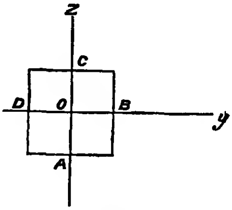

Fig. 37.

Let the coordinates of $O$, the centre of gravity of the element, be $x_0$, $y_0$, $z_0$, and let $G_0$, $H_0$ be the values of $G$ and of $H$ at this point.

The coordinates of $A$, the middle point of the first side of the rectangle, are $y_0$ and $z_0 - \frac{1}{2}dz$. The corresponding value of $G$ is

$$G = G_0 - \frac{1}{2}\frac{dG}{dz}dz + \&c., \tag{8}$$

and the part of the value of $p$ which arises from the side $A$ is approximately

$$G_0 dy - \frac{1}{2}\frac{dG}{dz}dy\,dz. \tag{9}$$

Similarly, for $B$, $\quad H_0 dz + \frac{1}{2}\frac{dH}{dy}dy\,dz,$

for $C$, $\quad -G_0 dy - \frac{1}{2}\frac{dG}{dz}dy\,dz,$

for $D$, $\quad -H_0 dz + \frac{1}{2}\frac{dH}{dy}dy\,dz.$

Adding these four quantities, we find the value of $p$ for the rectangle, viz.

$$p = \left(\frac{dH}{dy} - \frac{dG}{dz}\right)dy\,dz. \tag{10}$$

If we now assume three new quantities, $a$, $b$, $c$, such that

$$\left.\begin{aligned} a &= \frac{dH}{dy} - \frac{dG}{dz}, \\ b &= \frac{dF}{dz} - \frac{dH}{dx}, \\ c &= \frac{dG}{dx} - \frac{dF}{dy}, \end{aligned}\right\} \tag{A}$$

and consider these as the constituents of a new vector $\mathfrak{B}$, then by Theorem IV, Art. 24, we may express the line-integral of $\mathfrak{A}$ round any circuit in the form of the surface-integral of $\mathfrak{B}$ over a surface bounded by the circuit, thus

$$p = \int\left(F\frac{dx}{ds} + G\frac{dy}{ds} + H\frac{dz}{ds}\right)ds = \iint(la + mb + nc)\,dS, \tag{11}$$

or
$$p = \int T.\mathfrak{A} \cos \epsilon\, ds = \iint T.\mathfrak{B} \cos \eta\, dS, \tag{12}$$
where $\epsilon$ is the angle between $\mathfrak{A}$ and $ds$, and $\eta$ that between $\mathfrak{B}$ and the normal to $dS$, whose direction-cosines are $l$, $m$, $n$, and $T.\mathfrak{A}$, $T.\mathfrak{B}$ denote the numerical values of $\mathfrak{A}$ and $\mathfrak{B}$.

Comparing this result with equation (3), it is evident that the quantity $I$ in that equation is equal to $\mathfrak{B} \cos \eta$, or the resolved part of $\mathfrak{B}$ normal to $dS$.

592.] We have already seen (Arts. 490, 541) that, according to Faraday's theory, the phenomena of electromagnetic force and induction in a circuit depend on the variation of the number of lines of magnetic induction which pass through the circuit. Now the number of these lines is expressed mathematically by the surface-integral of the magnetic induction through any surface bounded by the circuit. Hence, we must regard the vector $\mathfrak{B}$ and its components $a$, $b$, $c$ as representing what we are already acquainted with as the magnetic induction and its components.

In the present investigation we propose to deduce the properties of this vector from the dynamical principles stated in the last chapter, with as few appeals to experiment as possible.

In identifying this vector, which has appeared as the result of a mathematical investigation, with the magnetic induction, the properties of which we learned from experiments on magnets, we do not depart from this method, for we introduce no new fact into the theory, we only give a name to a mathematical quantity, and the propriety of so doing is to be judged by the agreement of the relations of the mathematical quantity with those of the physical quantity indicated by the name.

The vector $\mathfrak{B}$, since it occurs in a surface-integral, belongs evidently to the category of fluxes described in Art. 12. The vector $\mathfrak{A}$, on the other hand, belongs to the category of forces, since it appears in a line-integral.

593.] We must here recall to mind the conventions about positive and negative quantities and directions, some of which were stated in Art. 23. We adopt the right-handed system of axes, so that if a right-handed screw is placed in the direction of the axis of $x$, and a nut on this screw is turned in the positive direction of rotation, that is, from the direction of $y$ to that of $z$, it will move along the screw in the positive direction of $x$.

We also consider vitreous electricity and austral magnetism as positive. The positive direction of an electric current, or of a line of electric induction, is the direction in which positive electricity moves or tends to move, and the positive direction of a line of magnetic induction is the direction in which a compass needle points with that end which turns to the north. See Fig. 24, Art. 498, and Fig. 25, Art. 501.

The student is recommended to select whatever method appears to him most effectual in order to fix these conventions securely in his memory, for it is far more difficult to remember a rule which determines in which of two previously indifferent ways a statement is to be made, than a rule which selects one way out of many.

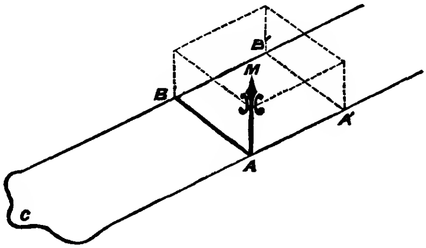


Fig. 38.

594.] We have next to deduce from dynamical principles the expressions for the electromagnetic force acting on a conductor carrying an electric current through the magnetic field, and for the electromotive force acting on the electricity within a body moving in the magnetic field. The mathematical method which we shall adopt may be compared with the experimental method used by Faraday* in exploring the field by means of a wire, and with what we have already done in Art. 490, by a method founded on experiments. What we have now to do is to determine the effect on the value of $p$, the electrokinetic momentum of the secondary circuit, due to given alterations of the form of that circuit.

Let $AA'$, $BB'$ be two parallel straight conductors connected by the conducting arc $C$, which may be of any form, and by a straight conductor $AB$, which is capable of sliding parallel to itself along the conducting rails $AA'$ and $BB'$.

* *Exp. Res.*, 3082, 3087, 3113.

Let the circuit thus formed be considered as the secondary circuit, and let the direction $ABC$ be assumed as the positive direction round it.

Let the sliding piece move parallel to itself from the position $AB$ to the position $A'B'$. We have to determine the variation of $p$, the electrokinetic momentum of the circuit, due to this displacement of the sliding piece.

The secondary circuit is changed from $ABC$ to $A'B'C$, hence, by Art. 587, $p\,(A'B'C)-p\,(ABC) = p\,(AA'B'B)$. (13)

We have therefore to determine the value of $p$ for the parallelogram $AA'B'B$. If this parallelogram is so small that we may neglect the variations of the direction and magnitude of the magnetic induction at different points of its plane, the value of $p$ is, by Art. 591, $\mathfrak{B}\cos\eta\,.\,AA'B'B$, where $\mathfrak{B}$ is the magnetic induction, and $\eta$ the angle which it makes with the positive direction of the normal to the parallelogram $AA'B'B$.

We may represent the result geometrically by the volume of the parallelepiped, whose base is the parallelogram $AA'B'B$, and one of whose edges is the line $AM$, which represents in direction and magnitude the magnetic induction $\mathfrak{B}$. If the parallelogram is in the plane of the paper, and if $AM$ is drawn upwards from the paper, or more generally, if the directions of the circuit $AB$, of the magnetic induction $AM$, and of the displacement $AA'$, form a right-handed system when taken in this cyclical order, the volume of the parallelepiped is to be taken positively.

The volume of this parallelepiped represents the increment of the value of $p$ for the secondary circuit due to the displacement of the sliding piece from $AB$ to $A'B'$.

### *Electromotive Force acting on the Sliding Piece.*

595.] The electromotive force produced in the secondary circuit by the motion of the sliding piece is, by Art. 579,

$$E = -\frac{dp}{dt}. \tag{14}$$

If we suppose $AA'$ to be the displacement in unit of time, then $AA'$ will represent the velocity, and the parallelepiped will represent $\frac{dp}{dt}$, and therefore, by equation (14), the electromotive force in the negative direction $BA$.

Hence, the electromotive force acting on the sliding piece $AB$, in consequence of its motion through the magnetic field, is represented by the volume of the parallelepiped, whose edges represent in direction and magnitude—the velocity, the magnetic induction, and the sliding piece itself, and is positive when these three directions are in right-handed cyclical order.

### *Electromagnetic Force acting on the Sliding Piece.*

596.] Let $i_2$ denote the current in the secondary circuit in the positive direction $ABC$, then the work done by the electromagnetic force on $AB$ while it slides from the position $AB$ to the position $A'B'$ is $(M'-M)\, i_1\, i_2$, where $M$ and $M'$ are the values of $M_{12}$ in the initial and final positions of $AB$. But $(M'-M)\, i_1$ is equal to $p'-p$, and this is represented by the volume of the parallelepiped on $AB$, $AM$, and $AA'$. Hence, if we draw a line parallel to $AB$ to represent the quantity $AB \,.\, i_2$, the parallelepiped contained by this line, by $AM$, the magnetic induction, and by $AA'$, the displacement, will represent the work done during this displacement.

For a given distance of displacement this will be greatest when the displacement is perpendicular to the parallelogram whose sides are $AB$ and $AM$. The electromagnetic force is therefore represented by the area of the parallelogram on $AB$ and $AM$ multiplied by $i_2$, and is in the direction of the normal to this parallelogram, drawn so that $AB$, $AM$, and the normal are in right-handed cyclical order.

### *Four Definitions of a Line of Magnetic Induction.*

597.] If the direction $AA'$, in which the motion of the sliding piece takes place, coincides with $AM$, the direction of the magnetic induction, the motion of the sliding piece will not call electromotive force into action, whatever be the direction of $AB$, and if $AB$ carries an electric current there will be no tendency to slide along $AA'$.

Again, if $AB$, the sliding piece, coincides in direction with $AM$, the direction of magnetic induction, there will be no electromotive force called into action by any motion of $AB$, and a current through $AB$ will not cause $AB$ to be acted on by mechanical force.

We may therefore define a line of magnetic induction in four different ways. It is a line such that

(1) If a conductor be moved along it parallel to itself it will experience no electromotive force.

(2) If a conductor carrying a current be free to move along a line of magnetic induction it will experience no tendency to do so.

(3) If a linear conductor coincide in direction with a line of magnetic induction, and be moved parallel to itself in any direction, it will experience no electromotive force in the direction of its length.

(4) If a linear conductor carrying an electric current coincide in direction with a line of magnetic induction it will not experience any mechanical force.

### *General Equations of Electromotive Intensity.*

598.] We have seen that $E$, the electromotive force due to induction acting on the secondary circuit, is equal to $-\frac{dp}{dt}$, where

$$p = \int\left(F\frac{dx}{ds} + G\frac{dy}{ds} + H\frac{dz}{ds}\right)ds. \tag{1}$$

To determine the value of $E$, let us differentiate the quantity under the integral sign with respect to $t$, remembering that if the secondary circuit is in motion, $x$, $y$, and $z$ are functions of the time. We obtain

$$\begin{aligned} E = &-\int\left(\frac{dF}{dt}\frac{dx}{ds} + \frac{dG}{dt}\frac{dy}{ds} + \frac{dH}{dt}\frac{dz}{ds}\right)ds \\ &-\int\left(\frac{dF}{dx}\frac{dx}{ds} + \frac{dG}{dx}\frac{dy}{ds} + \frac{dH}{dx}\frac{dz}{ds}\right)\frac{dx}{dt}ds \\ &-\int\left(\frac{dF}{dy}\frac{dx}{ds} + \frac{dG}{dy}\frac{dy}{ds} + \frac{dH}{dy}\frac{dz}{ds}\right)\frac{dy}{dt}ds \\ &-\int\left(\frac{dF}{dz}\frac{dx}{ds} + \frac{dG}{dz}\frac{dy}{ds} + \frac{dH}{dz}\frac{dz}{ds}\right)\frac{dz}{dt}ds \\ &-\int\left(F\frac{d^2x}{ds\,dt} + G\frac{d^2y}{ds\,dt} + H\frac{d^2z}{ds\,dt}\right)ds. \end{aligned} \tag{2}$$

Now consider the second line of the integral, and substitute from equations (A), Art. 591, the values of $\frac{dG}{dx}$ and $\frac{dH}{dx}$. This line then becomes,

$$-\int\left(c\frac{dy}{ds}-b\frac{dz}{ds}+\frac{dF}{dx}\frac{dx}{ds}+\frac{dF}{dy}\frac{dy}{ds}+\frac{dF}{dz}\frac{dz}{ds}\right)\frac{dx}{dt}ds,$$

which we may write

$$-\int\left(c\frac{dy}{ds}-b\frac{dz}{ds}+\frac{dF}{ds}\right)\frac{dx}{dt}ds.$$

Treating the third and fourth lines in the same way, and collecting the terms in $\frac{dx}{ds}$, $\frac{dy}{ds}$, and $\frac{dz}{ds}$, remembering that

$$\int\left(\frac{dF}{ds}\frac{dx}{dt}+F\frac{d^2x}{ds\,dt}\right)ds=F\frac{dx}{dt},\tag{3}$$

and therefore that the integral, when taken round the closed curve, vanishes,

$$\begin{aligned}E=&\int\left(c\frac{dy}{dt}-b\frac{dz}{dt}-\frac{dF}{dt}\right)\frac{dx}{ds}ds\\&+\int\left(a\frac{dz}{dt}-c\frac{dx}{dt}-\frac{dG}{dt}\right)\frac{dy}{ds}ds\\&+\int\left(b\frac{dx}{dt}-a\frac{dy}{dt}-\frac{dH}{dt}\right)\frac{dz}{ds}ds.\end{aligned}\tag{4}$$

We may write this expression in the form

$$E=\int\left(P\frac{dx}{ds}+Q\frac{dy}{ds}+R\frac{dz}{ds}\right)ds,\tag{5}$$

where

$$\left.\begin{aligned}P&=c\frac{dy}{dt}-b\frac{dz}{dt}-\frac{dF}{dt}-\frac{d\Psi}{dx},\\Q&=a\frac{dz}{dt}-c\frac{dx}{dt}-\frac{dG}{dt}-\frac{d\Psi}{dy},\\R&=b\frac{dx}{dt}-a\frac{dy}{dt}-\frac{dH}{dt}-\frac{d\Psi}{dz}.\end{aligned}\right\}\ \text{Equations of Electromotive Intensity.}\tag{B}$$

The terms involving the new quantity $\Psi$ are introduced for the sake of giving generality to the expressions for $P$, $Q$, $R$. They disappear from the integral when extended round the closed circuit. The quantity $\Psi$ is therefore indeterminate as far as regards the problem now before us, in which the electromotive force round the circuit is to be determined. We shall find, however, that when we know all the circumstances of the problem, we can assign a definite value to $\Psi$, and that it represents, according to a certain definition, the *electric potential* at the point $(x, y, z)$.

The quantity under the integral sign in equation (5) represents the electromotive intensity acting on the element $ds$ of the circuit.

If we denote by $T.\mathfrak{E}$, the numerical value of the resultant of $P$, $Q$, and $R$, and by $\epsilon$, the angle between the direction of this resultant and that of the element $ds$, we may write equation (5),

$$E = \int T.\mathfrak{E} \cos \epsilon \, ds. \tag{6}$$

The vector $\mathfrak{E}$ is the electromotive intensity at the moving element $ds$. Its direction and magnitude depend on the position and motion of $ds$, and on the variation of the magnetic field, but not on the direction of $ds$. Hence we may now disregard the circumstance that $ds$ forms part of a circuit, and consider it simply as a portion of a moving body, acted on by the electromotive intensity $\mathfrak{E}$. The electromotive intensity has already been defined in Art. 68. It is also called the resultant electrical intensity, being the force which would be experienced by a unit of positive electricity placed at that point. We have now obtained the most general value of this quantity in the case of a body moving in a magnetic field due to a variable electric system.

If the body is a conductor, the electromotive force will produce a current; if it is a dielectric, the electromotive force will produce only electric displacement.

The electromotive intensity, or the force on a particle, must be carefully distinguished from the electromotive force along an arc of a curve, the latter quantity being the line-integral of the former. See Art. 69.

599.] The electromotive intensity, the components of which are defined by equations (B), depends on three circumstances. The first of these is the motion of the particle through the magnetic field. The part of the force depending on this motion is expressed by the first two terms on the right of each equation. It depends on the velocity of the particle transverse to the lines of magnetic induction. If $\mathfrak{G}$ is a vector representing the velocity, and $\mathfrak{B}$ another representing the magnetic induction, then if $\mathfrak{E}_1$ is the part of the electromotive intensity depending on the motion,

$$\mathfrak{E}_1 = V.\mathfrak{G}\mathfrak{B}, \tag{7}$$

or, the electromotive intensity is the vector part of the product of the magnetic induction multiplied by the velocity, that is to say, the magnitude of the electromotive intensity is represented by the area of the parallelogram, whose sides represent the velocity and the magnetic induction, and its direction is the normal to this parallelogram, drawn so that the velocity, the magnetic induction, and the electromotive intensity are in right-handed cyclical order.

The third term in each of the equations (B) depends on the time-variation of the magnetic field. This may be due either to the time-variation of the electric current in the primary circuit, or to motion of the primary circuit. Let $\mathfrak{E}_2$ be the part of the electromotive intensity which depends on these terms. Its components are

$$-\frac{dF}{dt}, \quad -\frac{dG}{dt}, \quad \text{and} \quad -\frac{dH}{dt},$$

and these are the components of the vector, $-\dfrac{d\mathfrak{A}}{dt}$ or $-\dot{\mathfrak{A}}$.

Hence,
$$\mathfrak{E}_2 = -\dot{\mathfrak{A}}. \tag{8}$$

The last term of each equation (B) is due to the variation of the function $\Psi$ in different parts of the field. We may write the third part of the electromotive intensity, which is due to this cause,
$$\mathfrak{E}_3 = -\nabla\Psi. \tag{9}$$

The electromotive intensity, as defined by equations (B), may therefore be written in the quaternion form,

$$\mathfrak{E} = V.\,\mathfrak{G}\mathfrak{B} - \dot{\mathfrak{A}} - \nabla\Psi. \tag{10}$$

*On the Modification of the Equations of Electromotive Intensity when the Axes to which they are referred are moving in Space.*

600.] Let $x'$, $y'$, $z'$ be the coordinates of a point referred to a system of rectangular axes moving in space, and let $x, y, z$ be the coordinates of the same point referred to fixed axes.

Let the components of the velocity of the origin of the moving system be $u$, $v$, $w$, and those of its angular velocity $\omega_1$, $\omega_2$, $\omega_3$ referred to the fixed system of axes, and let us choose the fixed axes so as to coincide at the given instant with the moving ones, then the only quantities which will be different for the two systems of axes will be those differentiated with respect to the time. If $\dfrac{\delta x}{\delta t}$ denotes a component velocity at a point moving in rigid connexion with the moving axes, and $\dfrac{dx}{dt}$ and $\dfrac{dx'}{dt}$ those of any moving point, having the same instantaneous position, referred to the fixed and the moving axes respectively, then

$$\frac{dx}{dt} = \frac{\delta x}{\delta t} + \frac{dx'}{dt}, \tag{1}$$

with similar equations for the other components.

By the theory of the motion of a body of invariable form,

$$\left.\begin{aligned} \frac{\delta x}{\delta t} &= u + \omega_2 z - \omega_3 y, \\ \frac{\delta y}{\delta t} &= v + \omega_3 x - \omega_1 z, \\ \frac{\delta z}{\delta t} &= w + \omega_1 y - \omega_2 x. \end{aligned}\right\} \tag{2}$$

Since $F$ is a component of a directed quantity parallel to $x$, if $\frac{dF'}{dt}$ be the value of $\frac{dF}{dt}$ referred to the moving axes, it may be shewn that

$$\frac{dF'}{dt} = \frac{dF}{dx}\frac{\delta x}{\delta t} + \frac{dF}{dy}\frac{\delta y}{\delta t} + \frac{dF}{dz}\frac{\delta z}{\delta t} + G\omega_3 - H\omega_2 + \frac{dF}{dt}. \tag{3}$$

Substituting for $\frac{dF}{dy}$ and $\frac{dF}{dz}$ their values as deduced from the equations (A) of magnetic induction, and remembering that, by (2),

$$\frac{d}{dx}\frac{\delta x}{\delta t} = 0, \qquad \frac{d}{dx}\frac{\delta y}{\delta t} = \omega_3, \qquad \frac{d}{dx}\frac{\delta z}{\delta t} = -\omega_2, \tag{4}$$

we find

$$\begin{aligned} \frac{dF'}{dt} = \frac{dF}{dx}\frac{\delta x}{\delta t} + F\frac{d}{dx}\frac{\delta x}{\delta t} + \frac{dG}{dx}\frac{\delta y}{\delta t} + G\frac{d}{dx}\frac{\delta y}{\delta t} + \frac{dH}{dx}\frac{\delta z}{\delta t} + H\frac{d}{dx}\frac{\delta z}{\delta t} \\ - c\frac{\delta y}{\delta t} + b\frac{\delta z}{\delta t} + \frac{dF}{dt}. \end{aligned} \tag{5}$$

If we now put

$$-\Psi' = F\frac{\delta x}{\delta t} + G\frac{\delta y}{\delta t} + H\frac{\delta z}{\delta t}, \tag{6}$$

$$\frac{dF'}{dt} = -\frac{d\Psi'}{dx} - c\frac{\delta y}{\delta t} + b\frac{\delta z}{\delta t} + \frac{dF}{dt}. \tag{7}$$

The equation for $P$, the component of the electromotive intensity parallel to $x$, is, by (B),

$$P = c\frac{dy}{dt} - b\frac{dz}{dt} - \frac{dF}{dt} - \frac{d\Psi}{dx}, \tag{8}$$

referred to the fixed axes. Substituting the values of the quantities as referred to the moving axes, we have

$$P' = c\frac{dy'}{dt} - b\frac{dz'}{dt} - \frac{dF'}{dt} - \frac{d(\Psi+\Psi')}{dx}, \tag{9}$$

for the value of $P$ referred to the moving axes.

601.] It appears from this that the electromotive intensity is expressed by a formula of the same type, whether the motions of the conductors be referred to fixed axes or to axes moving in space, the only difference between the formulæ being that in the case of moving axes the electric potential $\Psi$ must be changed into $\Psi+\Psi'$.

In all cases in which a current is produced in a conducting circuit, the electromotive force is the line-integral

$$E = \int\left(P\frac{dx}{ds} + Q\frac{dy}{ds} + R\frac{dz}{ds}\right)ds, \tag{10}$$

taken round the curve. The value of $\Psi$ disappears from this integral, so that the introduction of $\Psi'$ has no influence on its value. In all phenomena, therefore, relating to closed circuits and the currents in them, it is indifferent whether the axes to which we refer the system be at rest or in motion. See Art. 668.

*On the Electromagnetic Force acting on a Conductor which carries an Electric Current through a Magnetic Field.*

602.] We have seen in the general investigation, Art. 583, that if $x_1$ is one of the variables which determine the position and form of the secondary circuit, and if $X_1$ is the force acting on the secondary circuit tending to increase this variable, then

$$X_1 = \frac{dM}{dx_1} i_1 i_2. \tag{1}$$

Since $i_1$ is independent of $x_1$, we may write

$$M i_1 = p = \int\left(F\frac{dx}{ds} + G\frac{dy}{ds} + H\frac{dz}{ds}\right)ds, \tag{2}$$

and we have for the value of $X_1$,

$$X_1 = i_2\frac{d}{dx_1}\int\left(F\frac{dx}{ds} + G\frac{dy}{ds} + H\frac{dz}{ds}\right)ds. \tag{3}$$

Now let us suppose that the displacement consists in moving every point of the circuit through a distance $\delta x$ in the direction of $x$, $\delta x$ being any continuous function of $s$, so that the different parts of the circuit move independently of each other, while the circuit remains continuous and closed.

Also let $X$ be the total force in the direction of $x$ acting on the part of the circuit from $s = 0$ to $s = s$, then the part corresponding to the element $ds$ will be $\frac{dX}{ds}ds$. We shall then have the following expression for the work done by the force during the displacement,

$$\int \frac{dX}{ds}\,\delta x\, ds = i_2 \int \frac{d}{d\,\delta x}\left(F\frac{dx}{ds} + G\frac{dy}{ds} + H\frac{dz}{ds}\right)\delta x\, ds, \tag{4}$$

where the integration is to be extended round the closed curve, remembering that $\delta x$ is an arbitrary function of $s$. We may therefore perform the differentiation with respect to $\delta x$ in the same way that we differentiated with respect to $t$ in Art. 598, remembering that

$$\frac{dx}{d\,\delta x} = 1, \quad \frac{dy}{d\,\delta x} = 0, \quad \text{and} \quad \frac{dz}{d\,\delta x} = 0. \tag{5}$$

We thus find

$$\int \frac{dX}{ds}\,\delta x\, ds = i_2 \int \left(c\frac{dy}{ds} - b\frac{dz}{ds}\right)\delta x ds + i_2 \int \frac{d}{ds}(F\delta x) ds. \tag{6}$$

The last term vanishes when the integration is extended round the closed curve, and since the equation must hold for all forms of the function $\delta x$, we must have

$$\frac{dX}{ds} = i_2\left(c\frac{dy}{ds} - b\frac{dz}{ds}\right), \tag{7}$$

an equation which gives the force parallel to $x$ on any unit element of the circuit. The forces parallel to $y$ and $z$ are

$$\frac{dY}{ds} = i_2\left(a\frac{dz}{ds} - c\frac{dx}{ds}\right), \tag{8}$$

$$\frac{dZ}{ds} = i_2\left(b\frac{dx}{ds} - a\frac{dy}{ds}\right). \tag{9}$$

The resultant force on the element is given in direction and magnitude by the quaternion expression $i_2 V . d\rho\mathfrak{B}$, where $i_2$ is the numerical measure of the current, and $d\rho$ and $\mathfrak{B}$ are vectors representing the element of the circuit and the magnetic induction, and the multiplication is to be understood in the Hamiltonian sense.

603.] If the conductor is to be treated not as a line but as a body, we must express the force on the element of length, and the current through the complete section, in terms of symbols denoting the force per unit of volume, and the current per unit of area.

Let $X$, $Y$, $Z$ now represent the components of the force referred to unit of volume, and $u, v, w$ those of the current referred to unit of area. Then, if $S$ represents the section of the conductor, which we shall suppose small, the volume of the element $ds$ will be $Sds$, and $u = \frac{i_2}{S}\frac{dx}{ds}$. Hence, equation (7) will become

$$\frac{XSds}{ds} = S(vc - wb), \tag{10}$$

or
Similarly
and

$$\left.\begin{aligned} X &= vc - wb, \\ Y &= wa - uc, \\ Z &= ub - va. \end{aligned}\right\} \quad \text{(Equations of Electromagnetic Force.)} \tag{C}$$

Here $X, Y, Z$ are the components of the electromagnetic force on an element of a conductor divided by the volume of that element; $u, v, w$ are the components of the electric current through the element referred to unit of area, and $a, b, c$ are the components of the magnetic induction at the element, which are also referred to unit of area.

If the vector $\mathfrak{F}$ represents in magnitude and direction the force acting on unit of volume of the conductor, and if $\mathfrak{C}$ represents the electric current flowing through it,

$$\mathfrak{F} = V.\mathfrak{C}\mathfrak{B}. \tag{11}$$

[The equations (B) of Art. 598 may be proved by the following method, derived from Professor Maxwell's Memoir on A Dynamical Theory of the Electromagnetic Field. *Phil. Trans.* 1865, pp. 459–512.

The time variation of $-p$ may be taken in two parts, one of which depends and the other does not depend on the motion of the circuit. The latter part is clearly

$$-\int\left(\frac{dF}{dt}dx + \frac{dG}{dt}dy + \frac{dH}{dt}dz\right).$$

To find the former let us consider an arc $\delta s$ forming part of a circuit, and let us imagine this arc to move along rails, which may be taken as parallel, with velocity $v$ whose components are $\dot{x}, \dot{y}, \dot{z}$, the rest of the circuit being meanwhile supposed stationary. We may then suppose that a small parallelogram is generated by the moving arc, the direction-cosines of the normal to which are

$$\lambda, \mu, \nu = \frac{n\dot{y} - m\dot{z}}{v\sin\theta}, \quad \frac{l\dot{z} - n\dot{x}}{v\sin\theta}, \quad \frac{m\dot{x} - l\dot{y}}{v\sin\theta},$$

where $l, m, n$ are the direction-cosines of $\delta s$, and $\theta$ is the angle between $v$ and $\delta s$.

To verify the signs of $\lambda, \mu, \nu$ we may put $m = -1$, $\dot{x} = v$; they then become 0, 0, $-1$ as they ought to do with a right-handed system of axes.

Now let $a, b, c$ be the components of magnetic induction, we then have, due to the motion of $\delta s$ in time $\delta t$,

$$\delta p = (a\lambda + b\mu + c\nu)\, v\,\delta t\,\delta s \sin\theta.$$

If we suppose each part of the circuit to move in a similar manner the resultant effect will be the motion of the circuit as a whole, the currents in the rails forming a balance in each case of two adjacent arcs. The time variation of $-p$ due to the motion of the circuit is therefore

$$-\int\{a(n\dot{y} - m\dot{z}) + \text{two similar terms}\}\, ds$$

taken round the circuit

$$= \int(c\dot{y} - b\dot{z})\, dx + \text{two similar integrals}.$$

The results in Art. 602 for the components of electromagnetic force may be deduced from the above expression for $\delta p$; viz. let the arc $\delta s$ be displaced in the direction $l'$, $m'$, $n'$ through a distance $\delta s'$, then

$$\delta p = \{l'(cm - bn) + \text{two similar terms}\}\, \delta s\, \delta s'.$$

Now let $X$ be the $x$-component of the force upon the arc $s$, then for unit current we find by Art. 596,

$$\frac{dX}{ds} = \frac{dp}{dx}$$
$$= cm - bn.]$$

*Equations of the Electromagnetic Field.*

{If we assume that electric currents always flow in closed circuits, we can without introducing the vector-potential deduce equations which will determine the state of the electromagnetic field.

For let $i$ be the strength of the current round any circuit which we shall assume to be at rest. The electrokinetic energy $T$ due to this current is

$$i\iint (la + mb + nc)\, dS,$$

where $dS$ is an element of a surface bounded by the current.

Hence $-\frac{d}{dt}\frac{dT}{di}$ the total electromotive force round the circuit tending to increase $i$ equals

$$-\iint\left(l\frac{da}{dt} + m\frac{db}{dt} + n\frac{dc}{dt}\right) dS;$$

hence if $X$, $Y$, $Z$ are the components of the electromotive intensity

$$\int (X dx + Y dy + Z dz) = -\iint\left(l\frac{da}{dt} + m\frac{db}{dt} + n\frac{dc}{dt}\right) dS; \tag{1}$$

but by Stokes' Theorem the left-hand side of this equation is equal to

$$\iint\left\{l\left(\frac{dZ}{dy} - \frac{dY}{dz}\right) + m\left(\frac{dX}{dz} - \frac{dZ}{dx}\right) + n\left(\frac{dY}{dx} - \frac{dX}{dy}\right)\right\} dS.$$

Equating this integral to the right-hand side of equation (1), we obtain, since the surface closing up the current is quite arbitrary,

$$\frac{dZ}{dy} - \frac{dY}{dz} = -\frac{da}{dt},$$
$$\frac{dX}{dz} - \frac{dZ}{dx} = -\frac{db}{dt},$$
$$\frac{dY}{dx} - \frac{dX}{dy} = -\frac{dc}{dt}.$$

These with the relations

$$4\pi u = \frac{d\gamma}{dy} - \frac{d\beta}{dz},$$
$$4\pi v = \frac{d\alpha}{dz} - \frac{d\gamma}{dx},$$
$$4\pi w = \frac{d\beta}{dx} - \frac{d\alpha}{dy},$$

$$u = \frac{X}{\sigma}, \quad v = \frac{Y}{\sigma}, \quad w = \frac{Z}{\sigma}$$

in a conductor whose specific resistance is $\sigma$;

or $$u = \frac{K}{4\pi}\frac{dX}{dt}, \quad v = \frac{K}{4\pi}\frac{dY}{dt}, \quad w = \frac{K}{4\pi}\frac{dZ}{dt}$$

in an insulator whose specific inductive capacity is $K$, are sufficient to determine the state of the electromagnetic field. The boundary conditions at any surface are that the magnetic induction normal to the surface should be continuous, and that the magnetic force parallel to the surface should also be continuous.

This method of investigating the electromagnetic field has the merit of simplicity. It has been strongly supported by Mr. Heaviside. It is not however so general as the method in the text, which could be applied even if the currents did not always flow in closed circuits.}

# CHAPTER IX.

## GENERAL EQUATIONS OF THE ELECTROMAGNETIC FIELD.

604.] In our theoretical discussion of electrodynamics we began by assuming that a system of circuits carrying electric currents is a dynamical system, in which the currents may be regarded as velocities, and in which the coordinates corresponding to these velocities do not themselves appear in the equations. It follows from this that the kinetic energy of the system, in so far as it depends on the currents, is a homogeneous quadratic function of the currents, in which the coefficients depend only on the form and relative position of the circuits. Assuming these coefficients to be known, by experiment or otherwise, we deduced, by purely dynamical reasoning, the laws of the induction of currents, and of electromagnetic attraction. In this investigation we introduced the conceptions of the electrokinetic energy of a system of currents, of the electromagnetic momentum of a circuit, and of the mutual potential of two circuits.

We then proceeded to explore the field by means of various configurations of the secondary circuit, and were thus led to the conception of a vector $\mathfrak{A}$, having a determinate magnitude and direction at any given point of the field. We called this vector the electromagnetic momentum at that point. This quantity may be considered as the time-integral of the electromotive intensity which would be produced at that point by the sudden removal of all the currents from the field. It is identical with the quantity already investigated in Art. 405 as the vector-potential of magnetic induction. Its components parallel to $x$, $y$, and $z$ are $F$, $G$, and $H$. The electromagnetic momentum of a circuit is the line-integral of $\mathfrak{A}$ round the circuit.

We then, by means of Theorem IV, Art. 24, transformed the line-integral of $\mathfrak{A}$ into the surface-integral of another vector, $\mathfrak{B}$, whose components are $a$, $b$, $c$, and we found that the phenomena of induction due to motion of a conductor, and those of electromagnetic force can be expressed in terms of $\mathfrak{B}$. We gave to $\mathfrak{B}$ the name of the magnetic induction, since its properties are identical with those of the lines of magnetic induction as investigated by Faraday.

We also established three sets of equations: the first set, (A), are those of magnetic induction, expressing it in terms of the electromagnetic momentum. The second set, (B), are those of electromotive intensity, expressing it in terms of the motion of the conductor across the lines of magnetic induction, and of the rate of variation of the electromagnetic momentum. The third set, (C), are the equations of electromagnetic force, expressing it in terms of the current and the magnetic induction.

The current in all these cases is to be understood as the actual current, which includes not only the current of conduction, but the current due to variation of the electric displacement.

The magnetic induction $\mathfrak{B}$ is the quantity which we have already considered in Art. 400. In an unmagnetized body it is identical with the force on a unit magnetic pole, but if the body is magnetized, either permanently or by induction, it is the force which would be exerted on a unit pole, if placed in a narrow crevasse in the body, the walls of which are perpendicular to the direction of magnetization. The components of $\mathfrak{B}$ are $a$, $b$, $c$.

It follows from the equations (A), by which $a$, $b$, $c$ are defined, that

$$\frac{da}{dx} + \frac{db}{dy} + \frac{dc}{dz} = 0.$$

This was shewn at Art. 403 to be a property of the magnetic induction.

605.] We have defined the magnetic force within a magnet, as distinguished from the magnetic induction, to be the force on a unit pole placed in a narrow crevasse cut parallel to the direction of magnetization. This quantity is denoted by $\mathfrak{H}$, and its components by $\alpha$, $\beta$, $\gamma$. See Art. 398.

If $\mathfrak{J}$ is the intensity of magnetization, and $A$, $B$, $C$ its components, then, by Art. 400,

$$\left.\begin{aligned} a &= \alpha + 4\pi A, \\ b &= \beta + 4\pi B, \\ c &= \gamma + 4\pi C. \end{aligned}\right\} \quad \text{(Equations of Magnetization.)} \qquad \text{(D)}$$

We may call these the equations of magnetization, and they indicate that in the electromagnetic system the magnetic induction $\mathfrak{B}$, considered as a vector, is the sum, in the Hamiltonian sense, of two vectors, the magnetic force $\mathfrak{H}$, and the magnetization $\mathfrak{J}$ multiplied by $4\pi$, or

$$\mathfrak{B} = \mathfrak{H} + 4\pi\mathfrak{J}.$$

In certain substances, the magnetization depends on the magnetic force, and this is expressed by the system of equations of induced magnetism given at Arts. 426 and 435.

606.] Up to this point of our investigation we have deduced everything from purely dynamical considerations, without any reference to quantitative experiments in electricity or magnetism. The only use we have made of experimental knowledge is to recognise, in the abstract quantities deduced from the theory, the concrete quantities discovered by experiment, and to denote them by names which indicate their physical relations rather than their mathematical generation.

In this way we have pointed out the existence of the electromagnetic momentum $\mathfrak{A}$ as a vector whose direction and magnitude vary from one part of space to another, and from this we have deduced, by a mathematical process, the magnetic induction, $\mathfrak{B}$, as a derived vector. We have not, however, obtained any data for determining either $\mathfrak{A}$ or $\mathfrak{B}$ from the distribution of currents in the field. For this purpose we must find the mathematical connexion between these quantities and the currents.

We begin by admitting the existence of permanent magnets, the mutual action of which satisfies the principle of the conservation of energy. We make no assumption with respect to the laws of magnetic force except that which follows from this principle, namely, that the force acting on a magnetic pole must be capable of being derived from a potential.

We then observe the action between currents and magnets, and we find that a current acts on a magnet in a manner apparently the same as another magnet would act if its strength, form, and position were properly adjusted, and that the magnet acts on the current in the same way as another current. These observations need not be supposed to be accompanied by actual measurements of the forces. They are not therefore to be considered as furnishing numerical data, but are useful only in suggesting questions for our consideration.

The question these observations suggest is, whether the magnetic field produced by electric currents, as it is similar to that produced by permanent magnets in many respects, resembles it also in being related to a potential?

The evidence that an electric circuit produces, in the space surrounding it, magnetic effects precisely the same as those produced by a magnetic shell bounded by the circuit, has been stated in Arts. 482–485.

We know that in the case of the magnetic shell there is a potential, which has a determinate value for all points outside the substance of the shell, but that the values of the potential at two neighbouring points, on opposite sides of the shell, differ by a finite quantity.

If the magnetic field in the neighbourhood of an electric current resembles that in the neighbourhood of a magnetic shell, the magnetic potential, as found by a line-integration of the magnetic force, will be the same for any two lines of integration, provided one of these lines can be transformed into the other by continuous motion without cutting the electric current.

If, however, one line of integration cannot be transformed into the other without cutting the current, the line-integral of the magnetic force along the one line will differ from that along the other by a quantity depending on the strength of the current. The magnetic potential due to an electric current is therefore a function having an infinite series of values with a common difference, the particular value depending on the course of the line of integration. Within the substance of the conductor, there is no such thing as a magnetic potential.

607.] Assuming that the magnetic action of a current has a magnetic potential of this kind, we proceed to express this result mathematically.

In the first place, the line-integral of the magnetic force round any closed curve is zero, provided the closed curve does not surround the electric current.

In the next place, if the current passes once, and only once, through the closed curve in the positive direction, the line-integral has a determinate value, which may be used as a measure of the strength of the current. For if the closed curve alters its form in any continuous manner without cutting the current, the line-integral will remain the same.

In electromagnetic measure, the line-integral of the magnetic force round a closed curve is numerically equal to the current through the closed curve multiplied by $4\pi$.

If we take for the closed curve the rectangle whose sides are $dy$ and $dz$, the line-integral of the magnetic force round the parallelogram is

$$\left(\frac{d\gamma}{dy} - \frac{d\beta}{dz}\right) dy\,dz,$$

and if $u$, $v$, $w$ are the components of the flow of electricity, the current through the parallelogram is

$$u\,dy\,dz.$$

Multiplying this by $4\pi$, and equating the result to the line-integral, we obtain the equation

$$4\pi u = \frac{d\gamma}{dy} - \frac{d\beta}{dz},$$

with the similar equations

$$\left.\begin{aligned} 4\pi v &= \frac{d\alpha}{dz} - \frac{d\gamma}{dx}, \\ 4\pi w &= \frac{d\beta}{dx} - \frac{d\alpha}{dy}, \end{aligned}\right\} \quad \text{(Equations of Electric Currents.)} \qquad \text{(E)}$$

which determine the magnitude and direction of the electric currents when the magnetic force at every point is given.

When there is no current, these equations are equivalent to the condition that

$$\alpha dx + \beta dy + \gamma dz = -D\Omega,$$

or that the magnetic force is derivable from a magnetic potential in all points of the field where there are no currents.

By differentiating the equations (E) with respect to $x$, $y$, and $z$ respectively, and adding the results, we obtain the equation

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0,$$

which indicates that the current whose components are $u$, $v$, $w$ is subject to the condition of motion of an incompressible fluid, and that it must necessarily flow in closed circuits.

This equation is true only if we take $u$, $v$, and $w$ as the components of that electric flow which is due to the variation of electric displacement as well as to true conduction.

We have very little experimental evidence relating to the direct electromagnetic action of currents due to the variation of electric displacement in dielectrics, but the extreme difficulty of reconciling the laws of electromagnetism with the existence of electric currents which are not closed is one reason among many why we must admit the existence of transient currents due to the variation of displacement. Their importance will be seen when we come to the electromagnetic theory of light.

608.] We have now determined the relations of the principal quantities concerned in the phenomena discovered by Örsted, Ampère, and Faraday. To connect these with the phenomena described in the former parts of this treatise, some additional relations are necessary.

When electromotive intensity acts on a material body, it produces in it two electrical effects, called by Faraday Induction and Conduction, the first being most conspicuous in dielectrics, and the second in conductors.

In this treatise, static electric induction is measured by what we have called the electric displacement, a directed quantity or vector which we have denoted by $\mathfrak{D}$, and its components by $f, g, h$.

In isotropic substances, the displacement is in the same direction as the electromotive intensity which produces it, and is proportional to it, at least for small values of this intensity. This may be expressed by the equation

$$\mathfrak{D} = \frac{1}{4\pi} K \mathfrak{E}, \qquad \text{(Equation of Electric Displacement.)} \qquad \text{(F)}$$

where $K$ is the dielectric capacity of the substance. See Art. 68.

In substances which are not isotropic, the components $f, g, h$ of the electric displacement $\mathfrak{D}$ are linear functions of the components $P, Q, R$ of the electromotive intensity $\mathfrak{E}$.

The form of the equations of electric displacement is similar to that of the equations of conduction as given in Art. 298.

These relations may be expressed by saying that $K$ is, in isotropic bodies, a scalar quantity, but in other bodies it is a linear and vector function, operating on the vector $\mathfrak{E}$.

609.] The other effect of electromotive intensity is conduction. The laws of conduction as the result of electromotive intensity were established by Ohm, and are explained in the second part of this treatise, Art. 241. They may be summed up in the equation

$$\mathfrak{K} = C\mathfrak{E}, \qquad \text{(Equation of Conductivity.)} \qquad \text{(G)}$$

where $\mathfrak{E}$ is the electromotive intensity at the point, $\mathfrak{K}$ is the density of the current of conduction, the components of which are $p$, $q$, and $r$, and $C$ is the conductivity of the substance, which in the case of isotropic substances, is a simple scalar quantity, but in other substances becomes a linear and vector function operating on the vector $\mathfrak{E}$. The form of this function is given in Cartesian coordinates in Art. 298.

610.] One of the chief peculiarities of this treatise is the doctrine which it asserts, that the true electric current $\mathfrak{C}$, that on which the electromagnetic phenomena depend, is not the same thing as $\mathfrak{K}$, the current of conduction, but that the time-variation of $\mathfrak{D}$, the electric displacement, must be taken into account in estimating the total movement of electricity, so that we must write,

$$\mathfrak{C} = \mathfrak{K} + \dot{\mathfrak{D}}, \qquad \text{(Equation of True Currents.)} \qquad \text{(H)}$$

or, in terms of the components,

$$\left.\begin{aligned} u &= p + \frac{df}{dt}, \\ v &= q + \frac{dg}{dt}, \\ w &= r + \frac{dh}{dt}. \end{aligned}\right\} \qquad \text{(H*)}$$

611.] Since both $\mathfrak{K}$ and $\mathfrak{D}$ depend on the electromotive intensity $\mathfrak{E}$, we may express the true current $\mathfrak{C}$ in terms of the electromotive intensity, thus

$$\mathfrak{C} = \left(C + \frac{1}{4\pi} K \frac{d}{dt}\right)\mathfrak{E}, \qquad \text{(I)}$$

or, in the case in which $C$ and $K$ are constants,

$$\left.\begin{aligned} u &= CP + \frac{1}{4\pi} K \frac{dP}{dt}, \\ v &= CQ + \frac{1}{4\pi} K \frac{dQ}{dt}, \\ w &= CR + \frac{1}{4\pi} K \frac{dR}{dt}. \end{aligned}\right\} \qquad \text{(I*)}$$

612.] The volume-density of the free electricity at any point is found from the components of electric displacement by the equation

$$\rho = \frac{df}{dx} + \frac{dg}{dy} + \frac{dh}{dz}. \tag{J}$$

613.] The surface-density of electricity is

$$\sigma = lf + mg + nh + l'f' + m'g' + n'h', \tag{K}$$

where $l, m, n$ are the direction-cosines of the normal drawn from the surface into the medium in which $f, g, h$ are the components of the displacement, and $l', m', n'$ are those of the normal drawn from the surface into the medium in which they are $f', g', h'$.

614.] When the magnetization of the medium is entirely induced by the magnetic force acting on it, we may write the equation of induced magnetization,

$$\mathfrak{B} = \mu \mathfrak{H}, \tag{L}$$

where $\mu$ is the coefficient of magnetic permeability, which may be considered a scalar quantity, or a linear and vector function operating on $\mathfrak{H}$, according as the medium is isotropic or not.

615.] These may be regarded as the principal relations among the quantities we have been considering. They may be combined so as to eliminate some of these quantities, but our object at present is not to obtain compactness in the mathematical formulae, but to express every relation of which we have any knowledge. To eliminate a quantity which expresses a useful idea would be rather a loss than a gain in this stage of our enquiry.

There is one result, however, which we may obtain by combining equations (A) and (E), and which is of very great importance.

If we suppose that no magnets exist in the field except in the form of electric circuits, the distinction which we have hitherto maintained between the magnetic force and the magnetic induction vanishes, because it is only in magnetized matter that these quantities differ from each other.

According to Ampère's hypothesis, which will be explained in Art. 833, the properties of what we call magnetized matter are due to molecular electric circuits, so that it is only when we regard the substance in large masses that our theory of magnetization is applicable, and if our mathematical methods are supposed capable of taking account of what goes on within the individual molecules, they will discover nothing but electric circuits, and we shall find the magnetic force and the magnetic induction everywhere identical. In order, however, to be able to make use of the electrostatic or of the electromagnetic system of measurement at pleasure we shall retain the coefficient $\mu$, remembering that its value is unity in the electromagnetic system.

616.] The components of the magnetic induction are by equations (A), Art. 591,

$$\left.\begin{aligned} a &= \frac{dH}{dy} - \frac{dG}{dz}, \\ b &= \frac{dF}{dz} - \frac{dH}{dx}, \\ c &= \frac{dG}{dx} - \frac{dF}{dy}. \end{aligned}\right\}$$

The components of the electric current are by equations (E), Art. 607, given by

$$\left.\begin{aligned} 4\pi u &= \frac{d\gamma}{dy} - \frac{d\beta}{dz}, \\ 4\pi v &= \frac{d\alpha}{dz} - \frac{d\gamma}{dx}, \\ 4\pi w &= \frac{d\beta}{dx} - \frac{d\alpha}{dy}. \end{aligned}\right\}$$

According to our hypothesis, $a$, $b$, $c$ are identical with $\mu\alpha$, $\mu\beta$, $\mu\gamma$ respectively. We therefore obtain {when $\mu$ is constant}

$$4\pi\mu u - \frac{d^2G}{dx\,dy} - \frac{d^2F}{dy^2} - \frac{d^2F}{dz^2} + \frac{d^2H}{dz\,dx}. \tag{1}$$

If we write
$$J = \frac{dF}{dx} + \frac{dG}{dy} + \frac{dH}{dz}, \tag{2}$$

and*
$$\nabla^2 = -\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}\right), \tag{3}$$

we may write equation (1),

Similarly,
$$\left.\begin{aligned} 4\pi\mu u &= \frac{dJ}{dx} + \nabla^2 F. \\ 4\pi\mu v &= \frac{dJ}{dy} + \nabla^2 G, \\ 4\pi\mu w &= \frac{dJ}{dz} + \nabla^2 H. \end{aligned}\right\} \tag{4}$$

* The negative sign is employed here in order to make our expressions consistent with those in which Quaternions are employed.

If we write
$$\left.\begin{aligned} F' &= \mu\iiint\frac{u}{r}\,dx\,dy\,dz,\\ G' &= \mu\iiint\frac{v}{r}\,dx\,dy\,dz,\\ H' &= \mu\iiint\frac{w}{r}\,dx\,dy\,dz,\end{aligned}\right\}\qquad(5)$$

$$\chi = \frac{1}{4\pi}\iiint\frac{J}{r}\,dx\,dy\,dz,\qquad(6)$$

where $r$ is the distance of the given point from the element $(x, y, z)$ and the integrations are to be extended over all space, then
$$\left.\begin{aligned} F &= F' - \frac{d\chi}{dx},\\ G &= G' - \frac{d\chi}{dy},\\ H &= H' - \frac{d\chi}{dz}.\end{aligned}\right\}\qquad(7)$$

The quantity $\chi$ disappears from the equations (A), and it is not related to any physical phenomenon. If we suppose it to be zero everywhere, $J$ will also be zero everywhere, and equations (5), omitting the accents, will give the true values of the components of $\mathfrak{A}$.

617.] We may therefore adopt, as a definition of $\mathfrak{A}$, that it is the vector-potential of the electric current, standing in the same relation to the electric current that the scalar potential stands to the matter of which it is the potential, and obtained by a similar process of integration, which may be thus described:—

From a given point let a vector be drawn, representing in magnitude and direction a given element of an electric current, divided by the numerical value of the distance of the element from the given point. Let this be done for every element of the electric current. The resultant of all the vectors thus found is the potential of the whole current. Since the current is a vector quantity, its potential is also a vector. See Art. 422.

When the distribution of electric currents is given, there is one, and only one, distribution of the values of $\mathfrak{A}$, such that $\mathfrak{A}$ is everywhere finite and continuous, and satisfies the equations
$$\nabla^2\mathfrak{A} = 4\pi\mu\mathfrak{C},\qquad S.\nabla\mathfrak{A} = 0,$$
and vanishes at an infinite distance from the electric system.

This value is that given by equations (5), which may be written in the quaternion form

$$\mathfrak{A} = \mu \iiint \frac{\mathfrak{C}}{r}\, dx\, dy\, dz.$$

*Quaternion Expressions for the Electromagnetic Equations.*

618.] In this treatise we have endeavoured to avoid any process demanding from the reader a knowledge of the Calculus of Quaternions. At the same time we have not scrupled to introduce the idea of a vector when it was necessary to do so. When we have had occasion to denote a vector by a symbol, we have used a German letter, the number of different vectors being so great that Hamilton's favourite symbols would have been exhausted at once. Whenever therefore a German letter is used it denotes a Hamiltonian vector, and indicates not only its magnitude but its direction. The constituents of a vector are denoted by Roman or Greek letters.

The principal vectors which we have to consider are

| | Symbol of Vector. | Constituents. |
|---|---|---|
| The radius vector of a point .............. | $\rho$ | $x\ y\ z$ |
| The electromagnetic momentum at a point | $\mathfrak{A}$ | $F\ G\ H$ |
| The magnetic induction ..................... | $\mathfrak{B}$ | $a\ b\ c$ |
| The (total) electric current.................. | $\mathfrak{C}$ | $u\ v\ w$ |
| The electric displacement .................. | $\mathfrak{D}$ | $f\ g\ h$ |
| The electromotive intensity .............. | $\mathfrak{E}$ | $P\ Q\ R$ |
| The mechanical force ........................ | $\mathfrak{F}$ | $X\ Y\ Z$ |
| The velocity of a point ..................... | $\mathfrak{G}$ or $\dot{\rho}$ | $\dot{x}\ \dot{y}\ \dot{z}$ |
| The magnetic force .......................... | $\mathfrak{H}$ | $\alpha\ \beta\ \gamma$ |
| The intensity of magnetization ........... | $\mathfrak{J}$ | $A\ B\ C$ |
| The current of conduction ................. | $\mathfrak{K}$ | $p\ q\ r$ |

We have also the following scalar functions:

The electric potential $\Psi$.

The magnetic potential (where it exists) $\Omega$.

The electric density $\mathfrak{e}$.

The density of magnetic 'matter' $\mathfrak{m}$.

Besides these we have the following quantities, indicating physical properties of the medium at each point:—

$C$, the conductivity for electric currents.

$K$, the dielectric inductive capacity.

$\mu$, the magnetic inductive capacity.

These quantities are, in isotropic media, mere scalar functions of $\rho$, but in general they are linear and vector operators on the vector functions to which they are applied. $K$ and $\mu$ are certainly always self-conjugate, and $C$ is probably so also.

619.] The equations (A) of magnetic induction, of which the first is,

$$a = \frac{dH}{dy} - \frac{dG}{dz},$$

may now be written $\mathfrak{B} = V.\nabla\,\mathfrak{A}$,

where $\nabla$ is the operator

$$i\frac{d}{dx} + j\frac{d}{dy} + k\frac{d}{dz},$$

and $V$ indicates that the vector part of the result of this operation is to be taken.

Since $\mathfrak{A}$ is subject to the condition $S.\nabla\,\mathfrak{A} = 0$, $\nabla\,\mathfrak{A}$ is a pure vector, and the symbol $V$ is unnecessary.

The equations (B) of electromotive force, of which the first is

$$P = c\dot{y} - b\dot{z} - \frac{dF}{dt} - \frac{d\Psi}{dx},$$

become $\mathfrak{E} = V.\,\mathfrak{G}\mathfrak{B} - \dot{\mathfrak{A}} - \nabla\Psi.$

The equations (C) of mechanical force, of which the first is

$$X = cv - bw + eP - m\frac{d\Omega}{dx},\,*$$

become $\mathfrak{F} = V\mathfrak{C}\mathfrak{B} + e\mathfrak{E} - m\nabla\,\Omega.$

The equations (D) of magnetization, of which the first is

$$a = \alpha + 4\pi A,$$

become $\mathfrak{B} = \mathfrak{H} + 4\pi\mathfrak{J}.$

The equations (E) of electric currents, of which the first is

$$4\pi u = \frac{d\gamma}{dy} - \frac{d\beta}{dz},$$

become $4\pi\mathfrak{C} = V.\nabla\,\mathfrak{H}.$

The equation of the current of conduction is, by Ohm's Law,

$$\mathfrak{K} = C\,\mathfrak{E}.$$

That of electric displacement is

$$\mathfrak{D} = \frac{1}{4\pi}K\mathfrak{E}.$$

* {In the 1st and 2nd editions of this work $-\frac{d\Psi}{dx}$ was written for $P$ in this equation. The correction is due to Prof. G. F. Fitzgerald *Trans. R. S. Dublin*, 1883.}

The equation of the total current, arising from the variation of the electric displacement as well as from conduction, is

$$\mathfrak{C} = \mathfrak{K} + \dot{\mathfrak{D}}.$$

When the magnetization arises from magnetic induction,

$$\mathfrak{B} = \mu \mathfrak{H}.$$

We have also, to determine the electric volume-density,

$$e = S . \nabla \mathfrak{D}.$$

To determine the magnetic volume-density,

$$m = S . \nabla \mathfrak{J}.$$

When the magnetic force can be derived from a potential,

$$\mathfrak{H} = -\nabla \Omega.$$

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## APPENDIX TO CHAPTER IX.

The expressions (5) are not in general accurate if the electromagnetic field contains substances of different magnetic permeabilities, for in that case, at the surface of separation of two surfaces of different magnetic permeabilities, there will in general be free magnetism; this will contribute terms to the expression for the vector potential which are given by equations (22), p. 30. The boundary equations at the surface separating two media whose magnetic permeabilities are $\mu_1$ and $\mu_2$, and where $F_1$, $G_1$, $H_1$ and $F_2$, $G_2$, $H_2$ denote the values of the components of the vector potential on the two sides of the surface of separation, $l$, $m$, $n$ the direction cosines of the normal to this surface; are (1), since the normal induction is continuous,

$$l\left(\frac{dH_1}{dy}-\frac{dG_1}{dz}\right)+m\left(\frac{dF_1}{dz}-\frac{dH_1}{dx}\right)+n\left(\frac{dG_1}{dx}-\frac{dF_1}{dy}\right)$$
$$=l\left(\frac{dH_2}{dy}-\frac{dG_2}{dz}\right)+m\left(\frac{dF_2}{dz}-\frac{dH_2}{dx}\right)+n\left(\frac{dG_2}{dx}-\frac{dF_2}{dy}\right),$$

and (2), since the magnetic force along the surface is continuous,

$$\frac{\frac{1}{\mu_1}\left(\frac{dH_1}{dy}-\frac{dG_1}{dz}\right)-\frac{1}{\mu_2}\left(\frac{dH_2}{dy}-\frac{dG_2}{dz}\right)}{l}$$
$$=\frac{\frac{1}{\mu_1}\left(\frac{dF_1}{dz}-\frac{dH_1}{dx}\right)-\frac{1}{\mu_2}\left(\frac{dF_2}{dz}-\frac{dH_2}{dx}\right)}{m}$$
$$=\frac{\frac{1}{\mu_1}\left(\frac{dG_1}{dx}-\frac{dF_1}{dy}\right)-\frac{1}{\mu_2}\left(\frac{dG_2}{dx}-\frac{dF_2}{dy}\right)}{n}.$$

The expressions (5) do not in general satisfy both these surface conditions. It is therefore best to regard $F$, $G$, $H$ as given by the equations

$$\nabla^2 F = 4\pi\mu u$$
$$\nabla^2 G = 4\pi\mu v$$
$$\nabla^2 H = 4\pi\mu w$$

and the preceding boundary conditions.}

{It does not appear legitimate to assume that $\Psi$ in equations (B) represents the electrostatic potential when the conductors are moving, for in deducing those equations Maxwell leaves out the term

$$-\frac{d}{ds}\left(F\frac{dx}{dt} + G\frac{dy}{dt} + H\frac{dz}{dt}\right),$$

since it vanishes when integrated round a closed circuit. If we insert this term, then $\Psi$ is no longer the electrostatic potential but is the sum of this potential, and

$$F\frac{dx}{dt} + G\frac{dy}{dt} + H\frac{dz}{dt}.$$

This has an important application to a problem which has attracted much attention, that of a sphere rotating with angular velocity $\omega$ about a vertical axis in a uniform magnetic field where the magnetic force is vertical and equal to $c$. Equations (B) become in this case, supposing the sphere to have settled down into a steady state,

$$P = c\omega x - \frac{d\Psi}{dx},$$

$$Q = c\omega y - \frac{d\Psi}{dy},$$

$$R = \quad - \frac{d\Psi}{dz}.$$

Since the sphere is a conductor and in a steady state, and since $\frac{P}{\sigma}, \frac{Q}{\sigma}, \frac{R}{\sigma}$ are the components of the current,

$$\frac{dP}{dx} + \frac{dQ}{dy} + \frac{dR}{dz} = 0;$$

hence
$$2c\omega = \frac{d^2\Psi}{dx^2} + \frac{d^2\Psi}{dy^2} + \frac{d^2\Psi}{dz^2}.$$

This equation has usually been interpreted to mean that throughout the sphere there is a distribution of electricity whose volume density is $-c\omega/2\pi$, but this is only legitimate if we assume that $\Psi$ is the electrostatic potential.

If in accordance with the investigation by which equations (B) were deduced we assume that, $\Phi$ being the electrostatic potential,

$$\Psi = \Phi + F\frac{dx}{dt} + G\frac{dy}{dt} + H\frac{dz}{dt},$$

or in this case $$\Psi = \Phi + \omega(Gx - Fy),$$
then, since
$$\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2}\right)\left(Gx - Fy\right) = 2\left(\frac{dG}{dx}-\frac{dF}{dy}\right),$$
$$= 2c,$$
we see that since
$$\frac{d^2\Psi}{dx^2}+\frac{d^2\Psi}{dy^2}+\frac{d^2\Psi}{dz^2} = 2c\omega,$$
$$\frac{d^2\Phi}{dx^2}+\frac{d^2\Phi}{dy^2}+\frac{d^2\Phi}{dz^2} = 0;$$
that is, there is no distribution of free electricity throughout the volume of the sphere.

There is therefore nothing in the equations of the electromagnetic field which would lead us to suppose that a rotating sphere contains free electricity.

## *Equations of the Electromagnetic Field expressed in Polar and Cylindrical Co-ordinates.*

If $F$, $G$, $H$ are the components of the vector potential along the radius vector, the meridian and a parallel of latitude respectively, $a$, $b$, $c$ the components of the magnetic induction, $\alpha$, $\beta$, $\gamma$ the components of the magnetic force, and $u$, $v$, $w$ the components of the current in those directions, then we can easily prove that

$$a = \frac{1}{r^2\sin\theta}\left\{\frac{d}{d\theta}(r\sin\theta H) - \frac{d}{d\phi}(rG)\right\},$$
$$b = \frac{1}{r\sin\theta}\left\{\frac{dF}{d\phi} - \frac{d}{dr}(r\sin\theta H)\right\},$$
$$c = \frac{1}{r}\left\{\frac{d}{dr}(rG) - \frac{dF}{d\theta}\right\};$$
$$4\pi u = \frac{1}{r^2\sin\theta}\left\{\frac{d}{d\theta}(r\sin\theta\gamma) - \frac{d}{d\phi}(r\beta)\right\},$$
$$4\pi v = \frac{1}{r\sin\theta}\left\{\frac{d\alpha}{d\phi} - \frac{d}{dr}(r\sin\theta\gamma)\right\},$$
$$4\pi w = \frac{1}{r}\left\{\frac{d}{dr}(r\beta) - \frac{d\alpha}{d\theta}\right\}.$$

If $P$, $Q$, $R$ are the components of the electromotive intensity along the radius vector, the meridian and a parallel of latitude,

$$\frac{da}{dt} = -\frac{1}{r^2\sin\theta}\left\{\frac{d}{d\theta}(r\sin\theta R) - \frac{d}{d\phi}(rQ)\right\},$$
$$\frac{db}{dt} = -\frac{1}{r\sin\theta}\left\{\frac{dP}{d\phi} - \frac{d}{r}(r\sin\theta R)\right\},$$
$$\frac{dc}{dt} = -\frac{1}{r}\left\{\frac{d}{dr}(rQ) - \frac{dP}{d\theta}\right\}.$$

If the cylindrical co-ordinates are $\rho$, $\theta$, $z$, and if $F$, $G$, $H$ are the components of the vector potential parallel to $\rho$, $\theta$, $z$; $a$, $b$, $c$ the components of the magnetic induction, $\alpha$, $\beta$, $\gamma$ the components of the magnetic force, and $u$, $v$, $w$ the components of the current in these directions, then

$$a = \frac{1}{\rho}\left\{\frac{dH}{d\theta} - \frac{d}{dz}(\rho G)\right\},$$

$$b = \frac{dF}{dz} - \frac{dH}{d\rho},$$

$$c = \frac{1}{\rho}\left\{\frac{d}{d\rho}(\rho G) - \frac{dF}{d\theta}\right\};$$

$$4\pi u = \frac{1}{\rho}\left\{\frac{d\gamma}{d\theta} - \frac{d}{dz}(\rho\beta)\right\},$$

$$4\pi v = \frac{d\alpha}{dz} - \frac{d\gamma}{d\rho},$$

$$4\pi w = \frac{1}{\rho}\left\{\frac{d}{d\rho}(\rho\beta) - \frac{d\alpha}{d\theta}\right\}.$$

If $P$, $Q$, $R$ are the components of the electromotive intensity parallel to $\rho$, $\theta$, $z$,

$$\frac{da}{dt} = -\frac{1}{\rho}\left\{\frac{dR}{d\theta} - \frac{d}{dz}(\rho Q)\right\},$$

$$\frac{db}{dt} = -\left\{\frac{dP}{dz} - \frac{dR}{d\rho}\right\},$$

$$\frac{dc}{dt} = -\frac{1}{\rho}\left\{\frac{d}{d\rho}(\rho Q) - \frac{dP}{d\theta}\right\}.$$

# CHAPTER X.

## DIMENSIONS OF ELECTRIC UNITS.

620.] Every electromagnetic quantity may be defined with reference to the fundamental units of Length, Mass, and Time. If we begin with the definition of the unit of electricity, as given in Art. 65, we may obtain definitions of the units of every other electromagnetic quantity, in virtue of the equations into which they enter along with quantities of electricity. The system of units thus obtained is called the Electrostatic System.

If, on the other hand, we begin with the definition of the unit magnetic pole, as given in Art. 374, we obtain a different system of units of the same set of quantities. This system of units is not consistent with the former system, and is called the Electromagnetic System.

We shall begin by stating those relations between the different units which are common to both systems, and we shall then form a table of the dimensions of the units according to each system.

621.] We shall arrange the primary quantities which we have to consider in pairs. In the first three pairs, the product of the two quantities in each pair is a quantity of energy or work. In the second three pairs, the product of each pair is a quantity of energy referred to unit of volume.

### First Three Pairs.

*Electrostatic Pair.*

| | Symbol. |
|---|---|
| (1) Quantity of electricity . . . . . . . | $e$ |
| (2) Electromotive force, or electric potential . . | $E$ |

*Magnetic Pair.* Symbol.

(3) Quantity of free magnetism, or strength of a pole $m$
(4) Magnetic potential . . . . . . $\Omega$

*Electrokinetic Pair.*

(5) Electrokinetic momentum of a circuit . . . $p$
(6) Electric current . . . . . . . $C$

SECOND THREE PAIRS.

*Electrostatic Pair.*

(7) Electric displacement (measured by surface-density) $\mathfrak{D}$
(8) Electromotive intensity . . . . . $\mathfrak{E}$

*Magnetic Pair.*

(9) Magnetic induction . . . . . . $\mathfrak{B}$
(10) Magnetic force . . . . . . . $\mathfrak{H}$

*Electrokinetic Pair.*

(11) Intensity of electric current at a point . . $\mathfrak{C}$
(12) Vector potential of electric currents . . . $\mathfrak{A}$

622. The following relations exist between these quantities. In the first place, since the dimensions of energy are $\left[\frac{L^2M}{T^2}\right]$, and those of energy referred to unit of volume $\left[\frac{M}{LT^2}\right]$, we have the following equations of dimensions:

$$[eE] = [m\Omega] = [pC] = \left[\frac{L^2M}{T^2}\right], \tag{1}$$

$$[\mathfrak{D}\mathfrak{E}] = [\mathfrak{B}\mathfrak{H}] = [\mathfrak{C}\mathfrak{A}] = \left[\frac{M}{LT^2}\right]. \tag{2}$$

Secondly, since $e$, $p$, and $\mathfrak{A}$ are the time-integrals of $C$, $E$, and $\mathfrak{E}$ respectively,

$$\left[\frac{e}{C}\right] = \left[\frac{p}{E}\right] = \left[\frac{\mathfrak{A}}{\mathfrak{E}}\right] = [T]. \tag{3}$$

Thirdly, since $E$, $\Omega$, and $p$ are the line-integrals of $\mathfrak{E}$, $\mathfrak{H}$, and $\mathfrak{A}$ respectively,

$$\left[\frac{E}{\mathfrak{E}}\right] = \left[\frac{\Omega}{\mathfrak{H}}\right] = \left[\frac{p}{\mathfrak{A}}\right] = [L].^* \tag{4}$$

Finally, since $e$, $C$, and $m$ are the surface-integrals of $\mathfrak{D}$, $\mathfrak{C}$, and $\mathfrak{B}$ respectively,

$$\left[\frac{e}{\mathfrak{D}}\right] = \left[\frac{C}{\mathfrak{C}}\right] = \left[\frac{m}{\mathfrak{B}}\right] = [L^2]. \tag{5}$$

623.] These fifteen equations are not independent, and in order to deduce the dimensions of the twelve units involved, we require one additional equation. If, however, we take either $e$ or $m$ as an independent unit, we can deduce the dimensions of the rest in terms of either of these.

$$
\begin{array}{lllll}
(1) & [e] & = [e] & & = \left[\frac{L^2M}{mT}\right]. \\
(2) & [E] & = \left[\frac{L^2M}{eT^2}\right] & & = \left[\frac{m}{T}\right]. \\
(3) \text{ and } (5) & [p] = [m] & = \left[\frac{L^2M}{eT}\right] & & = [m]. \\
(4) \text{ and } (6) & [C] = [\Omega] & = \left[\frac{e}{T}\right] & & = \left[\frac{L^2M}{mT^2}\right]. \\
(7) & [\mathfrak{D}] & = \left[\frac{e}{L^2}\right] & & = \left[\frac{M}{mT}\right]. \\
(8) & [\mathfrak{E}] & = \left[\frac{LM}{eT^2}\right] & & = \left[\frac{m}{LT}\right]. \\
(9) & [\mathfrak{B}] & = \left[\frac{M}{eT}\right] & & = \left[\frac{m}{L^2}\right]. \\
(10) & [\mathfrak{H}] & = \left[\frac{e}{LT}\right] & & = \left[\frac{LM}{mT^2}\right]. \\
(11) & [\mathfrak{C}] & = \left[\frac{e}{L^2T}\right] & & = \left[\frac{M}{mT^2}\right]. \\
(12) & [\mathfrak{A}] & = \left[\frac{LM}{eT}\right] & & = \left[\frac{m}{L}\right].
\end{array}
$$

* [We have also $\left[\frac{\mathfrak{A}}{\mathfrak{B}}\right] = [L]$.]

624.] The relations of the first ten of these quantities may be exhibited by means of the following arrangement:—

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $e$, | $\mathfrak{D}$, | $\mathfrak{H}$, | $C$ and $\Omega$. | $E$, | $\mathfrak{E}$, | $\mathfrak{B}$, | $m$ and $p$. |
| $m$ and $p$, | $\mathfrak{B}$, | $\mathfrak{E}$, | $E$. | $C$ and $\Omega$, | $\mathfrak{H}$, | $\mathfrak{D}$, | $e$. |

The quantities in the first line are derived from $e$ by the same operations as the corresponding quantities in the second line are derived from $m$. It will be seen that the order of the quantities in the first line is exactly the reverse of the order in the second line. The first four of each line have the first symbol in the numerator. The second four in each line have it in the denominator.

All the relations given above are true whatever system of units we adopt.

625.] The only systems of any scientific value are the electrostatic and the electromagnetic systems. The electrostatic system is founded on the definition of the unit of electricity, Arts. 41, 42, and may be deduced from the equation

$$\mathfrak{E} = \frac{e}{L^2},$$

which expresses that the resultant electric intensity $\mathfrak{E}$ at any point, due to the action of a quantity of electricity $e$ at a distance $L$, is found by dividing $e$ by $L^2$. Substituting in the equations of dimensions (1) and (8), we find

$$\left[\frac{LM}{eT^2}\right] = \left[\frac{e}{L^2}\right], \qquad \left[\frac{m}{LT}\right] = \left[\frac{M}{mT}\right],$$

whence $$[e] = [L^{\frac{3}{2}} M^{\frac{1}{2}} T^{-1}], \quad m = [L^{\frac{1}{2}} M^{\frac{1}{2}}],$$

in the electrostatic system.

The electromagnetic system is founded on a precisely similar definition of the unit of strength of a magnetic pole, Art. 374, leading to the equation

$$\mathfrak{H} = \frac{m}{L^2},$$

whence $$\left[\frac{e}{LT}\right] = \left[\frac{M}{eT}\right], \qquad \left[\frac{LM}{mT^2}\right] = \left[\frac{m}{L^2}\right],$$

and $$[e] = [L^{\frac{1}{2}} M^{\frac{1}{2}}], \qquad [m] = [L^{\frac{3}{2}} M^{\frac{1}{2}} T^{-1}],$$

in the electromagnetic system. From these results we find the dimensions of the other quantities.

626.] *Table of Dimensions.*

| | Symbol. | Dimensions in Electrostatic System. | Dimensions in Electromagnetic System. |
|---|---|---|---|
| Quantity of electricity | $e$ | $[L^{\frac{3}{2}} M^{\frac{1}{2}} T^{-1}]$ | $[L^{\frac{1}{2}} M^{\frac{1}{2}}]$. |
| Line-integral of electromotive intensity | $E$ | $[L^{\frac{1}{2}} M^{\frac{1}{2}} T^{-1}]$ | $[L^{\frac{3}{2}} M^{\frac{1}{2}} T^{-2}]$. |
| Quantity of magnetism / Electrokinetic momentum of a circuit | $\begin{Bmatrix} m \\ p \end{Bmatrix}$ | $[L^{\frac{1}{2}} M^{\frac{1}{2}}]$ | $[L^{\frac{3}{2}} M^{\frac{1}{2}} T^{-1}]$. |
| Electric current / Magnetic potential | $\begin{Bmatrix} C \\ \Omega \end{Bmatrix}$ | $[L^{\frac{3}{2}} M^{\frac{1}{2}} T^{-2}]$ | $[L^{\frac{1}{2}} M^{\frac{1}{2}} T^{-1}]$. |
| Electric displacement / Surface-density | $\mathfrak{D}$ | $[L^{-\frac{1}{2}} M^{\frac{1}{2}} T^{-1}]$ | $[L^{-\frac{3}{2}} M^{\frac{1}{2}}]$. |
| Electromotive intensity | $\mathfrak{E}$ | $[L^{-\frac{1}{2}} M^{\frac{1}{2}} T^{-1}]$ | $[L^{\frac{1}{2}} M^{\frac{1}{2}} T^{-2}]$. |
| Magnetic induction | $\mathfrak{B}$ | $[L^{-\frac{3}{2}} M^{\frac{1}{2}}]$ | $[L^{-\frac{1}{2}} M^{\frac{1}{2}} T^{-1}]$. |
| Magnetic force | $\mathfrak{H}$ | $[L^{\frac{1}{2}} M^{\frac{1}{2}} T^{-2}]$ | $[L^{-\frac{1}{2}} M^{\frac{1}{2}} T^{-1}]$. |
| Strength of current at a point | $\mathfrak{C}$ | $[L^{-\frac{1}{2}} M^{\frac{1}{2}} T^{-2}]$ | $[L^{-\frac{3}{2}} M^{\frac{1}{2}} T^{-1}]$. |
| Vector potential | $\mathfrak{A}$ | $[L^{-\frac{1}{2}} M^{\frac{1}{2}}]$ | $[L^{\frac{1}{2}} M^{\frac{1}{2}} T^{-1}]$. |

627.] We have already considered the products of the pairs of these quantities in the order in which they stand. Their ratios are in certain cases of scientific importance. Thus

| | Symbol. | Electrostatic System. | Electromagnetic System. |
|---|---|---|---|
| $\frac{e}{E}$ = capacity of an accumulator | $q$ | $[L]$ | $\left[\frac{T^2}{L}\right]$. |
| $\frac{p}{C}$ = coefficient of self-induction of a circuit, or electromagnetic capacity | $L$ | $\left[\frac{T^2}{L}\right]$ | $[L]$. |
| $\frac{\mathfrak{D}}{\mathfrak{E}}$ = specific inductive capacity of dielectric | $K$ | $[0]$ | $\left[\frac{T^2}{L^2}\right]$. |
| $\frac{\mathfrak{B}}{\mathfrak{H}}$ = magnetic inductive capacity | $\mu$ | $\left[\frac{T^2}{L^2}\right]$ | $[0]$. |
| $\frac{E}{C}$ = resistance of a conductor | $R$ | $\left[\frac{T}{L}\right]$ | $\left[\frac{L}{T}\right]$. |
| $\frac{\mathfrak{E}}{\mathfrak{C}}$ = specific resistance of a substance | $r$ | $[T]$ | $\left[\frac{L^2}{T}\right]$. |

628.] If the units of length, mass, and time are the same in the two systems, the number of electrostatic units of electricity contained in one electromagnetic unit is numerically equal to a certain velocity, the absolute value of which does not depend on the magnitude of the fundamental units employed. This velocity is an important physical quantity, which we shall denote by the symbol $v$.

*Number of Electrostatic Units in one Electromagnetic Unit.*

For $e$, $C$, $\Omega$, $\mathfrak{D}$, $\mathfrak{H}$, $\mathfrak{C}$,...... $v$.

For $m$, $p$, $E$, $\mathfrak{B}$, $\mathfrak{E}$, $\mathfrak{A}$,......$\frac{1}{v}$.

For electrostatic capacity, dielectric inductive capacity, and conductivity, $v^2$.

For electromagnetic capacity, magnetic inductive capacity, and resistance, $\frac{1}{v^2}$.

Several methods of determining the velocity $v$ will be given in Arts. 768–780.

In the electrostatic system the specific dielectric inductive capacity of air is assumed equal to unity. This quantity is therefore represented by $\frac{1}{v^2}$ in the electromagnetic system.

In the electromagnetic system the specific magnetic inductive capacity of air is assumed equal to unity. This quantity is therefore represented by $\frac{1}{v^2}$ in the electrostatic system.

*Practical System of Electric Units.*

629.] Of the two systems of units, the electromagnetic is of the greater use to those practical electricians who are occupied with electromagnetic telegraphs. If, however, the units of length, time, and mass are those commonly used in other scientific work, such as the mètre or the centimètre, the second, and the gramme, the units of resistance and of electromotive force will be so small that to express the quantities occurring in practice enormous numbers must be used, and the units of quantity and capacity will be so large that only exceedingly small fractions of them can ever occur in practice. Practical electricians have therefore adopted a set of electrical units deduced by the electromagnetic system from a large unit of length and a small unit of mass.

The unit of length used for this purpose is ten million of mètres, or approximately the length of a quadrant of a meridian of the earth.

The unit of time is, as before, one second.

The unit of mass is $10^{-11}$ grammes, or one hundred millionth part of a milligramme.

The electrical units derived from these fundamental units have been named after eminent electrical discoverers. Thus the practical unit of resistance is called the Ohm, and is represented by the resistance-coil issued by the British Association, and described in Art. 340. It is expressed in the electromagnetic system by a velocity of 10,000,000 metres per second.

The practical unit of electromotive force is called the Volt, and is not very different from that of a Daniell's cell. Mr. Latimer Clark has recently invented a very constant cell, whose electromotive force is almost exactly 1·454 Volts.

The practical unit of capacity is called the Farad. The quantity of electricity which flows through one Ohm under the electromotive force of one Volt during one second, is equal to the charge produced in a condenser whose capacity is one Farad by an electromotive force of one Volt.

The use of these names is found to be more convenient in practice than the constant repetition of the words 'electromagnetic units,' with the additional statement of the particular fundamental units on which they are founded.

When very large quantities are to be measured, a large unit is formed by multiplying the original unit by one million, and placing before its name the prefix *mega*.

In like manner by prefixing *micro* a small unit is formed, one millionth of the original unit.

The following table gives the values of these practical units in the different systems which have been at various times adopted.

| Fundamental Units. | Practical System. | B. A. Report, 1863. | Thomson. | Weber. |
|---|---|---|---|---|
| *Length,*<br>*Time,*<br>*Mass.* | *Earth's Quadrant,*<br>*Second,*<br>$10^{-11}$ *Gramme.* | *Metre,*<br>*Second,*<br>*Gramme.* | *Centimetre,*<br>*Second,*<br>*Gramme.* | *Millimetre,*<br>*Second,*<br>*Milligramme.* |
| Resistance | Ohm | $10^7$ | $10^9$ | $10^{10}$ |
| Electromotive force | Volt | $10^5$ | $10^8$ | $10^{11}$ |
| Capacity | Farad | $10^{-7}$ | $10^{-9}$ | $10^{-10}$ |
| Quantity | Farad (charged to a Volt.) | $10^{-2}$ | $10^{-1}$ | 10 |

# CHAPTER XI.

## ON ENERGY AND STRESS IN THE ELECTROMAGNETIC FIELD.

### *Electrostatic Energy.*

630.] THE energy of the system may be divided into the Potential Energy and the Kinetic Energy.

The potential energy due to electrification has been already considered in Art. 85. It may be written

$$W = \tfrac{1}{2}\Sigma(e\Psi), \tag{1}$$

where $e$ is the charge of electricity at a place where the electric potential is $\Psi$, and the summation is to be extended to every place where there is electrification.

If $f, g, h$ are the components of the electric displacement, the quantity of electricity in the element of volume $dx\,dy\,dz$ is

$$e = \left(\frac{df}{dx} + \frac{dg}{dy} + \frac{dh}{dz}\right) dx\,dy\,dz, \tag{2}$$

and

$$W = \tfrac{1}{2}\iiint \left(\frac{df}{dx} + \frac{dg}{dy} + \frac{dh}{dz}\right) \Psi\, dx\,dy\,dz, \tag{3}$$

where the integration is to be extended throughout all space.

631.] Integrating this expression by parts, and remembering that when the distance, $r$, from a given point of a finite electrified system becomes infinite, the potential $\Psi$ becomes an infinitely small quantity of the order $r^{-1}$, and that $f, g, h$ become infinitely small quantities of the order $r^{-2}$, the expression is reduced to

$$W = -\tfrac{1}{2}\iiint\left(f\frac{d\Psi}{dx} + g\frac{d\Psi}{dy} + h\frac{d\Psi}{dz}\right) dx\,dy\,dz, \tag{4}$$

where the integration is to be extended throughout all space.

If we now write $P$, $Q$, $R$ for the components of the electromotive intensity, instead of $-\frac{d\Psi}{dx}$, $-\frac{d\Psi}{dy}$ and $-\frac{d\Psi}{dz}$, we find

$$W = \tfrac{1}{2}\iiint(Pf+Qg+Rh)\,dx\,dy\,dz.* \qquad (5)$$

Hence, the electrostatic energy of the whole field will be the same if we suppose that it resides in every part of the field where electrical force and electrical displacement occur, instead of being confined to the places where free electricity is found.

The energy in unit of volume is half the product of the electromotive force and the electric displacement, multiplied by the cosine of the angle which these vectors include.

In Quaternion language it is $-\tfrac{1}{2}S.\mathfrak{E}\mathfrak{D}$.

## *Magnetic Energy.*

†632.] We may treat the energy due to magnetization in a way similar to that pursued in the case of electrification, Art. 85. If $A$, $B$, $C$ are the components of magnetization and $\alpha$, $\beta$, $\gamma$ the components of magnetic force, the potential energy of the system of magnets is then, by Art. 389,

$$-\tfrac{1}{2}\iiint(A\alpha+B\beta+C\gamma)\,dx\,dy\,dz, \qquad (6)$$

the integration being extended over the space occupied by magnetized matter. This part of the energy, however, will be included in the kinetic energy in the form in which we shall presently obtain it.

633.] We may transform this expression when there are no electric currents by the following method.

We know that
$$\frac{da}{dx}+\frac{db}{dy}+\frac{dc}{dz}=0. \qquad (7)$$

* {This expression for the electrostatic energy was deduced in the first volume on the assumption that the electrostatic force could be derived from a potential function. This proof will not hold when part of the electromotive intensity is due to electromagnetic induction. If however we take the view that this part of the energy arises from the polarized state of the dielectric and is per unit volume $\frac{1}{8\pi K}(f^2+g^2+h^2)$, the potential energy will then only depend on the polarization of the dielectric no matter how it is produced. Thus the energy will, since

$$\frac{f}{4\pi K}=P,\quad \frac{g}{4\pi K}=Q,\quad \frac{h}{4\pi K}=R,$$

be equal to $\tfrac{1}{2}(Pf+Qg+Rh)$ per unit volume.}

† See Appendix I at the end of this Chapter.

Hence, by Art. 97, if

$$a = -\frac{d\Omega}{dx}, \quad \beta = -\frac{d\Omega}{dy}, \quad \gamma = -\frac{d\Omega}{dz}, \tag{8}$$

as is always the case in magnetic phenomena where there are no currents,

$$\iiint (a\alpha + b\beta + c\gamma)\, dx\, dy\, dz = 0, \tag{9}$$

the integral being extended throughout all space, or

$$\iiint \{(\alpha + 4\pi A)\, \alpha + (\beta + 4\pi B)\, \beta + (\gamma + 4\pi C)\gamma\}\, dx\, dy\, dz = 0. \tag{10}$$

Hence, the energy due to a magnetic system

$$-\tfrac{1}{2}\iiint (A\alpha + B\beta + C\gamma)\, dx dy dz = \frac{1}{8\pi}\iiint (\alpha^2 + \beta^2 + \gamma^2)\, dx\, dy\, dz,$$

$$= \frac{1}{8\pi}\iiint \mathfrak{H}^2\, dx\, dy\, dz. \tag{11}$$

*Electrokinetic Energy.*

634.] We have already, in Art. 578, expressed the kinetic energy of a system of currents in the form

$$T = \tfrac{1}{2}\Sigma(pi), \tag{12}$$

where $p$ is the electromagnetic momentum of a circuit, and $i$ is the strength of the current flowing round it, and the summation extends to all the circuits.

But we have proved, in Art. 590, that $p$ may be expressed as a line-integral of the form

$$p = \int \left(F\frac{dx}{ds} + G\frac{dy}{ds} + H\frac{dz}{ds}\right) ds, \tag{13}$$

where $F$, $G$, $H$ are the components of the electromagnetic momentum, $\mathfrak{A}$, at the point $(x,y,z)$, and the integration is to be extended round the closed circuit $s$. We therefore find

$$T = \tfrac{1}{2}\Sigma i \int \left(F\frac{dx}{ds} + G\frac{dy}{ds} + H\frac{dz}{ds}\right) ds. \tag{14}$$

If $u$, $v$, $w$ are the components of the density of the current at any point of the conducting circuit, and if $S$ is the transverse section of the circuit, then we may write

$$i\frac{dx}{ds} = uS, \quad i\frac{dy}{ds} = vS, \quad i\frac{dz}{ds} = wS, \tag{15}$$

and we may also write the volume

$$S ds = dx dy dz,$$

and we now find

$$T = \tfrac{1}{2}\iiint (Fu + Gv + Hw)\,dx\,dy\,dz, \tag{16}$$

where the integration is to be extended to every part of space where there are electric currents.

635.] Let us now substitute for $u$, $v$, $w$ their values as given by the equations of electric currents (E), Art. 607, in terms of the components $\alpha$, $\beta$, $\gamma$ of the magnetic force. We then have

$$T = \frac{1}{8\pi}\iiint \left\{F\left(\frac{d\gamma}{dy} - \frac{d\beta}{dz}\right) + G\left(\frac{d\alpha}{dz} - \frac{d\gamma}{dx}\right) + H\left(\frac{d\beta}{dx} - \frac{d\alpha}{dy}\right)\right\} dx\,dy\,dz, \tag{17}$$

where the integration is extended over a portion of space including all the currents.

If we integrate this by parts, and remember that, at a great distance $r$ from the system, $\alpha$, $\beta$, and $\gamma$ are of the order of magnitude $r^{-3}$, {and that at a surface separating two media, $F$, $G$, $H$, and the tangential magnetic force are continuous,} we find that when the integration is extended throughout all space, the expression is reduced to

$$T = \frac{1}{8\pi}\iiint \left\{\alpha\left(\frac{dH}{dy} - \frac{dG}{dz}\right) + \beta\left(\frac{dF}{dz} - \frac{dH}{dx}\right) + \gamma\left(\frac{dG}{dx} - \frac{dF}{dy}\right)\right\} dx\,dy\,dz. \tag{18}$$

By the equations (A), Art. 591, of magnetic induction, we may substitute for the quantities in small brackets the components of magnetic induction $a$, $b$, $c$, so that the kinetic energy may be written

$$T = \frac{1}{8\pi}\iiint (a\alpha + b\beta + c\gamma)\,dx\,dy\,dz, \tag{19}$$

where the integration is to be extended throughout every part of space in which the magnetic force and magnetic induction have values differing from zero.

The quantity within brackets in this expression is the product of the magnetic induction into the resolved part of the magnetic force in its own direction.

In the language of quaternions this may be written more simply,

$$-S\,.\,\mathfrak{B}\mathfrak{H},$$

where $\mathfrak{B}$ is the magnetic induction, whose components are $a$, $b$, $c$, and $\mathfrak{H}$ is the magnetic force, whose components are $\alpha$, $\beta$, $\gamma$.

636.] The electrokinetic energy of the system may therefore be expressed either as an integral to be taken where there are electric currents, or as an integral to be taken over every part of the field in which magnetic force exists. The first integral, however, is the natural expression of the theory which supposes the currents to act upon each other directly at a distance, while the second is appropriate to the theory which endeavours to explain the action between the currents by means of some intermediate action in the space between them. As in this treatise we have adopted the latter method of investigation, we naturally adopt the second expression as giving the most significant form to the kinetic energy.

According to our hypothesis, we assume the kinetic energy to exist wherever there is magnetic force, that is, in general, in every part of the field. The amount of this energy per unit of volume is $-\frac{1}{8\pi}S.\mathfrak{B}\mathfrak{H}$, and this energy exists in the form of some kind of motion of the matter in every portion of space.

When we come to consider Faraday's discovery of the effect of magnetism on polarized light, we shall point out reasons for believing that wherever there are lines of magnetic force, there is a rotatory motion of matter round those lines. See Art. 821.

*Magnetic and Electrokinetic Energy compared.*

637.] We found in Art. 423 that the mutual potential energy of two magnetic shells, of strengths $\phi$ and $\phi'$, and bounded by the closed curves $s$ and $s'$ respectively, is

$$-\phi\phi'\iint\frac{\cos\epsilon}{r}\,ds\,ds',$$

where $\epsilon$ is the angle between the directions of $ds$ and $ds'$, and $r$ is the distance between them.

We also found in Art. 521 that the mutual energy of two circuits $s$ and $s'$, in which currents $i$ and $i'$ flow, is

$$i\,i'\iint\frac{\cos\epsilon}{r}\,ds\,ds'.$$

If $i$, $i'$ are equal to $\phi$, $\phi'$ respectively, the mechanical action between the magnetic shells is equal to that between the corresponding electric circuits, and in the same direction. In the case of the magnetic shells the force tends to diminish their mutual potential energy, in the case of the circuits it tends to increase their mutual energy, because this energy is kinetic.

It is impossible, by any arrangement of magnetized matter, to produce a system corresponding in all respects to an electric circuit, for the potential of the magnetic system is single valued at every point of space, whereas that of the electric system is many-valued.

But it is always possible, by a proper arrangement of infinitely small electric circuits, to produce a system corresponding in all respects to any magnetic system, provided the line of integration which we follow in calculating the potential is prevented from passing through any of these small circuits. This will be more fully explained in Art. 833.

The action of magnets at a distance is perfectly identical with that of electric currents. We therefore endeavour to trace both to the same cause, and since we cannot explain electric currents by means of magnets, we must adopt the other alternative, and explain magnets by means of molecular electric currents.

638.] In our investigation of magnetic phenomena, in Part III of this treatise, we made no attempt to account for magnetic action at a distance, but treated this action as a fundamental fact of experience. We therefore assumed that the energy of a magnetic system is potential energy, and that this energy is *diminished* when the parts of the system yield to the magnetic forces which act on them.

If, however, we regard magnets as deriving their properties from electric currents circulating within their molecules, their energy is kinetic, and the force between them is such that it tends to move them in a direction such that if the strengths of the currents were maintained constant the kinetic energy would *increase.*

This mode of explaining magnetism requires us also to abandon the method followed in Part III, in which we regarded the magnet as a continuous and homogeneous body, the minutest part of which has magnetic properties of the same kind as the whole.

We must now regard a magnet as containing a finite, though very great, number of electric circuits, so that it has essentially a molecular, as distinguished from a continuous structure.

If we suppose our mathematical machinery to be so coarse that our line of integration cannot thread a molecular circuit, and that an immense number of magnetic molecules are contained in our element of volume, we shall still arrive at results similar to those of Part III, but if we suppose our machinery of a finer order, and capable of investigating all that goes on in the interior of the molecules, we must give up the old theory of magnetism, and adopt that of Ampère, which admits of no magnets except those which consist of electric currents.

We must also regard both magnetic and electromagnetic energy as kinetic energy, and we must attribute to it the proper sign, as given in Art. 635.

In what follows, though we may occasionally, as in Art. 639, &c., attempt to carry out the old theory of magnetism, we shall find that we obtain a perfectly consistent system only when we abandon that theory and adopt Ampère's theory of molecular currents, as in Art. 644.

The energy of the field therefore consists of two parts only, the electrostatic or potential energy

$$W = \tfrac{1}{2}\iiint (Pf+Qg+Rh)\,dx\,dy\,dz,$$

and the electromagnetic or kinetic energy

$$T = \frac{1}{8\pi}\iiint (a\alpha+b\beta+c\gamma)\,dx\,dy\,dz.$$

## ON THE FORCES WHICH ACT ON AN ELEMENT OF A BODY PLACED IN THE ELECTROMAGNETIC FIELD.

### *Forces acting on a Magnetic Element.*

*639.] The potential energy of the element $dx\,dy\,dz$ of a body magnetized with an intensity whose components are $A$, $B$, $C$, and placed in a field of magnetic force whose components are $\alpha$, $\beta$, $\gamma$, is
$$-(A\alpha+B\beta+C\gamma)\,dx\,dy\,dz.$$

Hence, if the force urging the element to move without rotation in the direction of $x$ is $X_1\,dx\,dy\,dz$,

$$X_1 = A\frac{d\alpha}{dx}+B\frac{d\beta}{dx}+C\frac{d\gamma}{dx}, \tag{1}$$

and if the moment of the couple tending to turn the element about the axis of $x$ from $y$ towards $z$ is $L\,dx\,dy\,dz$,

$$L = B\gamma - C\beta. \tag{2}$$

The forces and the moments corresponding to the axes of $y$ and $z$ may be written down by making the proper substitutions.

640.] If the magnetized body carries an electric current, of which the components are $u$, $v$, $w$, then, by equations ($C$), Art. 603,

* See Appendix II at the end of this Chapter.

there will be an additional electromagnetic force whose components are $X_2$, $Y_2$, $Z_2$, of which $X_2$ is given by

$$X_2 = vc - wb. \tag{3}$$

Hence, the total force, $X$, arising from the magnetism of the molecule, as well as the current passing through it, is

$$X = A\frac{d\alpha}{dx} + B\frac{d\beta}{dx} + C\frac{d\gamma}{dx} + vc - wb. \tag{4}$$

The quantities $a, b, c$ are the components of magnetic induction, and are related to $\alpha, \beta, \gamma$, the components of magnetic force, by the equations given in Art. 400,

$$\left.\begin{aligned} a &= \alpha + 4\pi A, \\ b &= \beta + 4\pi B, \\ c &= \gamma + 4\pi C. \end{aligned}\right\} \tag{5}$$

The components of the current, $u, v, w$, can be expressed in terms of $\alpha, \beta, \gamma$ by the equations of Art. 607,

$$\left.\begin{aligned} 4\pi u &= \frac{d\gamma}{dy} - \frac{d\beta}{dz}, \\ 4\pi v &= \frac{d\alpha}{dz} - \frac{d\gamma}{dx}, \\ 4\pi w &= \frac{d\beta}{dx} - \frac{d\alpha}{dy}. \end{aligned}\right\} \tag{6}$$

Hence

$$\begin{aligned} X &= \frac{1}{4\pi}\left\{(a-\alpha)\frac{d\alpha}{dx} + (b-\beta)\frac{d\beta}{dx} + (c-\gamma)\frac{d\gamma}{dx} + b\left(\frac{d\alpha}{dy} - \frac{d\beta}{dx}\right) + c\left(\frac{d\alpha}{dz} - \frac{d\gamma}{dx}\right)\right\}, \\ &= \frac{1}{4\pi}\left\{a\frac{d\alpha}{dx} + b\frac{d\alpha}{dy} + c\frac{d\alpha}{dz} - \frac{1}{2}\frac{d}{dx}(\alpha^2+\beta^2+\gamma^2)\right\}. \end{aligned} \tag{7}$$

By Art. 403, $$\frac{da}{dx} + \frac{db}{dy} + \frac{dc}{dz} = 0. \tag{8}$$

Multiplying this equation, (8), by $\alpha$, and dividing by $4\pi$, we may add the result to (7), and we find

$$X = \frac{1}{4\pi}\left\{\frac{d}{dx}\left[a\alpha - \tfrac{1}{2}(\alpha^2+\beta^2+\gamma^2)\right] + \frac{d}{dy}[b\alpha] + \frac{d}{dz}[c\alpha]\right\}, \tag{9}$$

also, by (2), $$L = \frac{1}{4\pi}((b-\beta)\gamma - (c-\gamma)\beta), \tag{10}$$

$$= \frac{1}{4\pi}(b\gamma - c\beta), \tag{11}$$

where $X$ is the force referred to unit of volume in the direction of $x$, and $L$ is the moment of the forces (per unit volume) about this axis.

*On the Explanation of these Forces by the Hypothesis of a Medium in a State of Stress.*

641.] Let us denote a stress of any kind referred to unit of area by a symbol of the form $P_{hk}$, where the first suffix, $_h$, indicates that the normal to the surface on which the stress is supposed to act is parallel to the axis of $h$, and the second suffix, $_k$, indicates that the direction of the stress with which the part of the body on the positive side of the surface acts on the part on the negative side is parallel to the axis of $k$.

The directions of $h$ and $k$ may be the same, in which case the stress is a normal stress. They may be oblique to each other, in which case the stress is an oblique stress, or they may be perpendicular to each other, in which case the stress is a tangential stress.

The condition that the stresses shall not produce any tendency to rotation in the elementary portions of the body is

$$P_{hk} = P_{kh}.$$

In the case of a magnetized body, however, there is such a tendency to rotation, and therefore this condition, which holds in the ordinary theory of stress, is not fulfilled.

Let us consider the effect of the stresses on the six sides of the elementary portion of the body $dx\,dy\,dz$, taking the origin of coordinates at its centre of gravity.

On the positive face $dy\,dz$, for which the value of $x$ is $\frac{1}{2}dx$, the forces are—

$$\left.\begin{array}{ll}\text{Parallel to } x, & \left(P_{xx}+\frac{1}{2}\frac{dP_{xx}}{dx}dx\right)dy\,dz = X_{+x},\\[2ex] \text{Parallel to } y, & \left(P_{xy}+\frac{1}{2}\frac{dP_{xy}}{dx}dx\right)dy\,dz = Y_{+x},\\[2ex] \text{Parallel to } z, & \left(P_{xz}+\frac{1}{2}\frac{dP_{xz}}{dx}dx\right)dy\,dz = Z_{+x}.\end{array}\right\} \qquad (12)$$

The forces acting on the opposite side, $-X_{-x}$, $-Y_{-x}$, and $-Z_{-x}$, may be found from these by changing the sign of $dx$. We may express in the same way the systems of three forces acting on each of the other faces of the element, the direction of the force being indicated by the capital letter, and the face on which it acts by the suffix.

If $X\,dxdydz$ is the whole force parallel to $x$ acting on the element,

$$X\,dxdydz = X_{+x} + X_{+y} + X_{+z} + X_{-x} + X_{-y} + X_{-z},$$
$$= \left(\frac{dP_{xx}}{dx} + \frac{dP_{yx}}{dy} + \frac{dP_{zx}}{dz}\right) dxdydz,$$

whence
$$X = \frac{d}{dx}P_{xx} + \frac{d}{dy}P_{yx} + \frac{d}{dz}P_{zx}. \tag{13}$$

If $L\,dxdydz$ is the moment of the forces about the axis of $x$ tending to turn the element from $y$ to $z$,

$$L\,dxdydz = \tfrac{1}{2}\,dy\,(Z_{+y} - Z_{-y}) - \tfrac{1}{2}\,dz\,(Y_{+z} - Y_{-z}),$$
$$= (P_{yz} - P_{zy})\,dx\,dy\,dz,$$

whence
$$L = P_{yz} - P_{zy}. \tag{14}$$

Comparing the values of $X$ and $L$ given by equations (9) and (11) with those given by (13) and (14), we find that, if we make

$$\left.\begin{array}{ll}
P_{xx} = \dfrac{1}{4\pi}\{a\alpha - \tfrac{1}{2}(\alpha^2+\beta^2+\gamma^2)\}, & \\
P_{yy} = \dfrac{1}{4\pi}\{b\beta - \tfrac{1}{2}(\alpha^2+\beta^2+\gamma^2)\}, & \\
P_{zz} = \dfrac{1}{4\pi}\{c\gamma - \tfrac{1}{2}(\alpha^2+\beta^2+\gamma^2)\}, & \\
P_{yz} = \dfrac{1}{4\pi}b\gamma, & P_{zy} = \dfrac{1}{4\pi}c\beta, \\
P_{zx} = \dfrac{1}{4\pi}c\alpha, & P_{xz} = \dfrac{1}{4\pi}a\gamma, \\
P_{xy} = \dfrac{1}{4\pi}a\beta, & P_{yx} = \dfrac{1}{4\pi}b\alpha,
\end{array}\right\} \tag{15}$$

the force arising from a system of stress of which these are the components will be statically equivalent, in its effects on each element of the body, to the forces arising from the magnetization and electric currents.

642.] The nature of the stress of which these are the components may be easily found, by making the axis of $x$ bisect the angle between the directions of the magnetic force and the magnetic induction, and taking the axis of $y$ in the plane of these directions, and measured towards the side of the magnetic force.

If we put $\mathfrak{H}$ for the numerical value of the magnetic force, $\mathfrak{B}$ for that of the magnetic induction, and $2\epsilon$ for the angle between their directions,

$$\left.\begin{array}{lll} \alpha = \mathfrak{H}\cos\epsilon, & \beta = -\mathfrak{H}\sin\epsilon, & \gamma = 0, \\ a = \mathfrak{B}\cos\epsilon, & b = -\mathfrak{B}\sin\epsilon, & c = 0; \end{array}\right\} \tag{16}$$

$$\left.\begin{aligned} P_{xx} &= \frac{1}{4\pi}(+\mathfrak{B}\mathfrak{H}\cos^2\epsilon - \tfrac{1}{2}\mathfrak{H}^2), \\ P_{yy} &= \frac{1}{4\pi}(-\mathfrak{B}\mathfrak{H}\sin^2\epsilon - \tfrac{1}{2}\mathfrak{H}^2), \\ P_{zz} &= \frac{1}{4\pi}(-\tfrac{1}{2}\mathfrak{H}^2), \\ P_{yz} &= P_{zx} = P_{zy} = P_{xz} = 0, \\ P_{xy} &= \frac{1}{4\pi}\mathfrak{B}\mathfrak{H}\cos\epsilon\sin\epsilon, \\ P_{yx} &= -\frac{1}{4\pi}\mathfrak{B}\mathfrak{H}\cos\epsilon\sin\epsilon. \end{aligned}\right\} \tag{17}$$

Hence, the state of stress may be considered as compounded of—

(1) A pressure equal in all directions $= \frac{1}{8\pi}\mathfrak{H}^2$.

(2) A tension along the line bisecting the angle between the directions of the magnetic force and the magnetic induction

$$= \frac{1}{4\pi}\mathfrak{B}\mathfrak{H}\cos^2\epsilon.$$

(3) A pressure along the line bisecting the exterior angle between these directions $= \frac{1}{4\pi}\mathfrak{B}\mathfrak{H}\sin^2\epsilon$.

(4) A couple tending to turn every element of the substance in the plane of the two directions *from* the direction of magnetic induction *to* the direction of magnetic force $= \frac{1}{4\pi}\mathfrak{B}\mathfrak{H}\sin 2\epsilon$.

When the magnetic induction is in the same direction as the magnetic force, as it always is in fluids and non-magnetized solids, then $\epsilon = 0$, and making the axis of $x$ coincide with the direction of the magnetic force,

$$P_{xx} = \frac{1}{4\pi}(\mathfrak{B}\mathfrak{H} - \tfrac{1}{2}\mathfrak{H}^2), \qquad P_{yy} = P_{zz} = -\frac{1}{8\pi}\mathfrak{H}^2, \tag{18}$$

and the tangential stresses disappear.

The stress in this case is therefore a hydrostatic pressure $\frac{1}{8\pi}\mathfrak{H}^2$, combined with a longitudinal tension $\frac{1}{4\pi}\mathfrak{B}\mathfrak{H}$ along the lines of force.

643.] When there is no magnetization, $\mathfrak{B} = \mathfrak{H}$, and the stress is still further simplified, being a tension along the lines of force equal to $\frac{1}{8\pi}\mathfrak{H}^2$, combined with a pressure in all directions at right angles to the line of force, numerically equal also to $\frac{1}{8\pi}\mathfrak{H}^2$. The components of stress in this important case are

$$\left.\begin{aligned} P_{xx} &= \frac{1}{8\pi}(a^2-\beta^2-\gamma^2), \\ P_{yy} &= \frac{1}{8\pi}(\beta^2-\gamma^2-a^2), \\ P_{zz} &= \frac{1}{8\pi}(\gamma^2-a^2-\beta^2), \\ P_{yz} = P_{zy} &= \frac{1}{4\pi}\beta\gamma, \\ P_{zx} = P_{xz} &= \frac{1}{4\pi}\gamma a, \\ P_{xy} = P_{yx} &= \frac{1}{4\pi}a\beta. \end{aligned}\right\} \qquad (19)$$

The $x$-component of the force arising from these stresses on an element of the medium referred to unit of volume is

$$\begin{aligned} X &= \frac{d}{dx}P_{xx} + \frac{d}{dy}P_{yx} + \frac{d}{dz}P_{zx}, \\ &= \frac{1}{4\pi}\left\{a\frac{da}{dx} - \beta\frac{d\beta}{dx} - \gamma\frac{d\gamma}{dx}\right\} + \frac{1}{4\pi}\left\{a\frac{d\beta}{dy} + \beta\frac{da}{dy}\right\} + \frac{1}{4\pi}\left\{a\frac{d\gamma}{dz} + \gamma\frac{da}{dz}\right\}, \\ &= \frac{1}{4\pi}a\left(\frac{da}{dx} + \frac{d\beta}{dy} + \frac{d\gamma}{dz}\right) + \frac{1}{4\pi}\gamma\left(\frac{da}{dz} - \frac{d\gamma}{dx}\right) - \frac{1}{4\pi}\beta\left(\frac{d\beta}{dx} - \frac{da}{dy}\right). \end{aligned}$$

Now
$$\begin{aligned} \frac{da}{dx} + \frac{d\beta}{dy} + \frac{d\gamma}{dz} &= 4\pi\mathfrak{m}, \\ \frac{da}{dz} - \frac{d\gamma}{dx} &= 4\pi v, \\ \frac{d\beta}{dx} - \frac{da}{dy} &= 4\pi w, \end{aligned}$$

where $\mathfrak{m}$ is the density of austral magnetic matter referred to unit of volume, and $v$ and $w$ are the intensities of electric currents perpendicular to $y$ and $z$ respectively. Hence,

Similarly
$$\left.\begin{aligned} X &= \alpha\mathfrak{m} + v\gamma - w\beta. \\ Y &= \beta\mathfrak{m} + w\alpha - u\gamma, \\ Z &= \gamma\mathfrak{m} + u\beta - v\alpha. \end{aligned}\right\} \quad \begin{array}{c}\text{(Equations of}\\ \text{Electromagnetic}\\ \text{Force.)}\end{array} \tag{20}$$

644.] If we adopt the theories of Ampère and Weber as to the nature of magnetic and diamagnetic bodies, and assume that magnetic and diamagnetic polarity are due to molecular electric currents, we get rid of imaginary magnetic matter, and find that everywhere $\mathfrak{m} = 0$, and

$$\frac{d\alpha}{dx} + \frac{d\beta}{dy} + \frac{d\gamma}{dz} = 0, \tag{21}$$

so that the equations of electromagnetic force become

$$\left.\begin{aligned} X &= v\gamma - w\beta, \\ Y &= w\alpha - u\gamma, \\ Z &= u\beta - v\alpha. \end{aligned}\right\} \tag{22}$$

These are the components of the mechanical force referred to unit of volume of the substance. The components of the magnetic force are $\alpha$, $\beta$, $\gamma$, and those of the electric current are $u$, $v$, $w$. These equations are identical with those already established. (Equations (C), Art. 603.)

645.] In explaining the electromagnetic force by means of a state of stress in a medium, we are only following out the conception of Faraday*, that the lines of magnetic force tend to shorten themselves, and that they repel each other when placed side by side. All that we have done is to express the value of the tension along the lines, and the pressure at right angles to them, in mathematical language, and to prove that the state of stress thus assumed to exist in the medium will actually produce the observed forces on the conductors which carry electric currents.

We have asserted nothing as yet with respect to the mode in which this state of stress is originated and maintained in the medium. We have merely shewn that it is possible to conceive the mutual action of electric currents to depend on a particular kind of stress in the surrounding medium, instead of being a direct and immediate action at a distance.

Any further explanation of the state of stress, by means of the motion of the medium or otherwise, must be regarded as

* *Exp. Res.*, 3266, 3267, 3268.

a separate and independent part of the theory, which may stand or fall without affecting our present position. See Art. 832.

In the first part of this treatise, Art. 108, we shewed that the observed electrostatic forces may be conceived as operating through the intervention of a state of stress in the surrounding medium. We have now done the same for the electromagnetic forces, and it remains to be seen whether the conception of a medium capable of supporting these states of stress is consistent with other known phenomena, or whether we shall have to put it aside as unfruitful.

In a field in which electrostatic as well as electromagnetic action is taking place, we must suppose the electrostatic stress described in Part I to be superposed on the electromagnetic stress which we have been considering.

646.] If we suppose the total terrestrial magnetic force to be 10 British units (grain, foot, second), as it is nearly in Britain, then the tension along the lines of force is 0·128 grains weight per square foot. The greatest magnetic tension produced by Joule * by means of electromagnets was about 140 pounds weight on the square inch.

* Sturgeon's *Annals of Electricity*, vol. v. p. 187 (1840); or *Philosophical Magazine*, Dec. 1851.

# APPENDIX I.

[The following note, derived from a letter written by Professor Clerk Maxwell to Professor Chrystal, is important in connexion with Arts. 389 and 632:—

In Art. 389 the energy due to the presence of a magnet whose magnetization components are $A_1$, $B_1$, $C_1$, placed in a field whose magnetic force components are $\alpha_2$, $\beta_2$, $\gamma_2$, is

$$-\iiint (A_1 \alpha_2 + B_1 \beta_2 + C_1 \gamma_2)\, dx\, dy\, dz,$$

where the integration is confined to the magnet in virtue of $A_1$, $B_1$, $C_1$ being zero everywhere else.

But the whole energy is of the form

$$-\tfrac{1}{2}\iiint \{ (A_1 + A_2)(\alpha_1 + \alpha_2) + \&c. \}\, dx\, dy\, dz,$$

the integration extending to every part of space where there are magnetized bodies, and $A_2$, $B_2$, $C_2$ denoting the components of magnetization at any point exterior to the magnet.

The whole energy thus consists of four parts:—

$$-\tfrac{1}{2}\iiint (A_1 \alpha_1 + \&c.)\, dx\, dy\, dz, \tag{1}$$

which is constant if the magnetization of the magnet is rigid;

$$-\tfrac{1}{2}\iiint (A_2 \alpha_1 + \&c.)\, dx\, dy\, dz, \tag{2}$$

which is equal, by Green's Theorem, to

$$-\tfrac{1}{2}\iiint (A_1 \alpha_2 + \&c.)\, dx\, dy\, dz, \tag{3}$$

and

$$-\tfrac{1}{2}\iiint (A_2 \alpha_2 + \&c.)\, dx\, dy\, dz, \tag{4}$$

which last we may suppose to arise from rigid magnetizations and therefore to be constant.

Hence the variable part of the energy of the moveable magnet, as rigidly magnetized, is the sum of the expressions (2) and (3), viz.,

$$-\iiint (A_1 \alpha_2 + B_1 \beta_2 + C_1 \gamma_2)\, dx\, dy\, dz.$$

Remembering that the displacement of the magnet alters the values of $\alpha_2$, $\beta_2$, $\gamma_2$, but not those of $A_1$, $B_1$, $C_1$, we find for the component of the force on the magnet in any direction $\phi$—

$$\iiint \left( A_1 \frac{d\alpha_2}{d\phi} + B_1 \frac{d\beta_2}{d\phi} + C_1 \frac{d\gamma_2}{d\phi} \right) dx\, dy\, dz.$$

If instead of a magnet we have a body magnetized by induction, the expression for the force must be the same, viz., writing $A_1 = \kappa\alpha$, &c., we have

$$\iiint \kappa \left( \alpha \frac{d\alpha_2}{d\phi} + \beta \frac{d\beta_2}{d\phi} + \gamma \frac{d\gamma_2}{d\phi} \right) dx\, dy\, dz.$$

In this expression $\alpha$ is put for $\alpha_1 + \alpha_2$, &c., but if either the magnetized body be small or $\kappa$ be small we may neglect $\alpha_1$ in comparison with $\alpha_2$, and the expression for the force becomes, as in Art. 440,

$$\frac{d}{d\phi}\, \tfrac{1}{2} \iiint \kappa (\alpha^2 + \beta^2 + \gamma^2)\, dx\, dy\, dz.$$

The work done by the magnetic forces while a body of small inductive capacity, magnetized inductively, is carried off to infinity is only half of that for the same body rigidly magnetized to the same original strength, for as the induced magnet is carried off it loses its strength.]

# APPENDIX II.

[Objection has been taken to the expression contained in Art. 639 for the potential energy per unit volume of the medium arising from magnetic forces, for the reason that in finding that expression in Art. 389 we assumed the force components $\alpha$, $\beta$, $\gamma$ to be derivable from a potential, whereas in Arts. 639, 640 this is not the case. This objection extends to the expression for the force $X$, which is the space variation of the energy. The purpose of this note is to bring forward some considerations tending to confirm the accuracy of the text.]

{The force on a piece of magnetic substance carrying a current may for convenience of calculation be divided into two parts, (1) the force on the element in consequence of the presence of the current, (2) the force due to the magnetism in the element. The first part will be the same as the force on an element of a non-magnetic substance, the components being respectively,

$$\left.\begin{aligned} &\gamma v-\beta w, \\ &\alpha w-\gamma u, \\ &\beta u-\alpha v, \end{aligned}\right\{ \begin{array}{l} u,\ v,\ w \text{ being components} \\ \text{of current, } \alpha,\ \beta,\ \gamma \text{ those} \\ \text{of magnetic force.} \end{array}$$

To calculate the second force imagine a long narrow cylinder cut out of the magnetic substance, the axis of the cylinder being parallel to the direction of magnetization.

If $I$ is the intensity of magnetization the force parallel to $x$ on the magnet per unit volume is

$$I\frac{d\alpha}{ds},$$

or, if $A$, $B$, $C$ are the components of $I$,

$$A\frac{d\alpha}{dx}+B\frac{d\alpha}{dy}+C\frac{d\alpha}{dz},$$

or

$$A\frac{d\alpha}{dx}+B\left(\frac{d\beta}{dx}-4\pi w\right)+C\left(\frac{d\gamma}{dx}+4\pi v\right).$$

The total force on the element parallel to $x$ is therefore

$$\gamma v-\beta w+A\frac{d\alpha}{dx}+B\left(\frac{d\beta}{dx}-4\pi w\right)+C\left(\frac{d\gamma}{dx}+4\pi v\right),$$

or

$$v(\gamma+4\pi C)-w(\beta+4\pi B)+A\frac{d\alpha}{dx}+B\frac{d\beta}{dx}+C\frac{d\gamma}{dx},$$

i. e.

$$vc-wb+A\frac{d\alpha}{dx}+B\frac{d\beta}{dx}+C\frac{d\gamma}{dx},$$

the expression in the text.}

# CHAPTER XII.

## CURRENT-SHEETS.

647.] A CURRENT-SHEET is an infinitely thin stratum of conducting matter, bounded on both sides by insulating media, so that electric currents may flow in the sheet, but cannot escape from it except at certain points called Electrodes, where currents are made to enter or to leave the sheet.

In order to conduct a finite electric current, a real sheet must have a finite thickness, and ought therefore to be considered a conductor of three dimensions. In many cases, however, it is practically convenient to deduce the electric properties of a real conducting sheet, or of a thin layer of coiled wire, from those of a current-sheet as defined above.

We may therefore regard a surface of any form as a current-sheet. Having selected one side of this surface as the positive side, we shall always suppose any lines drawn on the surface to be looked at from the positive side of the surface. In the case of a closed surface we shall consider the outside as positive. See Art. 294, where, however, the direction of the current is defined as seen from the *negative* side of the sheet.

### *The Current-function.*

648.] Let a fixed point $A$ on the surface be chosen as origin, and let a line be drawn on the surface from $A$ to another point $P$. Let the quantity of electricity which in unit of time crosses this line from left to right be $\phi$, then $\phi$ is called the Current-function at the point $P$.

The current-function depends only on the position of the point $P$ and is the same for any two forms of the line $AP$, provided this line can be transformed by continuous motion from one form to the other without passing through an electrode. For the two forms of the line will enclose an area within which there is no electrode, and therefore the same quantity of electricity which enters the area across one of the lines must issue across the other.

If $s$ denote the length of the line $AP$, the current across $ds$ from left to right will be $\frac{d\phi}{ds}ds$.

If $\phi$ is constant for any curve, there is no current across it. Such a curve is called a Current-line or a Stream-line.

649.] Let $\psi$ be the electric potential at any point of the sheet, then the electromotive force along any element $ds$ of a curve will be

$$-\frac{d\psi}{ds}ds,$$

provided no electromotive force exists except that which arises from differences of potential.

If $\psi$ is constant for any curve, the curve is called an Equipotential Line.

650.] We may now suppose that the position of a point on the sheet is defined by the values of $\phi$ and $\psi$ at that point. Let $ds_1$ be the length of the element of the equipotential line $\psi$ intercepted between the two current lines $\phi$ and $\phi + d\phi$, and let $ds_2$ be the length of the element of the current line $\phi$ intercepted between the two equipotential lines $\psi$ and $\psi + d\psi$. We may consider $ds_1$ and $ds_2$ as the sides of the element $d\phi\, d\psi$ of the sheet. The electromotive force $-d\psi$ in the direction of $ds_2$ produces the current $d\phi$ across $ds_1$.

Let the resistance of a portion of the sheet whose length is $ds_2$, and whose breadth is $ds_1$, be

$$\sigma\frac{ds_2}{ds_1},$$

where $\sigma$ is the specific resistance of the sheet referred to unit of area, then

$$d\psi = \sigma\frac{ds_2}{ds_1}d\phi,$$

whence

$$\frac{ds_1}{d\phi} = \sigma\frac{ds_2}{d\psi}.$$

651.] If the sheet is of a substance which conducts equally well in all directions, $ds_1$ is perpendicular to $ds_2$. In the case

of a sheet of uniform resistance $\sigma$ is constant, and if we make $\psi = \sigma\psi'$, we shall have
$$\frac{\delta s_1}{\delta s_2} = \frac{\delta \phi}{\delta \psi'},$$
and the stream-lines and equipotential lines will cut the surface into little squares.

It follows from this that if $\phi_1$ and $\psi_1'$ are conjugate functions (Art. 183) of $\phi$ and $\psi'$, the curves $\phi_1$ may be stream-lines in the sheet for which the curves $\psi_1'$ are the corresponding equipotential lines. One case, of course, is that in which $\phi_1 = \psi'$ and $\psi_1' = -\phi$. In this case the equipotential lines become current-lines, and the current-lines equipotential lines *.

If we have obtained the solution of the distribution of electric currents in a uniform sheet of any form for any particular case, we may deduce the distribution in any other case by a proper transformation of the conjugate functions, according to the method given in Art. 190.

652.] We have next to determine the magnetic action of a current-sheet in which the current is entirely confined to the sheet, there being no electrodes to convey the current to or from the sheet.

In this case the current-function $\phi$ has a determinate value at every point, and the stream-lines are closed curves which do not intersect each other, though any one stream-line may intersect itself.

Consider the annular portion of the sheet between the stream-lines $\phi$ and $\phi + \delta\phi$. This part of the sheet is a conducting circuit in which a current of strength $\delta\phi$ circulates in the positive direction round that part of the sheet for which $\phi$ is greater than the given value. The magnetic effect of this circuit is the same as that of a magnetic shell of strength $\delta\phi$ at any point not included in the substance of the shell. Let us suppose that the shell coincides with that part of the current-sheet for which $\phi$ has a greater value than it has at the given stream-line.

By drawing all the successive stream-lines, beginning with that for which $\phi$ has the greatest value, and ending with that for which its value is least, we shall divide the current-sheet into a series of circuits. Substituting for each circuit its corresponding magnetic shell, we find that the magnetic effect of the

* See Thomson, *Camb. Math. Journ.*, vol. iii. p. 286.

current-sheet at any point not included in the thickness of the sheet is the same as that of a complex magnetic shell, whose strength at any point is $C+\phi$, where $C$ is a constant.

If the current-sheet is bounded, then we must make $C+\phi = 0$ at the bounding curve. If the sheet forms a closed or an infinite surface, there is nothing to determine the value of the constant $C$.

653.] The magnetic potential at any point on either side of the current-sheet is given, as in Art. 415, by the expression

$$\Omega = \iint \frac{1}{r^2}\phi \cos\theta\, dS,$$

where $r$ is the distance of the given point from the element of surface $dS$, and $\theta$ is the angle between the direction of $r$, and that of the normal drawn from the positive side of $dS$.

This expression gives the magnetic potential for all points not included in the thickness of the current-sheet, and we know that for points within a conductor carrying a current there is no such thing as a magnetic potential.

The value of $\Omega$ is discontinuous at the current-sheet, for if $\Omega_1$ is its value at a point just within the current-sheet, and $\Omega_2$ its value at a point close to the first but just outside the current-sheet,

$$\Omega_2 = \Omega_1 + 4\pi\phi,$$

where $\phi$ is the current-function at that point of the sheet.

The value of the component of magnetic force normal to the sheet is continuous, being the same on both sides of the sheet. The component of the magnetic force parallel to the current-lines is also continuous, but the tangential component perpendicular to the current-lines is discontinuous at the sheet. If $s$ is the length of a curve drawn on the sheet, the component of magnetic force in the direction of $ds$ is, for the negative side, $-\frac{d\Omega_1}{ds}$, and for the positive side, $-\frac{d\Omega_2}{ds} = -\frac{d\Omega_1}{ds} - 4\pi\frac{d\phi}{ds}$.

The component of the magnetic force on the positive side therefore exceeds that on the negative side by $-4\pi\frac{d\phi}{ds}$. At a given point this quantity will be a maximum when $ds$ is perpendicular to the current-lines.

*On the Induction of Electric Currents in a Sheet of Infinite Conductivity.*

654.] It was shewn in Art. 579 that in any circuit

$$E = \frac{dp}{dt} + Ri,$$

where $E$ is the impressed electromotive force, $p$ the electrokinetic momentum of the circuit, $R$ the resistance of the circuit, and $i$ the current round it. If there is no impressed electromotive force and no resistance, then $\frac{dp}{dt} = 0$, or $p$ is constant.

Now $p$, the electrokinetic momentum of the circuit, was shewn in Art. 588 to be measured by the surface-integral of magnetic induction through the circuit. Hence, in the case of a current-sheet of no resistance, the surface-integral of magnetic induction through any closed curve drawn on the surface must be constant, and this implies that the normal component of magnetic induction remains constant at every point of the current-sheet.

655.] If, therefore, by the motion of magnets or variations of currents in the neighbourhood, the magnetic field is in any way altered, electric currents will be set up in the current-sheet, such that their magnetic effect, combined with that of the magnets or currents in the field, will maintain the normal component of magnetic induction at every point of the sheet unchanged. If at first there is no magnetic action, and no currents in the sheet, then the normal component of magnetic induction will always be zero at every point of the sheet.

The sheet may therefore be regarded as impervious to magnetic induction, and the lines of magnetic induction will be deflected by the sheet exactly in the same way as the lines of flow of an electric current in an infinite and uniform conducting mass would be deflected by the introduction of a sheet of the same form made of a substance of infinite resistance.

If the sheet forms a closed or an infinite surface, no magnetic actions which may take place on one side of the sheet will produce any magnetic effect on the other side.

*Theory of a Plane Current-sheet.*

656.] We have seen that the external magnetic action of a current-sheet is equivalent to that of a magnetic shell whose strength at any point is numerically equal to $\phi$, the current-function. When the sheet is a plane one, we may express all the quantities required for the determination of electromagnetic effects in terms of a single function, $P$, which is the potential due to a sheet of imaginary matter spread over the plane with a surface-density $\phi$. The value of $P$ is of course

$$P = \iint \frac{\phi}{r} dx'\, dy', \tag{1}$$

where $r$ is the distance from the point $(x, y, z)$ for which $P$ is calculated, to the point $(x', y', 0)$ in the plane of the sheet, at which the element $dx'dy'$ is taken.

To find the magnetic potential, we may regard the magnetic shell as consisting of two surfaces parallel to the plane of $xy$, the first, whose equation is $z = \frac{1}{2}c$, having the surface-density $\frac{\phi}{c}$, and the second, whose equation is $z = -\frac{1}{2}c$, having the surface-density $-\frac{\phi}{c}$.

The potentials due to these surfaces will be

$$\frac{1}{c} P_{\left(z-\frac{c}{2}\right)} \quad \text{and} \quad -\frac{1}{c} P_{\left(z+\frac{c}{2}\right)}$$

respectively, where the suffixes indicate that $z - \frac{c}{2}$ is put for $z$ in the first expression, and $z + \frac{c}{2}$ for $z$ in the second. Expanding these expressions by Taylor's Theorem, adding them, and then making $c$ infinitely small, we obtain for the magnetic potential due to the sheet at any point external to it,

$$\Omega = -\frac{dP}{dz}. \tag{2}$$

657.] The quantity $P$ is symmetrical with respect to the plane of the sheet, and is therefore the same when $-z$ is substituted for $z$.

$\Omega$, the magnetic potential, changes sign when $-z$ is put for $z$.

At the positive surface of the sheet

$$\Omega = -\frac{dP}{dz} = 2\pi\phi. \tag{3}$$

At the negative surface of the sheet

$$\Omega = -\frac{dP}{dz} = -2\pi\phi. \tag{4}$$

Within the sheet, if its magnetic effects arise from the magnetization of its substance, the magnetic potential varies continuously from $2\pi\phi$ at the positive surface to $-2\pi\phi$ at the negative surface.

If the sheet contains electric currents, the magnetic force within it does not satisfy the condition of having a potential. The magnetic force within the sheet is, however, perfectly determinate.

The normal component,

$$\gamma = -\frac{d\Omega}{dz} = \frac{d^2P}{dz^2}, \tag{5}$$

is the same on both sides of the sheet and throughout its substance.

If $\alpha$ and $\beta$ be the components of the magnetic force parallel to $x$ and to $y$ at the positive surface, and $\alpha'$, $\beta'$ those on the negative surface,

$$\alpha = -2\pi\frac{d\phi}{dx} = -\alpha', \tag{6}$$

$$\beta = -2\pi\frac{d\phi}{dy} = -\beta'. \tag{7}$$

Within the sheet the components vary continuously from $\alpha$ and $\beta$ to $\alpha'$ and $\beta'$.

The equations

$$\left.\begin{aligned} \frac{dH}{dy} - \frac{dG}{dz} &= -\frac{d\Omega}{dx}, \\ \frac{dF}{dz} - \frac{dH}{dx} &= -\frac{d\Omega}{dy}, \\ \frac{dG}{dx} - \frac{dF}{dy} &= -\frac{d\Omega}{dz}, \end{aligned}\right\} \tag{8}$$

which connect the components $F$, $G$, $H$ of the vector-potential due to the current-sheet with the scalar potential $\Omega$, are satisfied if we make

$$F = \frac{dP}{dy}, \qquad G = -\frac{dP}{dx}, \qquad H = 0. \tag{9}$$

We may also obtain these values by direct integration, thus for $F$ {we have by Art. 616 if $\mu$ is everywhere equal to unity},

$$F = \iint \frac{u}{r}\,dx'\,dy' = \iint \frac{1}{r}\frac{d\phi}{dy'}\,dx'\,dy',$$

$$= \int \frac{\phi}{r}\,dx' - \iint \phi\,\frac{d}{dy'}\frac{1}{r}\,dx'\,dy'.$$

Since the integration is to be estimated over the infinite plane sheet, and since the first term vanishes at infinity, the expression is reduced to the second term; and by substituting

$$\frac{d}{dy}\frac{1}{r} \text{ for } -\frac{d}{dy'}\frac{1}{r},$$

and remembering that $\phi$ depends on $x'$ and $y'$, and not on $x, y, z$, we obtain

$$F = \frac{d}{dy}\iint \frac{\phi}{r}\, dx'\, dy',$$

$$= \frac{dP}{dy}, \text{ by (1).}$$

If $\Omega'$ is the magnetic potential due to any magnetic or electric system external to the sheet, we may write

$$P' = -\int \Omega'\, dz, \tag{10}$$

and we shall then have

$$F' = \frac{dP'}{dy}, \qquad G' = -\frac{dP'}{dx}, \qquad H' = 0, \tag{11}$$

for the components of the vector-potential due to this system.

658.] Let us now determine the electromotive intensity at any point of the sheet, supposing the sheet fixed.

Let $X$ and $Y$ be the components of the electromotive intensity parallel to $x$ and $y$ respectively, then, by Art. 598, we have {writing $\psi$ for $\Psi$}

$$X = -\frac{d}{dt}(F+F') - \frac{d\psi}{dx}, \tag{12}$$

$$Y = -\frac{d}{dt}(G+G') - \frac{d\psi}{dy}. \tag{13}$$

If the electric resistance of the sheet is uniform and equal to $\sigma$,

$$X = \sigma u, \qquad Y = \sigma v, \tag{14}$$

where $u$ and $v$ are the components of the current, and if $\phi$ is the current-function,

$$u = \frac{d\phi}{dy}, \qquad v = -\frac{d\phi}{dx}. \tag{15}$$

But, by equation (3), $\quad 2\pi\phi = -\dfrac{dP}{dz}$

at the positive surface of the current-sheet. Hence, equations (12) and (13) may be written

$$-\frac{\sigma}{2\pi}\frac{d^2P}{dy\,dz} = -\frac{d^2}{dy\,dt}(P+P') - \frac{d\psi}{dx}, \tag{16}$$

$$\frac{\sigma}{2\pi}\frac{d^2P}{dx\,dz} = \frac{d^2}{dx\,dt}(P+P') - \frac{d\psi}{dy}, \tag{17}$$

where the values of the expressions are those corresponding to the positive surface of the sheet.

If we differentiate the first of these equations with respect to $x$, and the second with respect to $y$, and add the results, we obtain

$$\frac{d^2\psi}{dx^2}+\frac{d^2\psi}{dy^2}=0. \tag{18}$$

The only value of $\psi$ which satisfies this equation, and is finite and continuous at every point of the plane, and vanishes at an infinite distance, is

$$\psi = 0. \tag{19}$$

Hence the induction of electric currents in an infinite plane sheet of uniform conductivity is not accompanied with differences of electric potential in different parts of the sheet.

Substituting this value of $\psi$, and integrating equations (16), (17), we obtain

$$\frac{\sigma}{2\pi}\frac{dP}{dz}-\frac{dP}{dt}-\frac{dP'}{dt}=f(z,t). \tag{20}$$

Since the values of the currents in the sheet are found by differentiating with respect to $x$ or $y$, the arbitrary function of $z$ and $t$ will disappear. We shall therefore leave it out of account.

If we also write for $\frac{\sigma}{2\pi}$, the single symbol $R$, which represents a certain velocity, the equation between $P$ and $P'$ becomes

$$R\frac{dP}{dz}=\frac{dP}{dt}+\frac{dP'}{dt}. \tag{21}$$

659.] Let us first suppose that there is no external magnetic system acting on the current-sheet. We may therefore suppose $P'=0$. The case then becomes that of a system of electric currents in the sheet left to themselves, but acting on one another by their mutual induction, and at the same time losing their energy on account of the resistance of the sheet. The result is expressed by the equation

$$R\frac{dP}{dz}=\frac{dP}{dt}, \tag{22}$$

the solution of which is $P=F\{x, y, (z+Rt)\}$. (23)

*Hence, the value of $P$ at any point on the positive side

* [The equations (20) and (22) are proved to be true only at the surface of the sheet for which $z=0$. The expression (23) satisfies (22) generally, and therefore also at the surface of the sheet. It also satisfies the other conditions of the problem, and is therefore a solution. 'Any other solution must differ from this by a system of closed currents, depending on the initial state of the sheet, not due to any external cause, and which therefore must decay rapidly. Hence, since we assume an eternity of past time, this is the only solution of the problem.' See Professor Clerk Maxwell's Paper, *Royal Soc. Proc.*, xx. pp. 160–168.]

of the sheet whose coordinates are $x$, $y$, $z$, and at a time $t$, is equal to the value of $P$ at the point $x$, $y$, $(z+Rt)$ at the instant when $t = 0$.

If therefore a system of currents is excited in a uniform plane sheet of infinite extent and then left to itself, its magnetic effect at any point on the positive side of the sheet will be the same as if the system of currents had been maintained constant in the sheet, and the sheet moved in the direction of a normal from its negative side with the constant velocity $R$. The diminution of the electromagnetic forces, which arises from a decay of the currents in the real case, is accurately represented by the diminution of the forces on account of the increasing distance in the imaginary case.

660.] Integrating equation (21) with respect to $t$, we obtain

$$P + P' = \int R \frac{dP}{dz}\, dt. \tag{24}$$

If we suppose that at first $P$ and $P'$ are both zero, and that a magnet or electromagnet is suddenly magnetized or brought from an infinite distance, so as to change the value of $P'$ suddenly from zero to $P'$, then, since the time-integral in the second member of (24) vanishes with the time, we must have at the first instant $P = -P'$ at the surface of the sheet.

Hence, the system of currents excited in the sheet by the sudden introduction of the system to which $P'$ is due, is such that at the surface of the sheet it exactly neutralizes the magnetic effect of this system.

At the surface of the sheet, therefore, and consequently at all points on the negative side of it, the initial system of currents produces an effect exactly equal and opposite to that of the magnetic system on the positive side. We may express this by saying that the effect of the currents is equivalent to that of an *image* of the magnetic system, coinciding in position with that system, but opposite as regards the direction of its magnetization and of its electric currents. Such an image is called a *negative* image.

The effect of the currents in the sheet at a point on the positive side of it is equivalent to that of a positive image of the magnetic system on the negative side of the sheet, the lines joining corresponding points being bisected at right angles by the sheet.

The action at a point on either side of the sheet, due to the currents in the sheet, may therefore be regarded as due to an image of the magnetic system on the side of the sheet opposite to the point, this image being a positive or a negative image according as the point is on the positive or the negative side of the sheet.

661.] If the sheet is of infinite conductivity, $R = 0$, and the right-hand side of (24) is zero, so that the image will represent the effect of the currents in the sheet at any time.

In the case of a real sheet, the resistance $R$ has some finite value. The image just described will therefore represent the effect of the currents only during the first instant after the sudden introduction of the magnetic system. The currents will immediately begin to decay, and the effect of this decay will be accurately represented if we suppose the two images to move from their original positions, in the direction of normals drawn from the sheet, with the constant velocity $R$.

662.] We are now prepared to investigate the system of currents induced in the sheet by any system, $M$, of magnets or electromagnets on the positive side of the sheet, the position and strength of which vary in any manner.

Let $P'$, as before, be the function from which the direct action of this system is to be deduced by the equations (3), (9), &c., then $\frac{dP'}{dt}\,\delta t$ will be the function corresponding to the system represented by $\frac{dM}{dt}\,\delta t$. This quantity, which is the increment of $M$ in the time $\delta t$, may be regarded as itself representing a magnetic system.

If we suppose that at the time $t$ a positive image of the system $\frac{dM}{dt}\,\delta t$ is formed on the negative side of the sheet, the magnetic action at any point on the positive side of the sheet due to this image will be equivalent to that due to the currents in the sheet excited by the change in $M$ during the first instant after the change, and the image will continue to be equivalent to the currents in the sheet, if, as soon as it is formed, it begins to move in the negative direction of $z$ with the constant velocity $R$.

If we suppose that in every successive element of the time an image of this kind is formed, and that as soon as it is formed it begins to move away from the sheet with velocity $R$, we shall obtain the conception of a trail of images, the last of which is in process of formation, while all the rest are moving like a rigid body away from the sheet with velocity $R$.

663.] If $P'$ denotes any function whatever arising from the action of the magnetic system, we may find $P$, the corresponding function arising from the currents in the sheet, by the following process, which is merely the symbolical expression for the theory of the trail of images.

Let $P_\tau$ denote the value of $P$ (the function arising from the currents in the sheet) at the point $(x, y, z+R\tau)$, and at the time $t-\tau$, and let $P'_\tau$ denote the value of $P'$ (the function arising from the magnetic system) at the point $(x, y, -(z+R\tau))$, and at the time $t-\tau$. Then

$$\frac{dP_\tau}{d\tau} = R\frac{dP_\tau}{dz} - \frac{dP_\tau}{dt}, \tag{25}$$

and equation (21) becomes

$$\frac{dP_\tau}{d\tau} = \frac{dP'_\tau}{dt}, \tag{26}$$

and we obtain by integrating with respect to $\tau$ from $\tau = 0$ to $\tau = \infty$,

$$P = -\int_0^\infty \frac{dP'_\tau}{dt}\,d\tau \tag{27}$$

as the value of the function $P$, whence we obtain all the properties of the current-sheet by differentiation, as in equations (3), (9), &c.*

664.] As an example of the process here indicated, let us take

* {This proof may be arranged as follows: let $\mathfrak{P}_\tau$ be the value of $P$ at the time $t-\tau$ at the point $x, y, -(z+R\tau)$, the rest of the notation being the same as in the text. Then since $\mathfrak{P}_\tau$ is a function of $x, y, z+R\tau, t-\tau$ we have

$$\frac{d\mathfrak{P}_\tau}{d\tau} = R\frac{d\mathfrak{P}_\tau}{dz} - \frac{d\mathfrak{P}_\tau}{dt};$$

and since by the footnote on page 294 equation (21) is satisfied at all points in the field and not merely in the plane, we have

$$\frac{d\mathfrak{P}_\tau}{d\tau} = \frac{dP'_\tau}{dt},$$

hence

$$\mathfrak{P}_\tau = -\int_0^\infty \frac{dP'_\tau}{dt}\,d\tau;$$

but since $P$ has the same value at any point as at the image of the point in the plane sheet,

$$\mathfrak{P}_\tau = P_\tau,$$

hence

$$P_\tau = -\int_0^\infty \frac{dP'_\tau}{dt}\,d\tau.\}$$

the case of a single magnetic pole of strength unity, moving with uniform velocity in a straight line.

Let the coordinates of the pole at the time $t$ be

$$\xi = \mathfrak{u}t, \qquad \eta = 0, \qquad \zeta = c + \mathfrak{w}t.$$

The coordinates of the image of the pole formed at the time $t-\tau$ are

$$\xi = \mathfrak{u}(t-\tau), \qquad \eta = 0, \qquad \zeta = -(c+\mathfrak{w}(t-\tau)+R\tau),$$

and if $r$ is the distance of this image from the point $(x, y, z)$,

$$r^2 = \big(x-\mathfrak{u}(t-\tau)\big)^2 + y^2 + (z+c+\mathfrak{w}(t-\tau)+R\tau)^2.$$

To obtain the potential due to the trail of images we have to calculate

$$-\frac{d}{dt}\int_0^\infty \frac{d\tau}{r}.$$

If we write $$Q^2 = \mathfrak{u}^2 + (R-\mathfrak{w})^2,$$

$$\int_0^\infty \frac{d\tau}{r} = -\frac{1}{Q}\log\{Qr + \mathfrak{u}(x-\mathfrak{u}t) + (R-\mathfrak{w})(z+c+\mathfrak{w}t)\},$$

+ a term infinitely great which however will disappear on differentiation with regard to $t$, the value of $r$ in this expression being found by making $\tau = 0$ in the expression for $r$ given above.

Differentiating this expression with respect to $t$, and putting $t = 0$, we obtain the magnetic potential due to the trail of images,

$$\Omega = \frac{1}{Q}\,\frac{Q\dfrac{\mathfrak{w}(z+c)-\mathfrak{u}x}{r} - \mathfrak{u}^2 - \mathfrak{w}^2 + R\mathfrak{w}}{Qr + \mathfrak{u}x + (R-\mathfrak{w})(z+c)}.$$

By differentiating this expression with respect to $x$ or $z$, we obtain the components parallel to $x$ or $z$ respectively of the magnetic force at any point, and by putting $x = 0$, $z = c$, and $r = 2c$ in these expressions, we obtain the following values of the components of the force acting on the moving pole itself,

$$X = -\frac{1}{4c^2}\frac{\mathfrak{u}}{Q+R-\mathfrak{w}}\left\{1 + \frac{\mathfrak{w}}{Q} - \frac{\mathfrak{u}^2}{Q(Q+R-\mathfrak{w})}\right\},$$

$$Z = -\frac{1}{4c^2}\left\{\frac{\mathfrak{w}}{Q} - \frac{\mathfrak{u}^2}{Q(Q+R-\mathfrak{w})}\right\}^*.$$

665.] In these expressions we must remember that the motion is supposed to have been going on for an infinite time before the

* {These expressions may be written in the simpler forms

$$X = -\frac{1}{4c^2}\frac{R}{Q}\frac{\mathfrak{u}}{Q+R-\mathfrak{w}},$$

$$Z = \frac{1}{4c^2}\left(1-\frac{R}{Q}\right)\}.$$

time considered. Hence we must not take $\mathfrak{w}$ a positive quantity, for in that case the pole must have passed through the sheet within a finite time.

If we make $\mathfrak{u} = 0$, and $\mathfrak{w}$ negative, $X = 0$, and

$$Z = \frac{1}{4c^2}\frac{\mathfrak{w}}{R+\mathfrak{w}},$$

or the pole as it approaches the sheet is repelled from it.

If we make $\mathfrak{w} = 0$, we find $Q^2 = \mathfrak{u}^2 + R^2$,

$$X = -\frac{1}{4c^2}\frac{\mathfrak{u}R}{Q(Q+R)} \quad \text{and} \quad Z = \frac{1}{4c^2}\frac{\mathfrak{u}^2}{Q(Q+R)}.$$

The component $X$ represents a retarding force acting on the pole in the direction opposite to that of its own motion. For a given value of $R$, $X$ is a maximum when $\mathfrak{u} = 1{\cdot}27\,R$.

When the sheet is a non-conductor, $R = \infty$ and $X = 0$.

When the sheet is a perfect conductor, $R = 0$ and $X = 0$.

The component $Z$ represents a repulsion of the pole from the sheet. It increases as the velocity $\mathfrak{u}$ increases, and ultimately becomes $\frac{1}{4c^2}$ when the velocity is infinite. It has the same value when $R$ is zero.

666.] When the magnetic pole moves in a curve parallel to the sheet, the calculation becomes more complicated, but it is easy to see that the effect of the nearest portion of the trail of images is to produce a force acting on the pole in the direction opposite to that of its motion. The effect of the portion of the trail immediately behind this is of the same kind as that of a magnet with its axis parallel to the direction of motion of the pole at some time before. Since the nearest pole of this magnet is of the same name with the moving pole, the force will consist partly of a repulsion, and partly of a force parallel to the former direction of motion, but backwards. This may be resolved into a retarding force, and a force towards the concave side of the path of the moving pole.

667.] Our investigation does not enable us to solve the case in which the system of currents cannot be completely formed, on account of a discontinuity or boundary of the conducting sheet.

It is easy to see, however, that if the pole is moving parallel to the edge of the sheet, the currents on the side next the edge will be enfeebled. Hence the forces due to these currents will be less, and there will not only be a smaller retarding force, but, since the repulsive force is least on the side next the edge, the pole will be attracted towards the edge.

*Theory of Arago's Rotating Disk.*

668.] Arago discovered * that a magnet placed near a rotating metallic disk experiences a force tending to make it follow the motion of the disk, although when the disk is at rest there is no action between it and the magnet.

This action of a rotating disk was attributed to a new kind of induced magnetization, till Faraday † explained it by means of the electric currents induced in the disk on account of its motion through the field of magnetic force.

To determine the distribution of these induced currents, and their effect on the magnet, we might make use of the results already found for a conducting sheet at rest acted on by a moving magnet, availing ourselves of the method given in Art. 600 for treating the electromagnetic equations when referred to a moving system of axes. As this case, however, has a special importance, we shall treat it in a direct manner, beginning by assuming that the poles of the magnet are so far from the edge of the disk that the effect of the limitation of the conducting sheet may be neglected.

Making use of the same notation as in the preceding articles (656–667), we find {equations 13, § 598, writing $\psi$ for $\Psi$} for the components of the electromotive intensity parallel to $x$ and $y$ respectively,

$$\left.\begin{aligned} \sigma u &= \gamma\frac{dy}{dt} - \frac{d\psi}{dx}, \\ \sigma v &= -\gamma\frac{dx}{dt} - \frac{d\psi}{dy}, \end{aligned}\right\} \tag{1}$$

where $\gamma$ is the resolved part of the magnetic force normal to the disk.

If we now express $u$ and $v$ in terms of $\phi$, the current-function,

$$u = \frac{d\phi}{dy}, \qquad v = -\frac{d\phi}{dx}, \tag{2}$$

and if the disk is rotating about the axis of $z$ with the angular velocity $\omega$,

$$\frac{dy}{dt} = \omega x, \qquad \frac{dx}{dt} = -\omega y. \tag{3}$$

* *Annales de Chimie et de Physique*, Tome 32, pp. 213–223, 1826.
† *Exp. Res.*, 81.

Substituting these values in equations (1), we find

$$\sigma \frac{d\phi}{dy} = \gamma\omega x - \frac{d\psi}{dx}, \tag{4}$$

$$-\sigma \frac{d\phi}{dx} = \gamma\omega y - \frac{d\psi}{dy}. \tag{5}$$

Multiplying (4) by $x$ and (5) by $y$, and adding, we obtain

$$\sigma\left(x\frac{d\phi}{dy} - y\frac{d\phi}{dx}\right) = \gamma\omega\,(x^2+y^2) - \left(x\frac{d\psi}{dx} + y\frac{d\psi}{dy}\right). \tag{6}$$

Multiplying (4) by $y$ and (5) by $-x$, and adding, we obtain

$$\sigma\left(x\frac{d\phi}{dx} + y\frac{d\phi}{dy}\right) = x\frac{d\psi}{dy} - y\frac{d\psi}{dx}. \tag{7}$$

If we now express these equations in terms of $r$ and $\theta$, where

$$x = r\cos\theta, \qquad y = r\sin\theta, \tag{8}$$

they become

$$\sigma\frac{d\phi}{d\theta} = \gamma\omega r^2 - r\frac{d\psi}{dr}, \tag{9}$$

$$\sigma r\frac{d\phi}{dr} = \frac{d\psi}{d\theta}. \tag{10}$$

Equation (10) is satisfied if we assume any arbitrary function $\chi$ of $r$ and $\theta$, and make

$$\phi = \frac{d\chi}{d\theta}, \tag{11}$$

$$\psi = \sigma r\frac{d\chi}{dr}. \tag{12}$$

Substituting these values in equation (9), it becomes

$$\sigma\left(\frac{d^2\chi}{d\ ^2} + r\frac{d}{dr}\left(r\frac{d\chi}{dr}\right)\right) = \gamma\omega r^2. \tag{13}$$

Dividing by $\sigma r^2$, and restoring the coordinates $x$ and $y$, this becomes

$$\frac{d^2\chi}{dx^2} + \frac{d^2\chi}{dy^2} = \frac{\omega}{\sigma}\gamma. \tag{14}$$

This is the fundamental equation of the theory, and expresses the relation between the function, $\chi$, and the component, $\gamma$, of the magnetic force resolved normal to the disk.

Let $Q$ be the potential, at any point on the positive side of the disk, due to imaginary attracting matter distributed over the disk with the surface-density $\chi$.

At the positive surface of the disk

$$\frac{dQ}{dz} = -2\pi\chi. \tag{15}$$

Hence the first member of equation (14) becomes

$$\frac{d^2\chi}{dx^2}+\frac{d^2\chi}{dy^2}=-\frac{1}{2\pi}\frac{d}{dz}\left(\frac{d^2Q}{dx^2}+\frac{d^2Q}{dy^2}\right). \tag{16}$$

But since $Q$ satisfies Laplace's equation at all points external to the disk,

$$\frac{d^2Q}{dx^2}+\frac{d^2Q}{dy^2}=-\frac{d^2Q}{dz^2}, \tag{17}$$

and equation (14) becomes

$$\frac{\sigma}{2\pi}\frac{d^3Q}{dz^3}=\omega\gamma. \tag{18}$$

Again, since $Q$ is the potential due to the distribution $\chi$, the potential due to the distribution $\phi$, or $\frac{d\chi}{d\theta}$, will be $\frac{dQ}{d\theta}$. From this we obtain for the magnetic potential due to the currents in the disk,

$$\Omega_1=-\frac{d^2Q}{d\theta\, dz}, \tag{19}$$

and for the component of the magnetic force normal to the disk due to the currents,

$$\gamma_1=-\frac{d\Omega}{dz}=\frac{d^3Q}{d\theta\, dz^2}. \tag{20}$$

If $\Omega_2$ is the magnetic potential due to external magnets, and if we write

$$P'=-\int\Omega_2 dz, \tag{21}$$

the component of the magnetic force normal to the disk due to the magnets will be

$$\gamma_2=\frac{d^2P'}{dz^2}. \tag{22}$$

We may now write equation (18), remembering that

$$\gamma=\gamma_1+\gamma_2,$$

$$\frac{\sigma}{2\pi}\frac{d^3Q}{dz^3}-\omega\frac{d^3Q}{d\theta\, dz^2}=\omega\frac{d^2P'}{dz^2}. \tag{23}$$

Integrating twice with respect to $z$, and writing $R$ for $\frac{\sigma}{2\pi}$,

$$\left(R\frac{d}{dz}-\omega\frac{d}{d\theta}\right)Q=\omega P'. \tag{24}$$

If the values of $P$ and $Q$ are expressed in terms of $r$, the distance from the axis of the disk, and of $\xi$ and $\zeta$ two new variables such that

$$2\xi=z+\frac{R}{\omega}\theta, \qquad 2\zeta=z-\frac{R}{\omega}\theta, \tag{25}$$

equation (24) becomes, by integration with respect to $\zeta$,

$$Q = \int \frac{\omega}{R} P' \, d\zeta. \tag{26}$$

669.] The form of this expression taken in conjunction with the method of Art. 662 shews that the magnetic action of the currents in the disk is equivalent to that of a trail of images of the magnetic system in the form of a helix.

If the magnetic system consists of a single magnetic pole of strength unity, the helix will lie on the cylinder whose axis is that of the disk, and which passes through the magnetic pole. The helix will begin at the position of the optical image of the pole in the disk. The distance, parallel to the axis, between consecutive coils of the helix will be $2\pi\frac{R}{\omega}\cdot$ The magnetic effect of the trail will be the same as if this helix had been magnetized everywhere in the direction of a tangent to the cylinder perpendicular to its axis, with an intensity such that the magnetic moment of any small portion is numerically equal to the length of its projection on the disk.

The calculation of the effect on the magnetic pole would be complicated, but it is easy to see that it will consist of—

(1) A dragging force, parallel to the direction of motion of the disk.

(2) A repulsive force acting from the disk.

(3) A force towards the axis of the disk.

When the pole is near the edge of the disk, the third of these forces may be overcome by the force towards the edge of the disk, indicated in Art. 667 *.

All these forces were observed by Arago, and described by him in the *Annales de Chimie* for 1826. See also Felici, in Tortolini's *Annals*, iv, p. 173 (1853), and v, p. 35; and E. Jochmann, in *Crelle's Journal*, lxiii, pp. 158 and 329; also in Pogg. *Ann.* cxxii, p. 214 (1864). In the latter paper the equations necessary for determining the induction of the currents on themselves are given, but this part of the action is omitted in the subsequent calculation of results. The method of images given here was published in the *Proceedings of the Royal Society* for Feb. 15, 1872.

* {If $a$ is the distance of a pole from the axis of the disk, $c$ its height above the disk, we can prove that for small values of $\omega$, the dragging force on the pole is $m^2 a\omega/8c^2 R$, the repulsive force $m^2 a^2 \omega^2/8c^2 R^2$, the force towards the axis $m^2 a\omega^2/4cR^2$.}

*Spherical Current-Sheet.*

670.] Let $\phi$ be the current-function at any point $Q$ of a spherical current-sheet, and let $P$ be the potential at a given point, due to a sheet of imaginary matter distributed over the sphere with surface-density $\phi$, it is required to find the magnetic potential and the vector-potential of the current-sheet in terms of $P$.

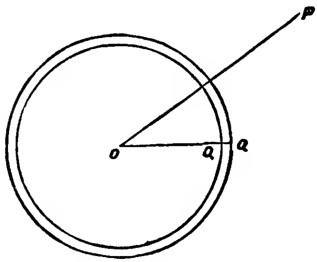

Fig. 39.

Let $a$ denote the radius of the sphere, $r$ the distance of the given point from the centre, and $p$ the reciprocal of the distance of the given point from the point $Q$ on the sphere at which the current-function is $\phi$.

The action of the current-sheet at any point not in its substance is identical with that of a magnetic shell whose strength at any point is numerically equal to the current-function.

The mutual potential of the magnetic shell and a unit pole placed at the point $P$ is, by Art. 410,

$$\Omega = \iint \phi \frac{dp}{da} dS.$$

Since $p$ is a homogeneous function of the degree $-1$ in $r$ and $a$,

$$a\frac{dp}{da} + r\frac{dp}{dr} = -p,$$

$$\text{or} \quad \frac{dp}{da} = -\frac{1}{a}\frac{d}{dr}(pr),$$

$$\text{and} \quad \Omega = -\iint \frac{\phi}{a}\frac{d}{dr}(pr) dS.$$

Since $r$ and $a$ are constant throughout the surface-integration,

$$\Omega = -\frac{1}{a}\frac{d}{dr}\left(r\iint \phi p dS\right).$$

But if $P$ is the potential due to a sheet of imaginary matter of surface-density $\phi$,

$$P = \iint \phi p dS,$$

and $\Omega$, the magnetic potential of the current-sheet, may be expressed in terms of $P$ in the form

$$\Omega = -\frac{1}{a}\frac{d}{dr}(Pr).$$

671.] We may determine $F$, the $x$-component of the vector-potential, from the expression given in Art. 416,

$$F = \iint \phi \left( m\frac{dp}{d\zeta} - n\frac{dp}{d\eta} \right) dS,$$

where $\xi, \eta, \zeta$ are the coordinates of the element $dS$, and $l, m, n$ are the direction-cosines of the normal.

Since the sheet is a sphere, the direction-cosines of the normal are

$$l = \frac{\xi}{a}, \qquad m = \frac{\eta}{a}, \qquad n = \frac{\zeta}{a},$$

But
$$\frac{dp}{d\zeta} = (z-\zeta)p^3 = -\frac{dp}{dz},$$

and
$$\frac{dp}{d\eta} = (y-\eta)p^3 = -\frac{dp}{dy},$$

so that
$$m\frac{dp}{d\zeta} - n\frac{dp}{d\eta} = \{\eta(z-\zeta) - \zeta(y-\eta)\}\frac{p^3}{a},$$

$$= \{z(\eta - y) - y(\zeta - z)\}\frac{p^3}{a},$$

$$= \frac{z}{a}\frac{dp}{dy} - \frac{y}{a}\frac{dp}{dz}.$$

Multiplying by $\phi dS$, and integrating over the surface of the sphere, we find

$$F = \frac{z}{a}\frac{dP}{dy} - \frac{y}{a}\frac{dP}{dz}.$$

Similarly
$$G = \frac{x}{a}\frac{dP}{dz} - \frac{z}{a}\frac{dP}{dx},$$

$$H = \frac{y}{a}\frac{dP}{dx} - \frac{x}{a}\frac{dP}{dy}.$$

The vector $\mathfrak{A}$, whose components are $F, G, H$, is evidently perpendicular to the radius vector $r$, and to the vector whose components are $\frac{dP}{dx}$, $\frac{dP}{dy}$, and $\frac{dP}{dz}$. If we determine the lines of intersection of the spherical surface whose radius is $r$, with the series of equipotential surfaces corresponding to values of $P$ in arithmetical progression, these lines will indicate by their direction the direction of $\mathfrak{A}$, and by their proximity the magnitude of this vector.

In the language of Quaternions,

$$\mathfrak{A} = \frac{1}{a} V . \rho \nabla P.$$

672.] If we assume as the value of $P$ within the sphere

$$P = A\left(\frac{r}{a}\right)^i Y_i,$$

where $Y_i$ is a spherical harmonic of degree $i$, then outside the sphere

$$P' = A\left(\frac{a}{r}\right)^{i+1} Y_i.$$

The current-function $\phi$ is since $\left(\frac{dP}{dr} - \frac{dP'}{dr}\right)_{r=a} = 4\pi\phi$, given by the equation

$$\phi = \frac{2i+1}{4\pi}\frac{1}{a}AY_i.$$

The magnetic potential within the sphere is

$$\Omega = -(i+1)\frac{1}{a}A\left(\frac{r}{a}\right)^i Y_i,$$

and outside

$$\Omega' = i\frac{1}{a}A\left(\frac{a}{r}\right)^{i+1} Y_i.$$

For example, let it be required to produce, by means of a wire coiled into the form of a spherical shell, a uniform magnetic force $M$ within the shell. The magnetic potential within the shell is, in this case, a solid harmonic of the first degree of the form

$$\Omega = -Mr\cos\theta,$$

where $M$ is the magnetic force. Hence $A = \frac{1}{2}a^2M$, and

$$\phi = \frac{3}{8\pi}Ma\cos\theta.$$

The current-function is therefore proportional to the distance from the equatorial plane of the sphere, and therefore the number of windings of the wire between any two small circles must be proportional to the distance between the planes of these circles.

If $N$ is the whole number of windings, and if $\gamma$ is the strength of the current in each winding,

$$\phi = \tfrac{1}{2}N\gamma\cos\theta.$$

Hence the magnetic force within the coil is

$$M = \frac{4\pi}{3}\frac{N\gamma}{a}.$$

673.] Let us next find the method of coiling the wire in order to produce within the sphere a magnetic potential of the form of a solid zonal harmonic of the second degree,

$$\Omega = -3\frac{1}{a}A\frac{r^2}{a^2}\left(\tfrac{3}{2}\cos^2\theta - \tfrac{1}{2}\right).$$

Here $$\phi = \frac{5}{4\pi}\frac{A}{a}(\tfrac{3}{2}\cos^2\theta - \tfrac{1}{2}).$$

If the whole number of windings is $N$, the number between the pole and the polar distance $\theta$ is $\frac{1}{2}N\sin^2\theta$.

The windings are closest at latitude 45°. At the equator the direction of winding changes, and in the other hemisphere the windings are in the contrary direction.

Let $\gamma$ be the strength of the current in the wire, then within the shell
$$\Omega = -\frac{4\pi}{5}N\gamma\frac{r^2}{a^2}(\tfrac{3}{2}\cos^2\theta - \tfrac{1}{2}).$$

Let us now consider a conductor in the form of a plane closed curve placed anywhere within the shell with its plane perpendicular to the axis. To determine its coefficient of induction we have to find the surface-integral of $-\frac{d\Omega}{dz}$ over the plane bounded by the curve, putting $\gamma = 1$.

Now $$\Omega = -\frac{4\pi}{5a^2}N\{z^2 - \tfrac{1}{2}(x^2+y^2)\},$$

and $$-\frac{d\Omega}{dz} = \frac{8\pi}{5a^2}Nz.$$

Hence, if $S$ is the area of the closed curve, its coefficient of induction is
$$M = \frac{8\pi}{5a^2}NSz.$$

If the current in this conductor is $\gamma'$, there will be, by Art. 583, a force $Z$, urging it in the direction of $z$, where
$$Z = \gamma\gamma'\frac{dM}{dz} = \frac{8\pi}{5a^2}NS\gamma\gamma',$$
and, since this is independent of $x$, $y$, $z$, the force is the same in whatever part of the shell the circuit is placed.

674.] The method given by Poisson, and described in Art. 437, may be applied to current-sheets by substituting for the body, supposed to be uniformly magnetized in the direction of $z$ with intensity $I$, a current-sheet having the form of its surface, and for which the current-function is
$$\phi = Iz. \tag{1}$$
The currents in the sheet will be in planes parallel to that of $xy$, and the strength of the current round a slice of thickness $dz$ will be $Idz$.

The magnetic potential due to this current-sheet at any point outside it will be

$$\Omega = -I\frac{dV}{dz}; \tag{2}$$

{where $V$ is the gravitation potential due to the sheet when the surface-density is unity.}

At any point inside the sheet it will be

$$\Omega = -4\pi Iz - I\frac{dV}{dz}. \tag{3}$$

The components of the vector-potential are

$$F = I\frac{dV}{dy}, \qquad G = -I\frac{dV}{dx}, \qquad H = 0. \tag{4}$$

These results can be applied to several cases occurring in practice.

675.] (1) A plane electric circuit of any form.

Let $V$ be the potential due to a plane sheet of any form of which the surface-density is unity, then, if for this sheet we substitute either a magnetic shell of strength $I$ or an electric current of strength $I$ round its boundary, the values of $\Omega$ and of $F$, $G$, $H$ will be those given above.

(2) For a solid sphere of radius $a$,

$$V = \frac{4\pi}{3}\frac{a^3}{r} \text{ when } r \text{ is greater than } a, \tag{5}$$

and $$V = \frac{2\pi}{3}(3a^2 - r^2) \text{ when } r \text{ is less than } a. \tag{6}$$

Hence, if such a sphere is magnetized parallel to $z$ with intensity $I$, the magnetic potential will be

$$\Omega = \frac{4\pi}{3} I \frac{a^3}{r^3} z \text{ outside the sphere,} \tag{7}$$

and $$\Omega = \frac{4\pi}{3} Iz \text{ inside the sphere.} \tag{8}$$

If, instead of being magnetized, the sphere is coiled with wire in equidistant circles, the total strength of current between two small circles whose planes are at unit distance being $I$, then outside the sphere the value of $\Omega$ is as before, but within the sphere

$$\Omega = -\frac{8\pi}{3} Iz. \tag{9}$$

This is the case already discussed in Art. 672.

(3) The case of an ellipsoid uniformly magnetized parallel to a given line has been discussed in Art. 437.

If the ellipsoid is coiled with wire in parallel and equidistant planes, the magnetic force within the ellipsoid will be uniform.

## (4) *A Cylindric Magnet or Solenoid.*

676.] If the body is a cylinder having any form of section and bounded by planes perpendicular to its generating lines, and if $V_1$ is the potential at the point $(x, y, z)$ due to a plane area of surface-density unity coinciding with the positive end of the solenoid, and $V_2$ the potential at the same point due to a plane area of surface-density unity coinciding with the negative end, then, if the cylinder is uniformly and longitudinally magnetized with intensity unity, the potential at the point $(x, y, z)$ will be

$$\Omega = V_1 - V_2. \tag{10}$$

If the cylinder, instead of being a magnetized body, is uniformly lapped with wire, so that there are $n$ windings of wire in unit of length, and if a current, $\gamma$, is made to flow through this wire, the magnetic potential outside the solenoid is as before,

$$\Omega = n\gamma(V_1 - V_2), \tag{11}$$

but within the space bounded by the solenoid and its plane ends

$$\Omega = n\gamma(-4\pi z + V_1 - V_2). \tag{12}$$

The magnetic potential is discontinuous at the plane ends of the solenoid, but the magnetic force is continuous.

If $r_1$, $r_2$, the distances of the centres of inertia of the positive and negative plane ends respectively from the point $(x, y, z)$, are very great compared with the transverse dimensions of the solenoid, we may write

$$V_1 = \frac{A}{r_1}, \qquad V_2 = \frac{A}{r_2}, \tag{13}$$

where $A$ is the area of either section.

The magnetic force outside the solenoid is therefore very small, and the force inside the solenoid approximates to a force parallel to the axis in the positive direction and equal to $4\pi n\gamma$.

If the section of the solenoid is a circle of radius $a$, the values of $V_1$ and $V_2$ may be expressed in the series of spherical harmonics given in Thomson and Tait's *Natural Philosophy*, Art. 546, Ex. II.,

$$V = 2\pi\left\{-rP_1 + a + \tfrac{1}{2}\frac{r^2}{a}P_2 - \frac{1.1}{2.4}\frac{r^4}{a^3}P_4 + \frac{1.1.3}{2.4.6}\frac{r^6}{a^5}I_6 - \&c.\right\} \text{ when } r < a, \tag{14}$$

$$V = 2\pi\left\{\tfrac{1}{2}\frac{a^2}{r} - \frac{1.1}{2.4}\frac{a^4}{r^3}P_2 + \frac{1.1.3}{2.4.6}\frac{a^6}{r^5}P_4 - \&c.\right\} \text{ when } r > a. \tag{15}$$

In these expressions $r$ is the distance of the point $(x, y, z)$ from the centre of one of the circular ends of the solenoid, and the zonal harmonics, $P_1$, $P_2$, &c., are those corresponding to the angle $\theta$ which $r$ makes with the axis of the cylinder.

The differential coefficient with respect to $z$ of the first of these expressions is discontinuous when $\theta = \frac{\pi}{2}$, but we must remember that within the solenoid we must add to the magnetic force deduced from this expression a longitudinal force $4\pi n\gamma$.

677.] Let us now consider a solenoid so long that in the part of space which we consider, the terms depending on the distance from the ends may be neglected.

The magnetic induction through any closed curve drawn within the solenoid is $4\pi n\gamma A'$, where $A'$ is the area of the projection of the curve on a plane normal to the axis of the solenoid.

If the closed curve is outside the solenoid, then, if it encloses the solenoid, the magnetic induction through it is $4\pi n\gamma A$, where $A$ is the area of the section of the solenoid. If the closed curve does not surround the solenoid, the magnetic induction through it is zero.

If a wire be wound $n'$ times round the solenoid, the coefficient of induction between it and the solenoid is

$$M = 4\pi nn'A. \tag{16}$$

By supposing these windings to coincide with $n$ windings of the solenoid, we find that the coefficient of self-induction of unit of length of the solenoid, taken at a sufficient distance from its extremities, is

$$L = 4\pi n^2 A. \tag{17}$$

Near the ends of a solenoid we must take into account the terms depending on the imaginary distribution of magnetism on the plane ends of the solenoid. The effect of these terms is to make the coefficient of induction between the solenoid and a circuit which surrounds it less than the value $4\pi nA$, which it has when the circuit surrounds a very long solenoid at a great distance from either end.

Let us take the case of two circular and coaxal solenoids of the same length $l$. Let the radius of the outer solenoid be $c_1$, and let it be wound with wire so as to have $n_1$ windings in unit of length. Let the radius of the inner solenoid be $c_2$, and let the number of windings in unit of length be $n_2$, then the coefficient of induction between the solenoids, if we neglect the effect of the ends, is

$$M = Gg, \tag{18}$$

where

$$G = 4\pi n_1, \tag{19}$$

and

$$g = \pi c_2^2 l n_2. \tag{20}$$

678.] To determine the effect of the positive end of the solenoid we must calculate the coefficient of induction on the outer solenoid due to the circular disk which forms the end of the inner solenoid. For this purpose we take the second expression for $V$, as given in equation (15), and differentiate it with respect to $r$. This gives the magnetic force in the direction of the radius. We then multiply this expression by $2\pi r^2 d\mu$, and integrate it with respect to $\mu$ from $\mu = 1$ to $\mu = \dfrac{z}{\sqrt{z^2+c_1^2}}$. This gives the coefficient of induction with respect to a single winding of the outer solenoid at a distance $z$ from the positive end. We then multiply this by $dz$ and integrate with respect to $z$ from $z = l$ to $z = 0$. Finally, we multiply the result by $n_1 n_2$, and so find the effect of one of the ends in diminishing the coefficient of induction.

We thus find for $M$, the value of the coefficient of mutual induction between the two cylinders,

$$M = 4\pi^2 n_1 n_2 c_2^2 (l - 2c_1 a), \tag{21}$$

where

$$a = \frac{1}{2}\frac{c_1 + l - r}{c_1} - \frac{1.3}{2.4}\cdot\frac{1}{2.3}\frac{c_2^2}{c_1^2}\left(1 - \frac{c_1^3}{r^3}\right)$$
$$+ \frac{1.3.5}{2.4.6}\cdot\frac{1}{4.5}\frac{c_2^4}{c_1^4}\left(-\frac{1}{2} - 2\frac{c_1^5}{r^5} + \frac{5}{2}\frac{c_1^7}{r^7}\right) + \&c., \tag{22}$$

where $r$ is put, for brevity, instead of $\sqrt{l^2 + c_1^2}$.

It appears from this, that in calculating the mutual induction of two coaxal solenoids, we must use in the expression (20) instead of the true length $l$ the corrected length $l - 2c_1 a$, in which a portion equal to $ac_1$ is supposed to be cut off at each end. When the solenoid is very long compared with its external radius,

$$a = \frac{1}{2} - \frac{1}{16}\frac{c_2^2}{c_1^2} - \frac{1}{128}\frac{c_2^4}{c_1^4} + \&c. \tag{23}$$

679.] When a solenoid consists of a number of layers of wire of such a diameter that there are $n$ layers in unit of length, the number of layers in the thickness $dr$ is $n\,dr$, and we have

$$G = 4\pi\int n^2 dr, \quad \text{and} \quad g = \pi l \int n^2 r^2 dr. \tag{24}$$

If the thickness of the wire is constant, and if the induction take place between an external coil whose outer and inner radii are $x$ and $y$ respectively, and an inner coil whose outer and inner radii are $y$ and $z$, then, neglecting the effect of the ends,

$$Gg = \tfrac{4}{3}\pi^2 l n_1^2 n_2^2 (x-y)(y^3-z^3). \tag{25}$$

That this may be a maximum, $x$ and $z$ being given, and $y$ variable,

$$x = \tfrac{4}{3}y - \tfrac{1}{3}\frac{z^3}{y^2}. \tag{26}$$

This equation gives the best relation between the depths of the primary and secondary coil for an induction-machine without an iron core.

If there is an iron core of radius $z$, then $G$ remains as before, but

$$g = \pi l \int n^2 (r^2 + 4\pi\kappa z^2)\,dr, \tag{27}$$

$$= \pi l n^2 \Big(\frac{y^3 - z^3}{3} + 4\pi\kappa z^2 (y-z)\Big). \tag{28}$$

If $y$ is given, the value of $z$ which gives the maximum value of $g$ is

$$z = \tfrac{2}{3} y \frac{12\pi\kappa}{12\pi\kappa + 1}. \tag{29}$$

When, as in the case of iron, $\kappa$ is a large number, $z = \frac{2}{3}y$, nearly.

If we now make $x$ constant, and $y$ and $z$ variable, we obtain the maximum value of $Gg$, $\kappa$ being large,

$$x : y : z :: 4 : 3 : 2. \tag{30}$$

The coefficient of self-induction of a long solenoid whose outer and inner radii are $x$ and $y$, having a long iron core whose radius is $z$, is per unit length

$$4\pi \int_y^x \Big\{ \pi \int_\rho^x n^2 (\rho^2 + 4\pi\kappa z^2)\,dr + \pi \int_y^\rho n^2 (r^2 + 4\pi\kappa z^2)\,dr \Big\} n^2\,d\rho,$$

$$= \tfrac{2}{3}\pi^2 n^4 (x-y)^2 (x^2 + 2xy + 3y^2 + 24\pi\kappa z^2). \tag{31}$$

680.] We have hitherto supposed the wire to be of uniform thickness. We shall now determine the law according to which the thickness must vary in the different layers in order that, for a given value of the resistance of the primary or the secondary coil, the value of the coefficient of mutual induction may be a maximum.

Let the resistance of unit of length of a wire, such that $n$ windings occupy unit of length of the solenoid, be $\rho n^2$.

The resistance of the whole solenoid is

$$R = 2\pi\rho l \int n^4 r dr. \tag{32}$$

The condition that, with a given value of $R$, $G$ may be a maximum is $\frac{dG}{dr} = C\frac{dR}{dr}$, where $C$ is some constant.

This gives $n^2$ proportional to $\frac{1}{r}$, or the thickness of the wire of the exterior coil must be proportional to the square root of the radius of the layer.

In order that, for a given value of $R$, $g$ may be a maximum

$$n^2 = C\left(r + \frac{4\pi\kappa z^2}{r}\right). \tag{33}$$

Hence, if there is no iron core, the thickness of the wire of the interior coil should be inversely as the square root of the radius of the layer, but if there is a core of iron having a high capacity for magnetization, the thickness of the wire should be more nearly directly proportional to the square root of the radius.

*An Endless Solenoid.*

681.] If a solid be generated by the revolution of a plane area $A$ about an axis in its own plane, not cutting it, it will have the form of a ring. If this ring be coiled with wire, so that the windings of the coil are in planes passing through the axis of the ring, then, if $n$ is the whole number of windings, the current-function of the layer of wire is $\phi = \frac{1}{2\pi}n\gamma\theta$, where $\theta$ is the angle of azimuth about the axis of the ring.

If $\Omega$ is the magnetic potential inside the ring and $\Omega'$ that outside, then $\Omega - \Omega' = -4\pi\phi + C = -2n\gamma\theta + C.$

Outside the ring, $\Omega'$ must satisfy Laplace's equation, and must vanish at an infinite distance. From the nature of the problem it must be a function of $\theta$ only. The only value of $\Omega'$ which fulfils these conditions is zero. Hence

$$\Omega' = 0, \qquad \Omega = -2n\gamma\theta + C.$$

The magnetic force at any point within the ring is perpendicular to the plane passing through the axis, and is equal to $2n\gamma\frac{1}{r}$, where $r$ is the distance from the axis. Outside the ring there is no magnetic force.

If the form of a closed curve be given by the coordinates $z$, $r$, and $\theta$ of its tracing point as functions of $s$, its length from a fixed point, the magnetic induction through the closed curve may be found by integration round it of the vector-potential, the components of which are

$$F = 2n\gamma\frac{xz}{r^2}, \qquad G = 2n\gamma\frac{yz}{r^2}, \qquad H = 0.$$

We thus find
$$2n\gamma\int_0^s \frac{z}{r}\frac{dr}{ds}\,ds$$

taken round the curve, provided the curve is wholly inside the ring. If the curve lies wholly without the ring, but embraces it, the magnetic induction through it is

$$2n\gamma\int_0^{s'} \frac{z'}{r'}\frac{dr'}{ds'}\,ds' = 2n\gamma a,$$

where $a$ is the linear quantity $\int_0^{s'} \frac{z'}{r'}\frac{dr'}{ds'}\,ds'$, and the accented coordinates refer not to the closed curve, but to a single winding of the solenoid.

The magnetic induction through any closed curve embracing the ring is therefore the same, and equal to $2n\gamma a$. If the closed curve does not embrace the ring, the magnetic induction through it is zero.

Let a second wire be coiled in any manner round the ring not necessarily in contact with it, so as to embrace it $n'$ times. The induction through this wire is $2nn'\gamma a$, and therefore $M$, the coefficient of induction of the one coil on the other, is $M = 2nn'a$.

Since this is quite independent of the particular form or position of the second wire, the wires, if traversed by electric currents, will experience no mechanical force acting between them. By making the second wire coincide with the first, we obtain for the coefficient of self-induction of the ring-coil

$$L = 2n^2 a.$$

## CHAPTER XIII.

### PARALLEL CURRENTS.

#### *Cylindrical Conductors.*

682.] In a very important class of electrical arrangements the current is conducted through round wires of nearly uniform section, and either straight, or such that the radius of curvature of the axis of the wire is very great compared with the radius of the transverse section of the wire. In order to be prepared to deal mathematically with such arrangements, we shall begin with the case in which the circuit consists of two very long parallel conductors, with two pieces joining their ends, and we shall confine our attention to a part of the circuit which is so far from the ends of the conductors that the fact of their not being infinitely long does not introduce any sensible change in the distribution of force.

We shall take the axis of $z$ parallel to the direction of the conductors, then, from the symmetry of the arrangements in the part of the field considered, everything will depend on $H$, the component of the vector-potential parallel to $z$.

The components of magnetic induction become, by equations (A),

$$a = \frac{dH}{dy}, \tag{1}$$

$$b = -\frac{dH}{dx}, \tag{2}$$

$$c = 0.$$

For the sake of generality we shall suppose the coefficient of magnetic induction to be $\mu$, so that $a = \mu\alpha$, $b = \mu\beta$, where $\alpha$ and $\beta$ are the components of the magnetic force.

The equations (E) of electric currents, Art. 607, give

$$u = 0, \qquad v = 0, \qquad 4\pi w = \frac{d\beta}{dx} - \frac{d\alpha}{dy}. \tag{3}$$

683.] If the current is a function of $r$, the distance from the axis of $z$, and if we write

$$x = r\cos\theta, \quad \text{and} \quad y = r\sin\theta, \tag{4}$$

and $\beta$ for the magnetic force, in the direction in which $\theta$ is measured perpendicular to the plane through the axis of $z$, we have

$$4\pi w = \frac{d\beta}{dr} + \frac{1}{r}\beta = \frac{1}{r}\frac{d}{dr}(\beta r). \tag{5}$$

If $C$ is the whole current flowing through a section bounded by a circle in the plane $xy$, whose centre is the origin and whose radius is $r$,

$$C = \int_0^r 2\pi r w\, dr = \tfrac{1}{2}\beta r. \tag{6}$$

It appears, therefore, that the magnetic force at a given point due to a current arranged in cylindrical strata, whose common axis is the axis of $z$, depends only on the total strength of the current flowing through the strata which lie between the given point and the axis, and not on the distribution of the current among the different cylindrical strata.

For instance, let the conductor be a uniform wire of radius $a$, and let the total current through it be $C$, then, if the current is uniformly distributed through all parts of the section, $w$ will be constant, and

$$C = \pi w a^2. \tag{7}$$

The current flowing through a circular section of radius $r$, $r$ being less than $a$, is $C' = \pi w r^2$. Hence at any point within the wire,

$$\beta = \frac{2C'}{r} = 2C\frac{r}{a^2}. \tag{8}$$

Outside the wire

$$\beta = 2\frac{C}{r}. \tag{9}$$

In the substance of the wire there is no magnetic potential, for within a conductor carrying an electric current the magnetic force does not fulfil the condition of having a potential.

Outside the wire the magnetic potential is

$$\Omega = -2C\theta. \tag{10}$$

Let us suppose that instead of a wire the conductor is a metal tube whose external and internal radii are $a_1$ and $a_2$, then, if $C$ is the current through the tubular conductor,

$$C = \pi w(a_1^2 - a_2^2). \tag{11}$$

The magnetic force within the tube is zero. In the metal of the tube, where $r$ is between $a_1$ and $a_2$,

$$\beta = 2C\frac{1}{a_1^2 - a_2^2}\left(r - \frac{a_2^2}{r}\right), \tag{12}$$

and outside the tube,

$$\beta = 2\frac{C}{r}, \tag{13}$$

the same as when the current flows through a solid wire.

684.] The magnetic induction at any point is $b = \mu\beta$, and since, by equation (2),

$$b = -\frac{dH}{dr}, \tag{14}$$

$$H = -\int \mu\beta\, dr. \tag{15}$$

The value of $H$ outside the tube is

$$A - 2\mu_0 C \log r, \tag{16}$$

where $\mu_0$ is the value of $\mu$ in the space outside the tube, and $A$ is a constant, the value of which depends on the position of the return current.

In the substance of the tube,

$$H = A - 2\mu_0 C \log a_1 + \frac{\mu C}{a_1^2 - a_2^2}\left(a_1^2 - r^2 + 2a_2^2 \log\frac{r}{a_1}\right). \tag{17}$$

In the space within the tube $H$ is constant, and

$$H = A - 2\mu_0 C \log a_1 + \mu C\left(1 + \frac{2a_2^2}{a_1^2 - a_2^2}\log\frac{a_2}{a_1}\right). \tag{18}$$

685.] Let the circuit be completed by a return current, flowing in a tube or wire parallel to the first, the axes of the two currents being at a distance $b$. To determine the kinetic energy of the system we have to calculate the integral

$$T = \tfrac{1}{2}\iiint Hw\, dx\, dy\, dz. \tag{19}$$

If we confine our attention to that part of the system which lies between two planes perpendicular to the axes of the conductors, and distant $l$ from each other, the expression becomes

$$T = \tfrac{1}{2} l \iint Hw\, dx\, dy. \tag{20}$$

If we distinguish by an accent the quantities belonging to the return current, we may write this

$$\frac{2T}{l} = \iint Hw'\, dx'\, dy' + \iint H'w\, dx\, dy + \iint Hw\, dx\, dy + \iint H'w'\, dx'\, dy'. \tag{21}$$

Since the action of the current on any point outside the tube is the same as if the same current had been concentrated at the axis of the tube, the mean value of $H$ for the section of the return current is $A - 2\mu_0 C \log b$, and the mean value of $H'$ for the section of the positive current is $A' - 2\mu_0 C' \log b$.

Hence, in the expression for $T$, the first two terms may be written $AC' - 2\mu_0 CC' \log b$, and $A'C - 2\mu_0 CC' \log b$.

Integrating the two latter terms in the ordinary way, and adding the results, remembering that $C + C' = 0$, we obtain the value of the kinetic energy $T$. Writing this $\frac{1}{2}LC^2$, where $L$ is the coefficient of self-induction of the system of two conductors, we find as the value of $L$ for length $l$ of the system

$$\frac{L}{l} = 2\mu_0 \log \frac{b^2}{a_1 a_1'} + \tfrac{1}{2}\mu \left[\frac{a_1^2 - 3a_2^2}{a_1^2 - a_2^2} + \frac{4a_2^4}{(a_1^2 - a_2^2)^2} \log \frac{a_1}{a_2}\right]$$
$$+ \tfrac{1}{2}\mu' \left[\frac{a_1'^2 - 3a_2'^2}{a_1'^2 - a_2'^2} + \frac{4a_2'^4}{(a_1'^2 - a_2'^2)^2} \log \frac{a_1'}{a_2'}\right]. \qquad (22)$$

If the conductors are solid wires, $a_2$ and $a_2'$ are zero, and

$$\frac{L}{l} = 2\mu_0 \log \frac{b^2}{a_1 a_1'} + \tfrac{1}{2}(\mu + \mu').^* \qquad (23)$$

It is only in the case of iron wires that we need take account of the magnetic induction in calculating their self-induction. In other cases we may make $\mu_0$, $\mu$, and $\mu'$ all equal to unity. The smaller the radii of the wires, and the greater the distance between them, the greater is the self-induction.

*To find the Repulsion, $X$, between the Two Portions of Wire.*

686.] By Art. 580 we obtain for the force tending to increase $b$,

$$X = \tfrac{1}{2}\frac{dL}{db}C^2,$$
$$= 2\mu_0 \frac{l}{b} C^2, \qquad (24)$$

which agrees with Ampère's formula, when $\mu_0 = 1$, as in air.

687.] If the length of the wires is great compared with the distance between them, we may use the coefficient of self-induction to determine the tension of the wires arising from the action of the current.

* {If the wires are magnetic, the magnetism induced in them will disturb the magnetic field and we cannot apply the preceding reasoning. Equations (22), (23) and (25) are only strictly true when $\mu = \mu' = \mu_0$.}

If $Z$ is this tension,

$$Z = \tfrac{1}{2}\frac{dL}{dl}C^2,$$

$$= C^2\left\{\mu_0 \log\frac{b^2}{a_1 a_1'} + \frac{\mu+\mu'}{4}\right\}. \tag{25}$$

In one of Ampère's experiments the parallel conductors consist of two troughs of mercury connected with each other by a floating bridge of wire. When a current is made to enter at the extremity of one of the troughs, to flow along it till it reaches one extremity of the floating wire, to pass into the other trough through the floating bridge, and so to return along the second trough, the floating bridge moves along the troughs so as to lengthen the part of the mercury traversed by the current.

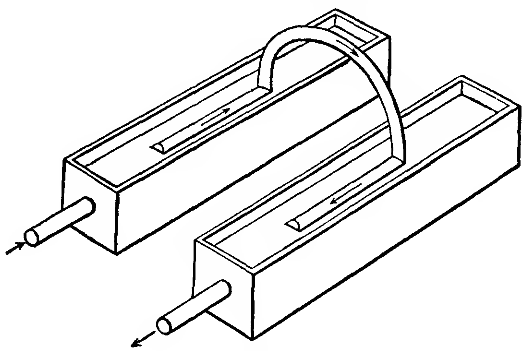

Fig. 40.

Professor Tait has simplified the electrical conditions of this experiment by substituting for the wire a floating siphon of glass filled with mercury, so that the current flows in mercury throughout its course.

This experiment is sometimes adduced to prove that two elements of a current in the same straight line repel one another, and thus to shew that Ampère's formula, which indicates such a repulsion of collinear elements, is more correct than that of Grassmann, which gives no action between two elements in the same straight line; Art. 526.

But it is manifest that since the formulae both of Ampère and of Grassmann give the same results for closed circuits, and since we have in the experiment only a closed circuit, no result of the experiment can favour one more than the other of these theories.

In fact, both formulae lead to the very same value of the repulsion as that already given, in which it appears that $b$, the distance between the parallel conductors, is an important element.

When the length of the conductors is not very great compared with their distance apart, the form of the value of $L$ becomes somewhat more complicated.

688.] As the distance between the conductors is diminished, the value of $L$ diminishes. The limit to this diminution is when the wires are in contact, or when $b = a_1 + a_1'$. In this case if $\mu_0 = \mu = \mu' = 1$,

$$L = 2\,l\,\{\log\frac{(a_1 + a_1')^2}{a_1 a_1'} + \tfrac{1}{2}\}. \tag{26}$$

This is a minimum when $a_1 = a_1'$, and then

$$\begin{aligned} L &= 2l\,(\log 4 + \tfrac{1}{2}), \\ &= 2l\,(1{\cdot}8863), \\ &= 3{\cdot}7726\,l. \end{aligned} \tag{27}$$

This is the smallest value of the self-induction of a round wire doubled on itself, the whole length of the wire being $2\,l$.

Since the two parts of the wire must be insulated from each other, the self-induction can never actually reach this limiting value. By using broad flat strips of metal instead of round wires the self-induction may be diminished indefinitely.

*On the Electromotive Force required to produce a Current of Varying Intensity along a Cylindrical Conductor.*

689.] When the current in a wire is of varying intensity, the electromotive force arising from the induction of the current on itself is different in different parts of the section of the wire, being in general a function of the distance from the axis of the wire as well as of the time. If we suppose the cylindrical conductor to consist of a bundle of wires all forming part of the same circuit, so that the current is compelled to be of uniform strength in every part of the section of the bundle, the method of calculation which we have hitherto used would be strictly applicable. If, however, we consider the cylindrical conductor as a solid mass in which electric currents are free to flow in obedience to electromotive force, the intensity of the current will not be the same at different distances from the axis of the cylinder, and the electromotive forces themselves will depend on the distribution of the current in the different cylindric strata of the wire.

The vector-potential $H$, the density of the current $w$, and the electromotive intensity at any point, must be considered as functions of the time and of the distance from the axis of the wire.

The total current, $C$, through the section of the wire, and the total electromotive force, $E$, acting round the circuit, are to be regarded as the variables, the relation between which we have to find.

Let us assume as the value of $H$,

$$H = S + T_0 + T_1 r^2 + \&c. + T_n r^{2n} + \dots, \tag{1}$$

where $S$, $T_0$, $T_1$, &c. are functions of the time.

Then, from the equation

$$\frac{d^2H}{dr^2} + \frac{1}{r}\frac{dH}{dr} = -4\pi w, \tag{2}$$

we find $$-\pi w = T_1 + \&c. + n^2 T_n r^{2n-2} + \dots. \tag{3}$$

If $\rho$ denotes the specific resistance of the substance per unit of volume, the electromotive intensity at any point is $\rho w$, and this may be expressed in terms of the electric potential and the vector-potential $H$ by equations (B), Art. 598,

$$\rho w = -\frac{d\Psi}{dz} - \frac{dH}{dt}, \tag{4}$$

or $$-\rho w = \frac{d\Psi}{dz} + \frac{dS}{dt} + \frac{dT_0}{dt} + \frac{dT_1}{dt} r^2 + \&c. + \frac{dT_n}{dt} r^{2n} + \dots. \tag{5}$$

Comparing the coefficients of like powers of $r$ in equations (3) and (5),

$$T_1 = \frac{\pi}{\rho}\left(\frac{d\Psi}{dz} + \frac{dS}{dt} + \frac{dT_0}{dt}\right), \tag{6}$$

$$T_2 = \frac{\pi}{\rho}\frac{1}{2^2}\frac{dT_1}{dt}, \tag{7}$$

$$T_n = \frac{\pi}{\rho}\frac{1}{n^2}\frac{dT_{n-1}}{dt}. \tag{8}$$

Hence we may write $$\frac{dS}{dt} = -\frac{d\Psi}{dz}, \tag{9}$$

$$T_0 = T, \qquad T_1 = \frac{\pi}{\rho}\frac{dT}{dt}, \dots \qquad T_n = \frac{\pi^n}{\rho^n}\frac{1}{(n\,!)^2}\frac{d^nT}{dt^n}. \tag{10}$$

690.] To find the total current $C$, we must integrate $w$ over the section of the wire whose radius is $a$,

$$C = 2\pi \int_0^a w r\,dr. \tag{11}$$

Substituting the value of $\pi w$ from equation (3), we obtain

$$C = -(T_1 a^2 + \&\text{c.} + n T_n a^{2n} + \ldots). \tag{12}$$

The value of $H$ at any point outside the wire depends only on the total current $C$, and not on the mode in which it is distributed within the wire. Hence we may assume that the value of $H$ at the surface of the wire is $AC$, where $A$ is a constant to be determined by calculation from the general form of the circuit. Putting $H = AC$ when $r = a$, we obtain

$$AC = S + T_0 + T_1 a^2 + \&\text{c.} + T_n a^{2n} + \ldots. \tag{13}$$

If we now write $\dfrac{\pi a^2}{\rho} = \alpha$, $\alpha$ is the value of the conductivity of unit of length of the wire, and we have

$$C = -\left(\alpha \frac{dT}{dt} + \frac{2\alpha^2}{1^2.\,2^2}\frac{d^2T}{dt^2} + \&\text{c.} + \frac{n\alpha^n}{(n!)^2}\frac{d^nT}{dt^n} + \&\text{c.}\right), \tag{14}$$

$$AC - S = T + \alpha\frac{dT}{dt} + \frac{\alpha^2}{1^2.\,2^2}\frac{d^2T}{dt^2} + \&\text{c.} + \frac{\alpha^n}{(n!)^2}\frac{d^nT}{dt^n} + \&\text{c.} \tag{15}$$

To eliminate $T$ from these equations we must first reverse the series (14). We thus find

$$\alpha\frac{dT}{dt} = -C + \tfrac{1}{2}\alpha\frac{dC}{dt} - \tfrac{1}{6}\alpha^2\frac{d^2C}{dt^2} + \tfrac{7}{144}\alpha^3\frac{d^3C}{dt^3} - \tfrac{39}{2880}\alpha^4\frac{d^4C}{dt^4} + \&\text{c.}$$

We have also from (14) and (15)

$$\alpha\left(A\frac{dC}{dt} - \frac{dS}{dt}\right) + C = \tfrac{1}{2}\alpha^2\frac{d^2T}{dt^2} + \tfrac{1}{6}\alpha^3\frac{d^3T}{dt^3} + \tfrac{1}{48}\alpha^4\frac{d^4T}{dt^4} + \tfrac{1}{720}\alpha^5\frac{d^5T}{dt^5} + \&\text{c.}$$

From the last two equations we find

$$\alpha\left(A\frac{dC}{dt} - \frac{dS}{dt}\right) + C + \tfrac{1}{2}\alpha\frac{dC}{dt} - \tfrac{1}{12}\alpha^2\frac{d^2C}{dt^2} + \tfrac{1}{48}\alpha^3\frac{d^3C}{dt^3} - \tfrac{1}{180}\alpha^4\frac{d^4C}{dt^4} + \&\text{c.} = 0. \tag{16}$$

If $l$ is the whole length of the circuit, $R$ its resistance, and $E$ the electromotive force due to other causes than the induction of the current on itself,

$$\frac{dS}{dt} = \frac{E}{l}, \qquad \alpha = \frac{l}{R}, \tag{17}$$

$$E = RC + l(A + \tfrac{1}{2})\frac{dC}{dt} - \tfrac{1}{12}\frac{l^2}{R}\frac{d^2C}{dt^2} + \tfrac{1}{48}\frac{l^3}{R^2}\frac{d^3C}{dt^3} - \tfrac{1}{180}\frac{l^4}{R^3}\frac{d^4C}{dt^4} + \&\text{c.} \tag{18}$$

The first term, $RC$, of the right-hand member of this equation expresses the electromotive force required to overcome the resistance according to Ohm's law.

The second term, $l(A+\frac{1}{2})\frac{dC}{dt}$, expresses the electromotive force which would be employed in increasing the electrokinetic momentum of the circuit, on the hypothesis that the current is of uniform strength at every point of the section of the wire.

The remaining terms express the correction of this value, arising from the fact that the current is not of uniform strength at different distances from the axis of the wire. The actual system of currents has a greater degree of freedom than the hypothetical system, in which the current is constrained to be of uniform strength throughout the section. Hence the electromotive force required to produce a rapid change in the strength of the current is somewhat less than it would be on this hypothesis.

The relation between the time-integral of the electromotive force and the time-integral of the current is

$$\int E\,dt = R\int C\,dt + l(A+\tfrac{1}{2})C - \tfrac{1}{12}\frac{l^2}{R}\frac{dC}{dt} + \&c. \tag{19}$$

If the current before the beginning of the time has a constant value $C_0$, and if during the time it rises to the value $C_1$, and remains constant at that value, then the terms involving the differential coefficients of $C$ vanish at both limits, and

$$\int E\,dt = R\int C\,dt + l(A+\tfrac{1}{2})(C_1 - C_0), \tag{20}$$

the same value of the electromotive impulse as if the current had been uniform throughout the wire *.

* {If the currents flowing through the wire are periodic and vary as $e^{ipt}$, the equation corresponding to (18) when $\mu$ is no longer assumed to be unity may be written

$$E = \left(R + \frac{1}{12}\frac{\mu^2 l^2 p^2}{R} - \frac{1}{180}\frac{\mu^4 l^4 p^4}{R^3} + \dots\right)C$$
$$+ \left\{\left(lA + \mu\frac{l}{2}\right) - \frac{1}{48}\frac{\mu^3 l^3 p^2}{R^2} + \dots\right\}\frac{dC}{dt}.$$

Thus the system behaves as if the resistance were

$$R + \frac{1}{12}\frac{\mu^2 l^2 p^2}{R} - \frac{1}{180}\frac{\mu^4 l^4 p^4}{R^3} + \dots$$

and the self-induction

$$lA + \mu\frac{l}{2} - \frac{1}{48}\frac{\mu^3 l^3 p^2}{R^2} \dots.$$

Thus the effective resistance is increased when the currents are oscillatory, and the self-induction is diminished. As Maxwell points out, this effect is due to the

## *On the Geometrical Mean Distance of Two Figures in a Plane**.

691.] In calculating the electromagnetic action of a current flowing in a straight conductor of any given section on the

* *Trans. R. S. Edin.*, 1871-2.

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alteration in the distribution of the current. When the current is alternating it is no longer equally distributed over the section of the conductor, but has a tendency to leave the middle and crowd towards the surface of the conductor, since by doing so it diminishes the self-induction and therefore the Kinetic Energy. The inertia of the system, in accordance with a general law of dynamics, makes the current tend to distribute itself so that while fulfilling the condition that the whole flow across any cross section is given, the Kinetic Energy is as small as possible; and this tendency gets more and more powerful as the rapidity with which the momentum of the system is reversed is increased. An inspection of equation {22}, Art. 685, will show that the self-induction of a system, and therefore the Kinetic Energy for a given current, is diminished by making the current denser near the surface of the wire than inside, for this corresponds to the case of the current flowing through tubes, and equation {22} shows that the self-induction for tubes is less than for solid wires of the same radius. As the rush of the current towards the side of the tube leaves it a smaller area to flow through, we can readily understand the increase in the resistance to alternating as compared with steady currents. As this subject is one of great importance some further results are given here, the proofs of which will be given in the Supplementary Volume. See also Rayleigh, *Phil. Mag. XXI.* p. 381.

The relation between the current and the electromotive force is expressed by the equation

$$\frac{E}{l} = -\frac{C\rho}{2\pi a^2}\,\frac{ina J_0(ina)}{J_0'(ina)} + A\frac{dC}{dt}, \tag{1}$$

where $n^2 = 4\pi\mu i p/\rho$, and $J_0$ is Bessel's function of zero order.

Since by the differential equation satisfied by this function

$$\frac{J_0''(x)}{J_0'(x)} + \frac{1}{x} + \frac{J_0(x)}{J_0'(x)} = 0,$$

we have

$$x\frac{J_0(x)}{J_0'(x)} = -1 - x\frac{d}{dx}\log J_0'(x),$$

$$= -2 + 2x^2 S_2 + 2x^4 S_4 + 2x^6 S_6 + \dots,$$

where $S_2, S_4, S_6 \dots$ are the sums of the reciprocals of the squares, fourth and sixth powers . . . of the roots of the equation

$$\frac{J_0'(x)}{x} = 0,$$

or

$$1 - \frac{x^2}{2.4} + \frac{x^4}{2.4.4.6} - \frac{x^6}{2.4.6.4.6.8} + \dots = 0.$$

Hence by Newton's method we find

$$S_2 = \frac{1}{4} \times \frac{1}{2},$$

$$S_4 = \frac{1}{4^2} \times \frac{1}{12},$$

$$S_6 = \frac{1}{4^3} \times \frac{1}{48},$$

$$S_8 = \frac{1}{4^4} \times \frac{1}{180},$$

$$S_{10} = \frac{1}{4^5} \times \frac{13}{8640},$$

. . . . . . . .

current in a parallel conductor whose section is also given, we have to find the integral

$$\iiiint \log r \, dx \, dy \, dx' \, dy',$$

where $dx\,dy$ is an element of the area of the first section, $dx'\,dy'$ an element of the second section, and $r$ the distance between these elements, the integration being extended first over every element of the first section, and then over every element of the second.

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Hence substituting in equation (1) this value for $\dfrac{ina\,J_0(ina)}{J_0'(ina)}$, we get

$$\frac{E}{l} = \frac{C\rho}{\pi a^2}\left\{1 + \frac{1}{12}\left(\frac{\pi\mu p a^2}{\rho}\right)^2 - \frac{1}{180}\left(\frac{\pi\mu p a^2}{\rho}\right)^4 + \dots\right\}$$
$$+ i\,Cp\left\{A + \frac{\mu}{2} - \frac{1}{48}\frac{\pi^2\mu^3p^2a^4}{\rho^2} + \frac{13}{8640}\frac{\pi^4\mu^5p^4a^8}{\rho^4} - \dots\right\},$$

which agrees with (18) when $\mu = 1$. This series is not convenient if $na$ is large, but in that case $J_0'(ina) = -i J_0(ina)$; Heine's Kugelfunctionen, p. 248, 2nd Edition. Hence when the rate of alternation is so rapid that $\mu p a^2/\rho$ is a large quantity,

$$\frac{E}{l} = \frac{C\rho}{2\pi a}n + AipC;$$

and since

$$n^2 = 4\frac{\pi\mu i p}{\rho},$$

$$\frac{E}{l} = \sqrt{\frac{\rho p\mu}{2\pi a^2}}\,C + ipC\left(A + \sqrt{\frac{\rho\mu}{2\pi a^2 p}}\right).$$

Thus the resistance per unit length is

$$\left\{\frac{\rho p\mu}{2\pi a^2}\right\}^{\frac{1}{2}},$$

and increases indefinitely as $p$ increases.

The self-induction per unit length is

$$A + \sqrt{\frac{\rho\mu}{2\pi a^2 p}},$$

and approaches the limit $A$ when $p$ is infinite.

The magnetic force at a point inside the wire may be shown to be

$$\frac{2C}{a}\frac{J_0'(inr)}{J_0'(ina)}.$$

When $na$ is large,

$$J_0'(ina) = -i\frac{e^{na}}{\sqrt{\pi 2 n a}};$$

so that if $r = a - x$, the magnetic force at a distance $x$ from the surface of the wire is

$$\frac{2C}{\sqrt{a(a-x)}}e^{-nx}.$$

Thus if $n$ be very large, the magnetic force, and therefore the intensity of the current, diminishes very rapidly as we recede from the surface, so that the inner portion of the wire is free from magnetic force and current. Since $\mu^{\frac{1}{2}}$ occurs in $n$, these effects will be much more apparent in iron wires than in those made of non-magnetic metals.}

If we now determine a line $R$, such that this integral is equal to

$$A_1 A_2 \log R,$$

where $A_1$ and $A_2$ are the areas of the two sections, the length of $R$ will be the same whatever unit of length we adopt, and whatever system of logarithms we use. If we suppose the sections divided into elements of equal size, then the logarithm of $R$, multiplied by the number of pairs of elements, will be equal to the sum of the logarithms of the distances of all the pairs of elements. Here $R$ may be considered as the geometrical mean of all the distances between pairs of elements. It is evident that the value of $R$ must be intermediate between the greatest and the least values of $r$.

If $R_A$ and $R_B$ are the geometrical mean distances of two figures, $A$ and $B$, from a third, $C$, and if $R_{A+B}$ is that of the sum of the two figures from $C$, then

$$(A+B)\log R_{A+B} = A\log R_A + B\log R_B.$$

By means of this relation we can determine $R$ for a compound figure when we know $R$ for the parts of the figure.

692.] EXAMPLES.*

(1) Let $R$ be the mean distance from the point $O$ to the line $AB$. Let $OP$ be perpendicular to $AB$, then

$$AB(\log R + 1) = AP \log OA + PB \log OB + OP\, \widehat{AOB}.$$

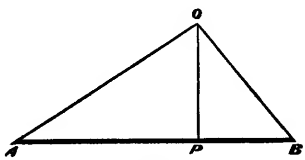



Fig. 41.

(2) For two lines (Fig. 42) of lengths $a$ and $b$ drawn perpendicular to the extremities of a line of length $c$ and on the same side of it:

$$ab(2\log R + 3) = (c^2 - (a-b)^2)\log\sqrt{c^2+(a-b)^2} + c^2\log c$$
$$+ (a^2 - c^2)\log\sqrt{a^2+c^2} + (b^2-c^2)\log\sqrt{b^2+c^2}$$
$$- c(a-b)\tan^{-1}\frac{a-b}{c} + ac\tan^{-1}\frac{a}{c} + bc\tan^{-1}\frac{b}{c}.$$

* {In these Examples all the logarithms are Napierian.}

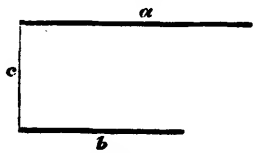



Fig. 42.

(3) For two lines, $PQ$ and $RS$ (Fig. 43), whose directions intersect at $O$;

$$\begin{aligned} PQ.RS(2\log R+3) = {} & \log PR(2\,OP.OR\sin^2 O - PR^2\cos O) \\ & + \log QS\,(2\,OQ.OS\sin^2 O - QS^2\cos O) \\ & - \log PS\,(2\,OP.OS\sin^2 O - PS^2\cos O) \\ & - \log QR(2\,OQ.OR\sin^2 O - QR^2\cos O) \\ & - \sin O\,\{OP^2.\widehat{SPR} - OQ^2.SQR + OR^2.\widehat{PRQ} - OS^2.\widehat{PSQ}\}. \end{aligned}$$

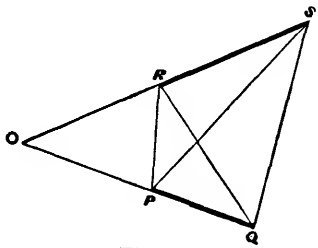



Fig. 43.

(4) For a point $O$ and a rectangular area $ABCD$ (Fig. 44). Let $OP$, $OQ$, $OR$, $OS$, be perpendiculars on the sides, then

$$\begin{aligned} AB.AD(2\log R+3) = {} & 2.OP.OQ\log OA + 2.OQ.OR\log OB \\ & + 2.OR.OS\log OC + 2.OS.OP\log OD \\ & + OP^2.\widehat{DOA} + OQ^2.\widehat{AOB} \\ & + OR^2.\widehat{BOC} + OS^2.\widehat{COD}. \end{aligned}$$

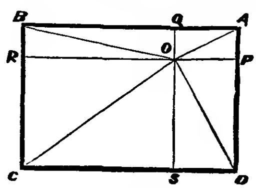



Fig. 44.

(5) It is not necessary that the two figures should be different, for we may find the geometrical mean of the distances between every pair of points in the same figure. Thus, for a straight line of length $a$,

$$\log R = \log a - \tfrac{3}{2},$$

or

$$R = ae^{-\frac{3}{2}},$$

$$R = 0{\cdot}22313\,a.$$

(6) For a rectangular area whose sides are $a$ and $b$,

$$\log R = \log\sqrt{a^2+b^2} - \tfrac{1}{6}\frac{a^2}{b^2}\log\sqrt{1+\frac{b^2}{a^2}} - \tfrac{1}{6}\frac{b^2}{a^2}\log\sqrt{1+\frac{a^2}{b^2}} + \tfrac{2}{3}\frac{a}{b}\tan^{-1}\frac{b}{a} + \tfrac{2}{3}\frac{b}{a}\tan^{-1}\frac{a}{b} - \tfrac{25}{12}.$$

When the rectangle is a square, whose side is $a$,

$$\log R = \log a + \tfrac{1}{3}\log 2 + \frac{\pi}{3} - \tfrac{25}{12},$$

$$R = 0{\cdot}44705\,a.$$

(7) The geometrical mean distance of a point from a circular line is equal to the greater of the two quantities, its distance from the centre of the circle, and the radius of the circle.

(8) Hence the geometrical mean distance of any figure from a ring bounded by two concentric circles is equal to its geometrical mean distance from the centre if it is entirely outside the ring, but if it is entirely within the ring

$$\log R = \frac{a_1^2\log a_1 - a_2^2\log a_2}{a_1^2 - a_2^2} - \tfrac{1}{2},$$

where $a_1$ and $a_2$ are the outer and inner radii of the ring. $R$ is in this case independent of the form of the figure within the ring.

(9) The geometrical mean distance of all pairs of points in the ring is found from the equation

$$\log R = \log a_1 - \frac{a_2^4}{(a_1^2 - a_2^2)^2}\log\frac{a_1}{a_2} + \tfrac{1}{4}\,\frac{3a_2^2 - a_1^2}{a_1^2 - a_2^2}.$$

For a circular area of radius $a$, this becomes

$$\log R = \log a - \tfrac{1}{4},$$

or

$$R = ae^{-\frac{1}{4}},$$

$$R = 0{\cdot}7788\,a.$$

For a circular line it becomes

$$R = a.$$

{For an elliptic area whose semi-axes are $a$, $b$,

$$\log R = \log\frac{a+b}{2} - \tfrac{1}{4}.\}$$

693.] In calculating the coefficient of self-induction of a coil of uniform section, the radius of curvature being great compared with the dimensions of the transverse section, we first determine the geometrical mean of the distances of every pair of points of the section by the method already described, and then we calculate the coefficient of mutual induction between two linear conductors of the given form, placed at this distance apart.

This will be the coefficient of self-induction when the total current in the coil is unity, and the current is uniform at all points of the section.

But if there are $n$ windings in the coil we must multiply the coefficient already obtained by $n^2$, and thus we shall obtain the coefficient of self-induction on the supposition that the windings of the conducting wire fill the whole section of the coil.

But the wire is cylindric, and is covered with insulating material, so that the current, instead of being uniformly distributed over the section, is concentrated in certain parts of it, and this increases the coefficient of self-induction. Besides this, the currents in the neighbouring wires have not the same action on the current in a given wire as a uniformly distributed current.

The corrections arising from these considerations may be determined by the method of the geometrical mean distance. They are proportional to the length of the whole wire of the coil, and may be expressed as numerical quantities, by which we must multiply the length of the wire in order to obtain the correction of the coefficient of self-induction.

Let the diameter of the wire be $d$. It is covered with insulating material, and wound into a coil. We shall suppose that the sections of the wires are in square order, as in Fig. 45,

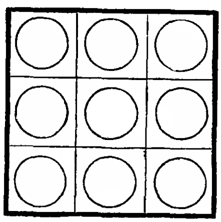


**Fig. 45.**

and that the distance between the axis of each wire and that of the next is $D$, whether in the direction of the breadth or the depth of the coil. $D$ is evidently greater than $d$.

We have first to determine the excess of self-induction of unit of length of a cylindric wire of diameter $d$ over that of unit of length of a square wire of side $D$, or

$$2 \log \frac{R \text{ for the square}}{R \text{ for the circle}}$$

$$= 2\left(\log \frac{D}{d} + \tfrac{4}{3}\log 2 + \frac{\pi}{3} - \tfrac{11}{6}\right)$$

$$= 2\left(\log \frac{D}{d} + 0{\cdot}1380606\right).$$

The inductive action of the eight nearest round wires on the wire under consideration is less than that of the corresponding eight square wires on the square wire in the middle by $2 \times (\cdot 01971)$*.

The corrections for the wires at a greater distance may be neglected, and the total correction may be written

$$2\left(\log_e \frac{D}{d} + 0{\cdot}11835\right).$$

The final value of the self-induction is therefore

$$L = n^2 M + 2l\left(\log_e \frac{D}{d} + 0{\cdot}11835\right),$$

where $n$ is the number of windings, and $l$ the length of the wire, $M$ the mutual induction of two circuits of the form of the mean wire of the coil placed at a distance $R$ from each other, where $R$ is the mean geometrical distance between pairs of points of the section. $D$ is the distance between consecutive wires, and $d$ the diameter of the wire.

* {To get this result notice that the mean distance for the round wires is the distance between their centres, the mean distance for two square wires placed side by side is $\cdot 99401\, D$, the mean distance for two squares corner to corner $1{\cdot}0011 \times \sqrt{2}\, D$. See Maxwell, *Trans. R. S. Edinburgh*, p. 733, 1871–72. Mr. Chree who has kindly re-calculated this correction finds that taking Maxwell's numbers as they stand it is $2 \times \cdot 019635$ instead of $2 \times \cdot 019671$. The work is as follows:

For 8 square wires

$$8 \log_{10} R = 4 \log_{10} (\cdot 99401\, D) + 4 \log_{10} (1{\cdot}0011\sqrt{2}\, D).$$

For 8 round wires

$$8 \log_{10} R_1 = 4 \log_{10} D + 4 \log_{10} \sqrt{2}\, D;$$

hence

$$8 \log_{10} \frac{R_1}{R} = \cdot 0085272\,;$$

and

$$8 \log_e \frac{R_1}{R} = \cdot 019635.$$

This makes the total correction

$$2\left\{\log_e \frac{D}{d} + 0{\cdot}118425\right\}.$$

It is possible however that in calculating this correction Maxwell may have used values for the mean distances, correct to more places of decimals than those given in his paper.}

# CHAPTER XIV.

## CIRCULAR CURRENTS.

### *Magnetic Potential due to a Circular Current.*

694.] THE magnetic potential at a given point, due to a circuit carrying a unit current, is numerically equal to the solid angle subtended by the circuit at that point; see Arts. 409, 485.

When the circuit is circular, the solid angle is that of a cone of the second degree, which, when the given point is on the axis of the circle, becomes a right cone. When the point is not on the axis, the cone is an elliptic cone, and its solid angle is numerically equal to the area of the spherical ellipse which it traces on a sphere whose radius is unity.

This area can be expressed in finite terms by means of elliptic integrals of the third kind. We shall find it more convenient to expand it in the form of an infinite series of spherical harmonics, for the facility with which mathematical operations may be performed on the general term of such a series more than counterbalances the trouble of calculating a number of terms sufficient to ensure practical accuracy.

For the sake of generality we shall assume the origin at any point on the axis of the circle, that is to say, on the line through the centre perpendicular to the plane of the circle.

Let $O$ (Fig. 46) be the centre of the circle, $C$ the point on the axis which we assume as origin, $H$ a point on the circle.

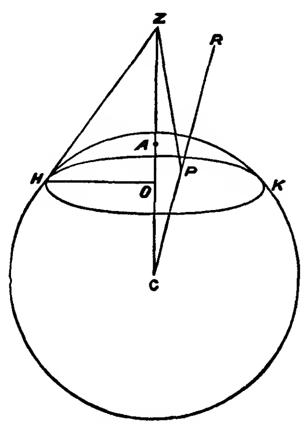


Fig. 46.

Describe a sphere with $C$ as centre, and $CH$ as radius. The circle will lie on this sphere, and will form a small circle of the sphere of angular radius $a$.

Let
$$CH = c,$$
$$OC = b = c\cos a,$$
$$OH = a = c\sin a.$$

Let $A$ be the pole of the sphere, and $Z$ any point on the axis, and let $CZ = z$.

Let $R$ be any point in space, and let $CR = r$, and $ACR = \theta$.

Let $P$ be the point where $CR$ cuts the sphere.

The magnetic potential due to the circular current is equal to that due to a magnetic shell of strength unity bounded by the current. As the form of the surface of the shell is indifferent, provided it is bounded by the circle, we may suppose it to coincide with the surface of the sphere.

We have shewn in Art. 670 that if $V$ is the potential due to a stratum of matter of surface-density unity, spread over the surface of the sphere within the small circle, the potential $\omega$ due to a magnetic shell of strength unity and bounded by the same circle is
$$\omega = -\frac{1}{c}\frac{d}{dr}(rV).$$

We have in the first place, therefore, to find $V$.

Let the given point be on the axis of the circle at $Z$, then the part of the potential at $Z$ due to an element $dS$ of the spherical surface at $P$ is
$$\frac{dS}{ZP}.$$

This may be expanded in one of the two series of spherical harmonics,
$$\frac{dS}{c}\left\{P_0 + P_1\frac{z}{c} + \&c. + P_i\frac{z^i}{c^i} + \&c.\right\},$$
or
$$\frac{dS}{z}\left\{P_0 + P_1\frac{c}{z} + \&c. + P_i\frac{c^i}{z^i} + \&c.\right\},$$
the first series being convergent when $z$ is less than $c$, and the second when $z$ is greater than $c$.

Writing
$$dS = -c^2 d\mu\, d\phi,$$
and integrating with respect to $\phi$ between the limits 0 and $2\pi$, and with respect to $\mu$ between the limits $\cos a$ and 1, we find
$$V = 2\pi c\left\{\int_{\cos a}^{1} P_0\, d\mu + \&c. + \frac{z^i}{c^i}\int_{\cos a}^{1} P_i\, d\mu + \&c.\right\}, \tag{1}$$
or
$$V' = 2\pi\frac{c^2}{z}\left\{\int_{\cos a}^{1} P_0\, d\mu + \&c. + \frac{c^i}{z^i}\int_{\cos a}^{1} P_i\, d\mu + \&c.\right\}. \tag{1'}$$

By the characteristic equation of $P_i$,

$$i(i+1)P_i + \frac{d}{d\mu}\left[(1-\mu^2)\frac{dP_i}{d\mu}\right] = 0.$$

Hence
$$\int_\mu^1 P_i d\mu = \frac{1-\mu^2}{i(i+1)}\frac{dP_i}{d\mu}. \tag{2}$$

This expression fails when $i = 0$, but since $P_0 = 1$,

$$\int_\mu^1 P_0 d\mu = 1-\mu. \tag{3}$$

As the function $\frac{dP_i}{d\mu}$ occurs in every part of this investigation we shall denote it by the abbreviated symbol $P_i'$. The values of $P_i'$ corresponding to several values of $i$ are given in Art. 698.

We are now able to write down the value of $V$ for any point $R$, whether on the axis or not, by substituting $r$ for $z$, and multiplying each term by the zonal harmonic of $\theta$ of the same order. For $V$ must be capable of expansion in a series of zonal harmonics of $\theta$ with proper coefficients. When $\theta = 0$ each of the zonal harmonics becomes equal to unity, and the point $R$ lies on the axis. Hence the coefficients are the terms of the expansion of $V$ for a point on the axis. We thus obtain the two series

$$V = 2\pi c\left\{1-\cos a + \&c. + \frac{\sin^2 a}{i(i+1)}\frac{r^i}{c^i}P_i'(a)P_i(\theta) + \&c.\right\}, \tag{4}$$

or
$$V' = 2\pi\frac{c^2}{r}\left\{1-\cos a + \&c. + \frac{\sin^2 a}{i(i+1)}\frac{c^i}{r^i}P_i'(a)P_i(\theta) + \&c.\right\}. \tag{4'}$$

695.] We may now find $\omega$, the magnetic potential of the circuit, by the method of Art. 670, from the equation

$$\omega = -\frac{1}{c}\frac{d}{dr}(Vr). \tag{5}$$

We thus obtain the two series

$$\omega = -2\pi\left\{1-\cos a + \&c. + \frac{\sin^2 a}{i}\frac{r^i}{c^i}P_i'(a)P_i(\theta) + \&c.\right\}, \tag{6}$$

or
$$\omega' = 2\pi\sin^2 a\left\{\tfrac{1}{2}\frac{c^2}{r^2}P_1'(a)P_1(\theta) + \&c. + \frac{1}{i+1}\frac{c^{i+1}}{r^{i+1}}P_i'(a)P_i(\theta) + \&c.\right\}. \tag{6'}$$

The series (6) is convergent for all values of $r$ less than $c$, and the series (6′) is convergent for all values of $r$ greater than $c$. At the surface of the sphere, where $r = c$, the two series give the same value for $\omega$ when $\theta$ is greater than $a$, that is, for points not occupied by the magnetic shell, but when $\theta$ is less than $a$, that is, at points on the magnetic shell,

$$\omega' = \omega + 4\pi. \tag{7}$$

If we assume $O$, the centre of the circle, as the origin of co-ordinates, we must put $a = \frac{\pi}{2}$, and the series become

$$\omega = -2\pi\left\{1 + \frac{r}{c}P_1(\theta) + \&\text{c.} + (-)^s \frac{1.3\ldots(2s-1)}{2.4\ldots 2s}\frac{r^{2s+1}}{c^{2s+1}}P_{2s+1}(\theta) + \&\text{c.}\right\}, \tag{8}$$

$$\omega = +2\pi\left\{\tfrac{1}{2}\frac{c^2}{r^2}P_1(\theta) + \&\text{c.} + (-)^s \frac{1.3\ldots(2s+1)}{2.4\ldots(2s+2)}\frac{c^{2s+2}}{r^{2s+2}}P_{2s+1}(\theta) + \&\text{c.}\right\}, \tag{8'}$$

where the orders of all the harmonics are odd *.

*On the Potential Energy of two Circular Currents.*

696.] Let us begin by supposing the two magnetic shells which are equivalent to the currents to be portions of two concentric spheres, their radii being $c_1$ and $c_2$, of which $c_1$ is the greater (Fig. 47). Let us also suppose that the axes of the two shells coincide, and that $a_1$ is the angle subtended by the radius of the first shell, and $a_2$ the angle subtended by the radius of the second shell at the centre $C$.

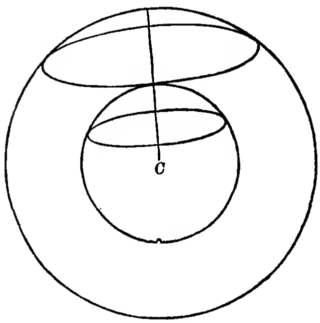



Fig. 47.

Let $\omega_1$ be the potential due to the first shell at any point within it, then the work required to carry the second shell to an infinite distance is the value of the surface-integral

$$M = -\iint \frac{d\omega_1}{dr}\, dS$$

* The value of the solid angle subtended by a circle may be obtained in a more direct way as follows :—

The solid angle subtended by the circle at the point $Z$ in the axis is easily shewn to be

$$\omega = 2\pi\left(1 - \frac{z - c\cos a}{HZ}\right).$$

Expanding this expression in spherical harmonics, we find

$$\omega = 2\pi\left\{(\cos a + 1) + (P_1(a)\cos a - P_0(a))\frac{z}{c} + \&\text{c.} + (P_i(a)\cos a - P_{i-1}(a))\frac{z^i}{c^i} + \&\text{c.}\right\},$$

$$\omega' = 2\pi\left\{(P_0(a)\cos a - P_1(a))\frac{c}{z} + \&\text{c.} + (P_i(a)\cos a - P_{i+1}(a))\frac{c^{i+1}}{z^{i+1}} + \&\text{c.}\right\},$$

for the expansions of $\omega$ for points on the axis for which $z$ is less than $c$ and greater than $c$ respectively. These results can easily be shewn to coincide with those in the text.

extended over the second shell. Hence

$$M = \int_{\mu_2}^{1} \frac{d\omega_1}{dr} 2\pi c_2^2 d\mu_2,$$

$$= 4\pi^2 \sin^2 a_1 c_2^2 \left\{ \frac{1}{c_1} P_1'(a_1) \int_{\mu_2}^{1} P_1(\theta) d\mu_2 + \&c. + \frac{c_2^{i-1}}{c^i} P_i'(a_1) \int_{\mu_2}^{1} P_i(\theta) d\mu_2 + \&c. \right\},$$

or, substituting the value of the integrals from equation (2), Art. 694,

$$M = 4\pi^2 \sin^2 a_1 \sin^2 a_2 c_2 \left\{ \tfrac{1}{2} \frac{c_2}{c_1} P_1'(a_1) P_1'(a_2) + \&c. + \frac{1}{i(i+1)} \frac{c_2^i}{c_1^i} P_i'(a_1) P_i'(a_2) + \&c. \right\}.\text{*}$$

697.] Let us next suppose that the axis of one of the shells is turned about $C$ as a centre, so that it now makes an angle $\theta$ with the axis of the other shell (Fig. 48). We have only to introduce the zonal harmonics of $\theta$ into this expression for $M$, and we find for the more general value of $M$,

$$M = 4\pi^2 \sin^2 a_1 \sin^2 a_2 c_2^2 \left\{ \tfrac{1}{2} \frac{c_2}{c_1} P_1'(a_1) P_1'(a_2) P_1(\theta) + \&c. \right.$$
$$\left. + \frac{1}{i(i+1)} \frac{c_2^i}{c_1^i} P'(a_1) P_i'(a_2) P_i(\theta) \right\}.\text{*}$$

This is the value of the potential energy due to the mutual action of two circular currents of unit strength, placed so that the normals through the centres of the circles meet in a point $C$ at an angle $\theta$, the distances of the circumferences of the circles from the point $C$ being $c_1$ and $c_2$, of which $c_1$ is the greater.

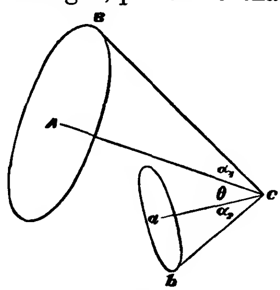


Fig. 48.

If any displacement $dx$ alters the value of $M$, then the force acting in the direction of the displacement is

$$X = \frac{dM}{dx}.$$

For instance, if the axis of one of the shells is free to turn about the point $C$, so as to cause $\theta$ to vary, then the moment of the force tending to increase $\theta$ is $\Theta$, where

$$\Theta = \frac{dM}{d\theta}.$$

* {This is easily proved by expressing the zonal harmonic $P_i(\theta)$, which occurs in the expression for $\omega_1$ in equation (6) as the sum of a series of zonal and tesseral harmonics, with $Ca$ for axis, and then using the formula

$$M = \int_{\mu_2}^{1} \frac{d\omega_1}{dr} 2\pi c_2^2 d\mu_2.\}$$

Performing the differentiation, and remembering that

$$\frac{dP_i(\theta)}{d\theta} = -\sin\theta\, P_i'(\theta),$$

where $P_i'$ has the same signification as in the former equations,

$$\Theta = -4\pi^2 \sin^2\alpha_1 \sin^2\alpha_2 \sin\theta\, c_2 \left\{ \tfrac{1}{2}\frac{c_2}{c_1} P_1'(\alpha_1) P_1'(\alpha_2) P_1'(\theta) + \&c. \right.$$
$$\left. + \frac{1}{i(i+1)}\frac{c_2{}^i}{c_1{}^i} P_i'(\alpha_1) P_i'(\alpha_2) P_i'(\theta) \right\}.$$

698.] As the values of $P_i'$ occur frequently in these calculations the following table of values of the first six degrees may be useful. In this table $\mu$ stands for $\cos\theta$, and $\nu$ for $\sin\theta$.

$$P_1' = 1,$$
$$P_2' = 3\mu,$$
$$P_3' = \tfrac{3}{2}(5\mu^2 - 1) = 6(\mu^2 - \tfrac{1}{4}\nu^2),$$
$$P_4' = \tfrac{5}{2}\mu(7\mu^2 - 3) = 10\mu(\mu^2 - \tfrac{3}{4}\nu^2),$$
$$P_5' = \tfrac{15}{8}(21\mu^4 - 14\mu^2 + 1) = 15(\mu^4 - \tfrac{3}{2}\mu^2\nu^2 + \tfrac{1}{8}\nu^4),$$
$$P_6' = \tfrac{21}{8}\mu(33\mu^4 - 30\mu^2 + 5) = 21\mu(\mu^4 - \tfrac{5}{2}\mu^2\nu^2 + \tfrac{5}{8}\nu^4).$$

699.] It is sometimes convenient to express the series for $M$ in terms of linear quantities as follows:—

Let $a$ be the radius of the smaller circuit, $b$ the distance of its plane from the origin, and $c = \sqrt{a^2 + b^2}$.

Let $A$, $B$, and $C$ be the corresponding quantities for the larger circuit.

The series for $M$ may then be written,

$$M = 1.2.\pi^2 \frac{A^2}{C^3} a^2 \cos\theta$$
$$+ 2.3.\pi^2 \frac{A^2 B}{C^5} a^2 b(\cos^2\theta - \tfrac{1}{2}\sin^2\theta)$$
$$+ 3.4.\pi^2 \frac{A^2(B^2 - \frac{1}{4}A^2)}{C^7} a^2(b^2 - \tfrac{1}{4}a^2)(\cos^3\theta - \tfrac{3}{2}\sin^2\theta\cos\theta)$$
$$+ \&c.$$

If we make $\theta = 0$, the two circles become parallel and on the same axis. To determine the attraction between them we may differentiate $M$ with respect to $b$. We thus find

$$\frac{dM}{db} = \pi^2 \frac{A^2 a^2}{C^4} \left\{ 2.3\frac{B}{C} + 2.3.4\frac{B^2 - \frac{1}{4}A^2}{C^3} b + \&c. \right\}.$$

700.] In calculating the effect of a coil of rectangular section we have to integrate the expressions already found with respect to $A$, the radius of the coil, and $B$, the distance of its plane from the origin, and to extend the integration over the breadth and depth of the coil.

In some cases direct integration is the most convenient, but there are others in which the following method of approximation leads to more useful results.

Let $P$ be any function of $x$ and $y$, and let it be required to find the value of $\bar{P}$ where

$$\bar{P}xy = \int_{-\frac{1}{2}x}^{+\frac{1}{2}x} \int_{-\frac{1}{2}y}^{+\frac{1}{2}y} P\,dx\,dy.$$

In this expression $\bar{P}$ is the mean value of $P$ within the limits of integration.

Let $P_0$ be the value of $P$ when $x = 0$ and $y = 0$, then, expanding $P$ by Taylor's Theorem,

$$P = P_0 + x\frac{dP_0}{dx} + y\frac{dP_0}{dy} + \tfrac{1}{2}x^2\frac{d^2P_0}{dx^2} + \&c.$$

Integrating this expression between the limits, and dividing the result by $xy$, we obtain as the value of $\bar{P}$,

$$\begin{aligned}\bar{P} &= P_0 + \tfrac{1}{24}\left(x^2\frac{d^2P_0}{dx^2} + y^2\frac{d^2P_0}{dy^2}\right)\\ &\quad + \tfrac{1}{1920}\left(x^4\frac{d^4P_0}{dx^4} + y^4\frac{d^4P_0}{dy^4}\right) + \tfrac{1}{576}x^2y^2\frac{d^4P_0}{dx^2\,dy^2} + \&c.\end{aligned}$$

In the case of the coil, let the outer and inner radii be $A + \frac{1}{2}\xi$, and $A - \frac{1}{2}\xi$ respectively, and let the distances of the planes of the windings from the origin lie between $B + \frac{1}{2}\eta$ and $B - \frac{1}{2}\eta$, then the breadth of the coil is $\eta$, and its depth $\xi$, these quantities being small compared with $A$ or $C$.

In order to calculate the magnetic effect of such a coil we may write the successive terms of the series (6) and (6′) of Art. 695 as follows:—

$$G_0 = \pi\frac{B}{C}\left(1 + \tfrac{1}{24}\frac{2A^2 - B^2}{C^4}\xi^2 - \tfrac{1}{8}\frac{A^2}{C^4}\eta^2 + \ldots\right),$$

$$G_1 = 2\pi\frac{A^2}{C^3}\left\{1 + \tfrac{1}{24}\left(\frac{2}{A^2} - 15\frac{B^2}{C^4}\right)\xi^2 + \tfrac{1}{8}\frac{4B^2 - A^2}{C^4}\eta^2 + \ldots\right\},$$

$$G_2 = 3\pi\frac{A^2B}{C^5}\left\{1 + \tfrac{1}{24}\left(\frac{2}{A^2} - \frac{25}{C^2} + \frac{35A^2}{C^4}\right)\xi^2 + \tfrac{5}{24}\frac{4B^2 - 3A^2}{C^4}\eta^2 + \ldots\right\},$$

$$G_3 = 4\pi\frac{A^2(B^2-\frac{1}{4}A^2)}{C^7} + \frac{\pi}{24}\frac{\xi^2}{C^{11}}\{C^4(8B^2-12A^2)+35A^2B^2(5A^2-4B^2)\}$$

$$+\frac{5}{8}\frac{\pi\eta^2}{C^{11}}A^2\{A^4-12A^2B^2+8B^4\},$$

&c., &c.;

$$g_1 = \pi a^2 + \tfrac{1}{12}\pi\xi^2,$$

$$g_2 = 2\pi a^2 b + \tfrac{1}{6}\pi b\xi^2,$$

$$g_3 = 3\pi a^2(b^2-\tfrac{1}{4}a^2) + \frac{\pi}{8}\xi^2(2b^2-3a^2) + \frac{\pi}{4}\eta^2 a^2,$$

&c., &c.

The quantities $G_0$, $G_1$, $G_2$, &c. belong to the large coil. The value of $\omega$ at points for which $r$ is less than $C$ is

$$\omega = -2\pi + 2G_0 - G_1 r P_1(\theta) - G_2 r^2 P_2(\theta) - \text{\&c.}$$

The quantities $g_1$, $g_2$, &c. belong to the small coil. The value of $\omega'$ at points for which $r$ is greater than $c$ is

$$\omega' = g_1\frac{1}{r^2}P_1(\theta) + g_2\frac{1}{r^3}P_2(\theta) + \text{\&c.}$$

The potential of the one coil with respect to the other when the total current through the section of each coil is unity is

$$M = G_1 g_1 P_1(\theta) + G_2 g_2 P_2(\theta) + \text{\&c.}$$

*To find M by Elliptic Integrals.*

701.] When the distance of the circumferences of the two circles is moderate as compared with the radius of the smaller, the series already given do not converge rapidly. In every case, however, we may find the value of $M$ for two parallel circles by elliptic integrals.

For let $b$ be the length of the line joining the centres of the circles, and let this line be perpendicular to the planes of the two circles, and let $A$ and $a$ be the radii of the circles, then

$$M = \iint\frac{\cos\epsilon}{r}\,ds\,ds',$$

the integration being extended round both curves.

In this case,

$$r^2 = A^2 + a^2 + b^2 - 2Aa\cos(\phi-\phi'),$$

$$\epsilon = \phi-\phi', \qquad ds = a\,d\phi, \qquad ds' = A\,d\phi',$$

$$M = \int_0^{2\pi} \int_0^{2\pi} \frac{Aa\cos(\phi-\phi')\,d\phi\,d\phi'}{\sqrt{A^2+a^2+b^2-2Aa\cos(\phi-\phi')}}$$

$$= -4\pi\sqrt{Aa}\left\{\left(c-\frac{2}{c}\right)F+\frac{2}{c}E\right\},$$

where $$c = \frac{2\sqrt{Aa}}{\sqrt{(A+a)^2+b^2}},$$

and $F$ and $E$ are complete elliptic integrals to modulus $c$.

From this, remembering that

$$\frac{dF}{dc} = \frac{1}{c(1-c^2)}\{E-(1-c^2)F\}, \qquad \frac{dE}{dc} = \frac{1}{c}(E-F),$$

and that $c$ is a function of $b$, we find

$$\frac{dM}{db} = \frac{\pi}{\sqrt{Aa}}\frac{bc}{1-c^2}\{(2-c^2)E-2(1-c^2)F\}.$$

If $r_1$ and $r_2$ denote the greatest and least values of $r$,

$$r_1^2 = (A+a)^2+b^2; \qquad r_2^2 = (A-a)^2+b^2,$$

and if an angle $\gamma$ be taken such that $\cos\gamma = \frac{r_2}{r_1}$,

$$\frac{dM}{db} = -\pi\frac{b\sin\gamma}{\sqrt{Aa}}\{2F_\gamma-(1+\sec^2\gamma)E_\gamma\},$$

where $F_\gamma$ and $E_\gamma$ denote the complete elliptic integrals of the first and second kind whose modulus is $\sin\gamma$.

If $A = a$, $\cot\gamma = \frac{b}{2a}$, and

$$\frac{dM}{db} = -2\pi\cos\gamma\,\{2F_\gamma-(1+\sec^2\gamma)E_\gamma\}.$$

The quantity $-\frac{dM}{db}$ represents the attraction between two parallel circular circuits, the current in each being unity.

On account of the importance of the quantity $M$ in electromagnetic calculations the values of $\log(M/4\pi\sqrt{Aa})$, which is a function of $c$ and therefore of $\gamma$ only, have been tabulated for intervals of 6′ in the value of the angle $\gamma$ between 60 and 90 degrees. The table will be found in an appendix to this chapter.

*Second Expression for M.*

An expression for $M$, which is sometimes more convenient, is got by making $c_1 = \dfrac{r_1 - r_2}{r_1 + r_2}$, in which case

$$*M = 8\pi\sqrt{Aa}\,\frac{1}{\sqrt{c_1}}\{F(c_1) - E(c_1)\}.$$

*To draw the Lines of Magnetic Force for a Circular Current.*

702.] The lines of magnetic force are evidently in planes, passing through the axis of the circle, and in each of these lines the value of $M$ is constant.

Calculate the value of $K_\theta = \dfrac{\sin\theta}{(F_{\sin\theta} - E_{\sin\theta})^2}$ from Legendre's tables for a sufficient number of values of $\theta$.

Draw rectangular axes of $x$ and $z$ on the paper {the origin being at the centre of the circle and the axis of $z$ the axis of the circle}, and, with centre at the point $x = \frac{1}{2}a(\sin\theta + \operatorname{cosec}\theta)$, draw a circle with radius $\frac{1}{2}a(\operatorname{cosec}\theta - \sin\theta)$. For all points of this circle the value of $c_1$ will be $\sin\theta$. Hence, for all points of this circle,

$$M = 8\pi\sqrt{Aa}\,\frac{1}{\sqrt{K_\theta}}, \quad \text{and} \quad A = \frac{1}{64\pi^2}\,\frac{M^2 K_\theta}{a}.$$

Now $A$ is the value of $x$ for which the value of $M$ was found. Hence, if we draw a line for which $x = A$, it will cut the circle in two points having the given value of $M$.

Giving $M$ a series of values in arithmetical progression, the values of $A$ will be as a series of squares. Drawing therefore a series of lines parallel to $z$, for which $x$ has the values found for $A$, the points where these lines cut the circle will be the points where the corresponding lines of force cut the circle.

* [The second expression for $M$ may be deduced from the first by means of the following transformations in Elliptic Integrals :—

If $\sqrt{1-c^2} = \dfrac{1-c_1}{1+c_1}$, or $c = \dfrac{2\sqrt{c_1}}{1+c_1}$,

then
$$F(c) = (1+c_1)F(c_1),$$
$$E(c) = \frac{2}{1+c_1}E(c_1) - (1-c_1)F(c_1).]$$

If we put $m = 8\pi a$, and $M = nm$, then

$$A = x = n^2 K_\theta a.$$

We may call $n$ the index of the line of force.

The forms of these lines are given in Fig. XVIII at the end of this volume. They are copied from a drawing given by Sir W. Thomson in his paper on 'Vortex Motion'.*

703.] If the position of a circle having a given axis is regarded as defined by $b$, the distance of its centre from a fixed point on the axis, and $a$, the radius of the circle, then $M$, the coefficient of induction of the circle with respect to any system whatever of magnets or currents, is subject to the following equation,

$$\frac{d^2M}{da^2} + \frac{d^2M}{db^2} - \frac{1}{a}\frac{dM}{da} = 0. \tag{1}$$

To prove this, let us consider the number of lines of magnetic force cut by the circle when $a$ or $b$ is made to vary.

(1) Let $a$ become $a+\delta a$, $b$ remaining constant. During this variation the circle, in expanding, sweeps over an annular surface in its own plane whose breadth is $\delta a$.

If $V$ is the magnetic potential at any point, and if the axis of $y$ be parallel to that of the circle, then the magnetic force perpendicular to the plane of the ring is $-\frac{dV}{dy}$.

To find the magnetic induction through the annular surface we have to integrate

$$-\int_0^{2\pi} a\,\delta a \frac{dV}{dy}\,d\theta,$$

where $\theta$ is the angular position of a point on the ring.

But this quantity represents the variation of $M$ due to the variation of $a$, or $\frac{dM}{da}\delta a$. Hence

$$\frac{dM}{da} = -\int_0^{2\pi} a\frac{dV}{dy}\,d\theta. \tag{2}$$

(2) Let $b$ become $b+\delta b$, while $a$ remains constant. During this variation the circle sweeps over a cylindric surface of radius $a$ and length $\delta b$, {and the lines of force which pass through this surface are those which cease to pass through the circle}.

The magnetic force perpendicular to this surface at any point is $-\frac{dV}{dr}$, where $r$ is the distance from the axis. Hence

$$\frac{dM}{db} = \int_0^{2\pi} a\frac{dV}{dr}\,d\theta. \tag{3}$$

* *Trans. R. S. Edin.*, vol. xxv. p. 217 (1869).

Differentiating equation (2) with respect to $a$, and (3) with respect to $b$, we get

$$\frac{d^2M}{da^2} = -\int_0^{2\pi} \frac{dV}{dy}\,d\theta - \int_0^{2\pi} a\frac{d^2V}{dr\,dy}\,d\theta, \tag{4}$$

$$\frac{d^2M}{db^2} = \int_0^{2\pi} a\frac{d^2V}{dr\,dy}\,d\theta. \tag{5}$$

Hence
$$\frac{d^2M}{da^2} + \frac{d^2M}{db^2} = -\int_0^{2\pi} \frac{dV}{dy}\,d\theta, \tag{6}$$

$$= \frac{1}{a}\frac{dM}{da},\text{ by (2).}$$

Transposing the last term we obtain equation (1).

*Coefficient of Induction of Two Parallel Circles when the Distance between the Arcs is small compared with the Radius of either Circle.*

704.] We might deduce the value of $M$ in this case from the expansion of the elliptic integrals already given when their modulus is nearly unity. The following method, however, is a more direct application of electrical principles.

*First Approximation.*

Let $a$ and $a+c$ be the radii of the circles and $b$ the distance between their planes, then the shortest distance between their circumferences is given by

$$r = \sqrt{c^2+b^2}.$$

We have to find the magnetic induction through the one circle due to a unit current in the other.

We shall begin by supposing the two circles to be in one plane. Consider a small element $\delta s$ of the circle whose radius is $a+c$. At a point in the plane of the circle, distant $\rho$ from the centre of $\delta s$, measured in a direction making an angle $\theta$ with the direction of $\delta s$, the magnetic force due to $\delta s$ is perpendicular to th plane and equal to

$$\frac{1}{\rho^2}\sin\theta\,\delta s.$$

To calculate the surface integral of this force over the space which lies within the circle of radius $a$ we must find the value of the integral

$$2\,\delta s\int_{\theta_1}^{\frac{1}{2}\pi}\int_{r_2}^{r_1}\frac{\sin\theta}{\rho}\,d\theta\,d\rho,$$

where $r_1$, $r_2$ are the roots of the equation

$$r^2 - 2(a+c)\sin\theta\, r + c^2 + 2ac = 0,$$

viz. $$r_1 = (a+c)\sin\theta + \sqrt{(a+c)^2\sin^2\theta - c^2 - 2ac},$$

$$r_2 = (a+c)\sin\theta - \sqrt{(a+c)^2\sin^2\theta - c^2 - 2ac},$$

and $$\sin^2\theta_1 = \frac{c^2+2ac}{(c+a)^2}.$$

When $c$ is small compared to $a$ we may put

$$r_1 = 2a\sin\theta,$$

$$r_2 = c/\sin\theta.$$

Integrating with regard to $\rho$ we have

$$2\delta s\int_{\theta_1}^{\frac{1}{2}\pi}\log\left(\frac{2a}{c}\sin^2\theta\right).\sin\theta d\theta =$$

$$2\delta s\left[\cos\theta\left\{2 - \log\left(\frac{2a}{c}\sin^2\theta\right)\right\} + 2\log\tan\frac{\theta}{2}\right]_{\theta_1}^{\frac{\pi}{2}}$$

$$= 2\delta s\left(\log_e\frac{8a}{c} - 2\right), \text{ nearly.}$$

We thus find for the whole induction

$$M_{ac} = 4\pi a\left(\log_e\frac{8a}{c} - 2\right).$$

Since the magnetic force at any point, the distance of which from a curved wire is small compared with the radius of curvature, is nearly the same as if the wire had been straight, we can (Art. 684) calculate the difference between the induction through the circle whose radius is $a-c$ and the circle $A$ by the formula

$$M_{aA} - M_{ac} = 4\pi a\{\log_e c - \log_e r\}.$$

Hence we find the value of the induction between $A$ and $a$ to be

$$M_{Aa} = 4\pi a(\log_e 8a - \log_e r - 2)$$

approximately, provided $r$ the shortest distance between the circles is small compared with $a$.

705.] Since the mutual induction between two windings of the same coil is a very important quantity in the calculation of experimental results, I shall now describe a method by which the approximation to the value of $M$ for this case can be carried to any required degree of accuracy.

We shall assume that the value of $M$ is of the form

$$M = 4\pi\left\{A\log_e\frac{8a}{r} + B\right\},$$

where $A = a + A_1 x + A_2\dfrac{x^2}{a} + A_2'\dfrac{y^2}{a} + A_3\dfrac{x^3}{a^2} + A_3'\dfrac{xy^2}{a^2} + \&c.$
$$+ a^{-(n-1)}\{x^n A_n + x^{n-2}y^2 A'_n + x^{n-4}y^4 A''_n + \dots\} + \&c.,$$

and $B = -2a + B_1 x + B_2\dfrac{x_2}{a} + B_2'\dfrac{y^2}{a} + B_3\dfrac{x^3}{a^2} + B_3'\dfrac{xy^2}{a^2} + \&c.,$

where $a$ and $a+x$ are the radii of the circles, and $y$ the distance between their planes.

We have to determine the values of the coefficients $A$ and $B$. It is manifest that only even powers of $y$ can occur in these quantities, because, if the sign of $y$ is reversed, the value of $M$ must remain the same.

We get another set of conditions from the reciprocal property of the coefficient of induction, which remains the same whichever circle we take as the primary circuit. The value of $M$ must therefore remain the same when we substitute $a+x$ for $a$, and $-x$ for $x$ in the above expressions.

We thus find the following conditions of reciprocity by equating the coefficients of similar combinations of $x$ and $y$,

$$\begin{aligned} &A_1 = 1 - A_1, && B_1 = 1 - 2 - B_1,\\ &A_3 = -A_2 - A_3, && B_3 = \tfrac{1}{3} - \tfrac{1}{2}A_1 + A_2 - B_2 - B_3,\\ &A_3' = -A_2' - A_3', && B_3' = \qquad A_2' - B_2' - B_3';\end{aligned}$$

$$(-)^n A_n = A_2 + (n-2)A_3 + \frac{(n-2)(n-3)}{1\,.\,2}A_4 + \&c. + A_n,$$

$$(-)^n B_n = -\frac{1}{n} + \frac{1}{n-1}A_1 - \frac{1}{n-2}A_2 + \&c. + (-)^n A_{n-1}$$
$$+ B_2 + (n-2)B_3 + \frac{(n-2)(n-3)}{1\,.\,2}B_4 + \&c. + B_n.$$

From the general equation of $M$, Art. 703,

$$\frac{d^2M}{dx^2} + \frac{d^2M}{dy^2} - \frac{1}{a+x}\frac{dM}{dx} = 0,$$

we obtain another set of conditions,

$$2A_2 + 2A'_2 = A_1,$$
$$2A_2 + 2A'_2 + 6A_3 + 2A'_3 = 2A_2;$$
$$n(n-1)A_n + (n+1)nA_{n+1} + 1\,.\,2A'_n + 1\,.\,2A'_{n+1} = nA_n,$$
$$*(n-1)(n-2)A'_n + n(n-1)A'_{n+1} + 2\,.\,3A''_n + 2\,.\,3A''_{n+1}$$
$$= (n-2)A'_n, \ \&c.;$$
$$4A_2 + \ A_1 = 2B_2 + 2B'_2 - \ B_1 = 4A'_2,$$
$$6A_3 + 3A_2 = 2B'_2 + 6B_3 + 2B'_3 = 6A'_3 + 3A'_2,$$

* {Mr. Chree finds that this equation should be $(n-2)(n-3)A'_n + (n-1)(n-2)A'_{n+1} + 3\,.\,4A''_n + 3\,.\,4A''_{n+1} = (n-2)A'_n$}.

$$(2n-1)A_n+(2n+2)A_{n+1} = (2n-1)A'_n+(2n+2)A'_{n+1}$$
$$= n(n-2)B_n+(n+1)nB_{n+1}+1.2B'_n+1.2B'_{n+1}.$$

Solving these equations and substituting the values of the coefficients, the series for $M$ becomes

$$*M = 4\pi a\log\frac{8a}{r}\left\{1+\tfrac{1}{2}\frac{x}{a}+\frac{x^2+3y^2}{16a^2}-\frac{x^3+3xy^2}{32a^3}+\&c.\right\}$$
$$+4\pi a\left\{-2-\tfrac{1}{2}\frac{x}{a}+\frac{3x^2-y^2}{16a^2}-\frac{x^3-6xy^2}{48a^3}+\&c.\right\}.$$

*To find the form of a coil for which the coefficient of self-induction is a maximum, the total length and thickness of the wire being given.*

706.] Omitting the corrections of Art. 705, we find by Art. 693

$$L = 4\pi n^2 a\left(\log\frac{8a}{R}-2\right),$$

where $n$ is the number of windings of the wire, $a$ is the mean radius of the coil, and $R$ is the geometrical mean distance of the transverse section of the coil from itself. See Art. 691. If this section is always similar to itself, $R$ is proportional to its linear dimensions, and $n$ varies as $R^2$.

Since the total length of the wire is $2\pi an$, $a$ varies inversely as $n$. Hence

$$\frac{dn}{n} = 2\frac{dR}{R}, \text{ and } \frac{da}{a} = -2\frac{dR}{R},$$

and we find the condition that $L$ may be a maximum

$$\log\frac{8a}{R} = \tfrac{7}{2}.$$

If the transverse section of the channel of the coil is circular, of radius $c$, then, by Art. 692,

$$\log\frac{R}{c} = -\tfrac{1}{4},$$

$$\text{and}\quad \log\frac{8a}{c} = \tfrac{13}{4},$$

whence $$a = 3{\cdot}22\,c;$$

* [This result may be obtained directly by the method suggested in Art. 704, viz. by the expansions of the elliptic integrals in the expression for $M$ found in Art. 701. See Cayley's *Elliptic Functions*, Art. 75.]

or, the mean radius of the coil should be 3·22 times the radius of the transverse section of the channel of the coil in order that such a coil may have the greatest coefficient of self-induction. This result was found by Gauss*.

If the channel in which the coil is wound has a square transverse section, the mean diameter of the coil should be 3·7 times the side of the square section of the channel.

* *Werke*, Göttingen edition, 1867, bd. v. p. 622.

# APPENDIX I.

*Table of the values of* $\log \dfrac{M}{4\pi\sqrt{Aa}}$ (Art. 701).

*The Logarithms are to base* 10.

| | $\log \frac{M}{4\pi\sqrt{Aa}}$. | | $\log \frac{M}{4\pi\sqrt{Aa}}$. | | $\log \frac{M}{4\pi\sqrt{Aa}}$. |
|---|---|---|---|---|---|
| 60° 0′ | $\bar{1}\cdot4994783$ | 63° 30′ | $\bar{1}\cdot5963782$ | 67° 0′ | $\bar{1}\cdot6927081$ |
| 6′ | $\bar{1}\cdot5022651$ | 36′ | $\bar{1}\cdot5991329$ | 6′ | $\bar{1}\cdot6954642$ |
| 12′ | $\bar{1}\cdot5050505$ | 42′ | $\bar{1}\cdot6018871$ | 12′ | $\bar{1}\cdot6982209$ |
| 18′ | $\bar{1}\cdot5078345$ | 48′ | $\bar{1}\cdot6046408$ | 18′ | $\bar{1}\cdot7009782$ |
| 24′ | $\bar{1}\cdot5106173$ | 54′ | $\bar{1}\cdot6073942$ | 24′ | $\bar{1}\cdot7037362$ |
| 30′ | $\bar{1}\cdot5133989$ | 64° 0′ | $\bar{1}\cdot6101472$ | 30′ | $\bar{1}\cdot7064949$ |
| 36′ | $\bar{1}\cdot5161791$ | 6′ | $\bar{1}\cdot6128998$ | 36′ | $\bar{1}\cdot7092544$ |
| 42′ | $\bar{1}\cdot5189582$ | 12′ | $\bar{1}\cdot6156522$ | 42′ | $\bar{1}\cdot7120146$ |
| 48′ | $\bar{1}\cdot5217361$ | 18′ | $\bar{1}\cdot6184042$ | 48′ | $\bar{1}\cdot7147756$ |
| 54′ | $\bar{1}\cdot5245128$ | 24′ | $\bar{1}\cdot6211560$ | 54′ | $\bar{1}\cdot7175375$ |
| 61° 0′ | $\bar{1}\cdot5272883$ | 30′ | $\bar{1}\cdot6239076$ | 68° 0′ | $\bar{1}\cdot7203003$ |
| 6′ | $\bar{1}\cdot5300628$ | 36′ | $\bar{1}\cdot6266589$ | 6′ | $\bar{1}\cdot7230640$ |
| 12′ | $\bar{1}\cdot5328361$ | 42′ | $\bar{1}\cdot6294101$ | 12′ | $\bar{1}\cdot7258286$ |
| 18′ | $\bar{1}\cdot5356084$ | 48′ | $\bar{1}\cdot6321612$ | 18′ | $\bar{1}\cdot7285942$ |
| 24′ | $\bar{1}\cdot5383796$ | 54′ | $\bar{1}\cdot6349121$ | 24′ | $\bar{1}\cdot7313609$ |
| 30′ | $\bar{1}\cdot5411498$ | 65° 0′ | $\bar{1}\cdot6376629$ | 30′ | $\bar{1}\cdot7341287$ |
| 36′ | $\bar{1}\cdot5439190$ | 6′ | $\bar{1}\cdot6404137$ | 36′ | $\bar{1}\cdot7368975$ |
| 42′ | $\bar{1}\cdot5466872$ | 12′ | $\bar{1}\cdot6431645$ | 42′ | $\bar{1}\cdot7396675$ |
| 48′ | $\bar{1}\cdot5494545$ | 18′ | $\bar{1}\cdot6459153$ | 48′ | $\bar{1}\cdot7424387$ |
| 54′ | $\bar{1}\cdot5522209$ | 24′ | $\bar{1}\cdot6486660$ | 54′ | $\bar{1}\cdot7452111$ |
| 62° 0′ | $\bar{1}\cdot5549864$ | 30′ | $\bar{1}\cdot6514169$ | 69° 0′ | $\bar{1}\cdot7479848$ |
| 6′ | $\bar{1}\cdot5577510$ | 36′ | $\bar{1}\cdot6541678$ | 6′ | $\bar{1}\cdot7507597$ |
| 12′ | $\bar{1}\cdot5605147$ | 42′ | $\bar{1}\cdot6569189$ | 12′ | $\bar{1}\cdot7535361$ |
| 18′ | $\bar{1}\cdot5632776$ | 48′ | $\bar{1}\cdot6596701$ | 18′ | $\bar{1}\cdot7563138$ |
| 24′ | $\bar{1}\cdot5660398$ | 54′ | $\bar{1}\cdot6624215$ | 24′ | $\bar{1}\cdot7590929$ |
| 30′ | $\bar{1}\cdot5688011$ | 66° 0′ | $\bar{1}\cdot6651732$ | 30′ | $\bar{1}\cdot7618735$ |
| 36′ | $\bar{1}\cdot5715618$ | 6′ | $\bar{1}\cdot6679250$ | 36′ | $\bar{1}\cdot7646556$ |
| 42′ | $\bar{1}\cdot5743217$ | 12′ | $\bar{1}\cdot6706772$ | 42′ | $\bar{1}\cdot7674392$ |
| 48′ | $\bar{1}\cdot5770809$ | 18′ | $\bar{1}\cdot6734296$ | 48′ | $\bar{1}\cdot7702245$ |
| 54′ | $\bar{1}\cdot5798394$ | 24′ | $\bar{1}\cdot6761824$ | 54′ | $\bar{1}\cdot7730114$ |
| 63° 0′ | $\bar{1}\cdot5825973$ | 30′ | $\bar{1}\cdot6789356$ | 70° 0′ | $\bar{1}\cdot7758000$ |
| 6′ | $\bar{1}\cdot5853546$ | 36′ | $\bar{1}\cdot6816891$ | 6′ | $\bar{1}\cdot7785903$ |
| 12′ | $\bar{1}\cdot5881113$ | 42′ | $\bar{1}\cdot6844431$ | 12′ | $\bar{1}\cdot7813823$ |
| 18′ | $\bar{1}\cdot5908675$ | 48′ | $\bar{1}\cdot6871976$ | 18′ | $\bar{1}\cdot7841762$ |
| 24′ | $\bar{1}\cdot5936231$ | 54′ | $\bar{1}\cdot6899526$ | 24′ | $\bar{1}\cdot7869720$ |

| | $\log \frac{M}{4\pi\sqrt{Aa}}$. | | $\log \frac{M}{4\pi\sqrt{Aa}}$. | | $\log \frac{M}{4\pi\sqrt{Aa}}$. |
|---|---|---|---|---|---|
| 70° 30′ | $\bar{1}$·7897696 | 75° 0′ | $\bar{1}$·9185141 | 79° 30′ | ·0576136 |
| 36′ | $\bar{1}$·7925692 | 6′ | $\bar{1}$·9214613 | 36′ | ·0609037 |
| 42′ | $\bar{1}$·7953709 | 12′ | $\bar{1}$·9244135 | 42′ | ·0642054 |
| 48′ | $\bar{1}$·7981745 | 18′ | $\bar{1}$·9273707 | 48′ | ·0675187 |
| 54′ | $\bar{1}$·8009803 | 24′ | $\bar{1}$·9303330 | 54′ | ·0708441 |
| 71° 0′ | $\bar{1}$·8037882 | 30′ | $\bar{1}$·9333005 | 80° 0′ | ·0741816 |
| 6′ | $\bar{1}$·8065983 | 36′ | $\bar{1}$·9362733 | 6′ | ·0775316 |
| 12′ | $\bar{1}$·8094107 | 42′ | $\bar{1}$·9392515 | 12′ | ·0808944 |
| 18′ | $\bar{1}$·8122253 | 48′ | $\bar{1}$·9422352 | 18′ | ·0842702 |
| 24′ | $\bar{1}$·8150423 | 54′ | $\bar{1}$·9452246 | 24′ | ·0876592 |
| 30′ | $\bar{1}$·8178617 | 76° 0′ | $\bar{1}$·9482196 | 30′ | ·0910619 |
| 36′ | $\bar{1}$·8206836 | 6′ | $\bar{1}$·9512205 | 36′ | ·0944784 |
| 42′ | $\bar{1}$·8235080 | 12′ | $\bar{1}$·9542272 | 42′ | ·0979091 |
| 48′ | $\bar{1}$·8263349 | 18′ | $\bar{1}$·9572400 | 48′ | ·1013542 |
| 54′ | $\bar{1}$·8291645 | 24′ | $\bar{1}$·9602590 | 54′ | ·1048142 |
| 72° 0′ | $\bar{1}$·8319967 | 30′ | $\bar{1}$·9632841 | 81° 0′ | ·1082893 |
| 6′ | $\bar{1}$·8348316 | 36′ | $\bar{1}$·9663157 | 6′ | ·1117799 |
| 12′ | $\bar{1}$·8376693 | 42′ | $\bar{1}$·9693537 | 12′ | ·1152863 |
| 18′ | $\bar{1}$·8405099 | 48′ | $\bar{1}$·9723983 | 18′ | ·1188089 |
| 24′ | $\bar{1}$·8433534 | 54′ | $\bar{1}$·9754497 | 24′ | ·1223481 |
| 30′ | $\bar{1}$·8461998 | 77° 0′ | $\bar{1}$·9785079 | 30′ | ·1259043 |
| 36′ | $\bar{1}$·8490493 | 6′ | $\bar{1}$·9815731 | 36′ | ·1294778 |
| 42′ | $\bar{1}$·8519018 | 12′ | $\bar{1}$·9846454 | 42′ | ·1330691 |
| 48′ | $\bar{1}$·8547575 | 18′ | $\bar{1}$·9877249 | 48′ | ·1366786 |
| 54′ | $\bar{1}$·8576164 | 24′ | $\bar{1}$·9908118 | 54′ | ·1403067 |
| 73° 0′ | $\bar{1}$·8604785 | 30′ | $\bar{1}$·9939062 | 82° 0′ | ·1439539 |
| 6′ | $\bar{1}$·8633440 | 36′ | $\bar{1}$·9970082 | 6′ | ·1476207 |
| 12′ | $\bar{1}$·8662129 | 42′ | ·0001181 | 12′ | ·1513075 |
| 18′ | $\bar{1}$·8690852 | 48′ | ·0032359 | 18′ | ·1550149 |
| 24′ | $\bar{1}$·8719611 | 54′ | ·0063618 | 24′ | ·1587434 |
| 30′ | $\bar{1}$·8748406 | 78° 0′ | ·0094959 | 30′ | ·1624935 |
| 36′ | $\bar{1}$·8777237 | 6′ | ·0126385 | 36′ | ·1662658 |
| 42′ | $\bar{1}$·8806106 | 12′ | ·0157896 | 42′ | ·1700609 |
| 48′ | $\bar{1}$·8835013 | 18′ | ·0189494 | 48′ | ·1738794 |
| 54′ | $\bar{1}$·8863958 | 24′ | ·0221181 | 54′ | ·1777219 |
| 74° 0′ | $\bar{1}$·8892943 | 30′ | ·0252959 | 83° 0′ | ·1815890 |
| 6′ | $\bar{1}$·8921969 | 36′ | ·0284830 | 6′ | ·1854815 |
| 12′ | $\bar{1}$·8951036 | 42′ | ·0316794 | 12′ | ·1894001 |
| 18′ | $\bar{1}$·8980144 | 48′ | ·0348855 | 18′ | ·1933455 |
| 24′ | $\bar{1}$·9009295 | 54′ | ·0381014 | 24′ | ·1973184 |
| 30′ | $\bar{1}$·9038489 | 79° 0′ | ·0413273 | 30′ | ·2013197 |
| 36′ | $\bar{1}$·9067728 | 6′ | ·0445633 | 36′ | ·2053502 |
| 42′ | $\bar{1}$·9097012 | 12′ | ·0478098 | 42′ | ·2094108 |
| 48′ | $\bar{1}$·9126341 | 18′ | ·0510668 | 48′ | ·2135026 |
| 54′ | $\bar{1}$·9155717 | 24′ | ·0543347 | 54′ | ·2176259 |

| | $\log \frac{M}{4\pi\sqrt{Aa}}$. | | $\log \frac{M}{4\pi\sqrt{Aa}}$. | | $\log \frac{M}{4\pi\sqrt{Aa}}$. |
|---|---|---|---|---|---|
| 84° 0′ | ·2217823 | 86° 0′ | ·3139097 | 88° 0′ | ·4385420 |
| 6′ | ·2259728 | 6′ | ·3191092 | 6′ | ·4465341 |
| 12′ | ·2301983 | 12′ | ·3243843 | 12′ | ·4548064 |
| 18′ | ·2344600 | 18′ | ·3297387 | 18′ | ·4633880 |
| 24′ | ·2387591 | 24′ | ·3351762 | 24′ | ·4723127 |
| 30′ | ·2430970 | 30′ | ·3407012 | 30′ | ·4816206 |
| 36′ | ·2474748 | 36′ | ·3463184 | 36′ | ·4913595 |
| 42′ | ·2518940 | 42′ | ·3520327 | 42′ | ·5015870 |
| 48′ | ·2563561 | 48′ | ·3578495 | 48′ | ·5123738 |
| 54′ | ·2608626 | 54′ | ·3637749 | 54′ | ·5238079 |
| 85° 0′ | ·2654152 | 87° 0′ | ·3698153 | 89° 0′ | ·5360007 |
| 6′ | ·2700156 | 6′ | ·3759777 | 6′ | ·5490969 |
| 12′ | ·2746655 | 12′ | ·3822700 | 12′ | ·5632886 |
| 18′ | ·2793670 | 18′ | ·3887006 | 18′ | ·5788406 |
| 24′ | ·2841221 | 24′ | ·3952792 | 24′ | ·5961320 |
| 30′ | ·2889329 | 30′ | ·4020162 | 30′ | ·6157370 |
| 36′ | ·2938018 | 36′ | ·4089234 | 36′ | ·6385907 |
| 42′ | ·2987312 | 42′ | ·4160138 | 42′ | ·6663883 |
| 48′ | ·3037238 | 48′ | ·4233022 | 48′ | ·7027765 |
| 54′ | ·3087823 | 54′ | ·4308053 | 54′ | ·7586941 |

## [APPENDIX II.

In the very important case of two circular coaxal coils Lord Rayleigh has suggested in the use of the foregoing tables a very convenient formula of approximation. The formula, applicable to any number of variables, occurs in Mr. Merrifield's Report on Quadratures and Interpolation to the British Association, 1880, and is attributed to the late Mr. H. J. Purkiss. In the present instance the number of variables is four.

Let $n$, $n'$ be the number of windings in the coils.

$a$, $a'$ the radii of their central windings.

$b$ the distance between their centres.

$2h$, $2h'$ the radial breadths of the coils.

$2k$, $2k'$ the axial breadths.

Also let $f(a, a', b)$ be the coefficient of mutual induction for the central windings. Then the coefficient of mutual induction of the two coils is

$$\frac{1}{6}nn'\left\{\begin{array}{l} f(a+h,\ a',\ b)+f(a-h,\ a',\ b) \\ \quad +f(a,\ a'+h',\ b)+f(a,\ a'-h',\ b) \\ \quad +f(a,\ a',\ b+k)+f(a,\ a',\ b-k) \\ \quad +f(a,\ a',\ b+k')+f(a,\ a',\ b-k') \\ \quad -2f(a,\ a',\ b). \end{array}\right.$$ ]

## {APPENDIX III.

*Self-induction of a circular coil of rectangular section.*

If $a$ denote the mean radius of a coil of $n$ windings whose axial breadth is $b$ and radial breadth is $c$, then the self-induction, as calculated by means of the series of Art. 705, has been shown by Weinstein Wied. Ann. xxi. 329 to be

$$L = 4\pi n^2 (a\lambda + \mu),$$

where, writing $x$ for $b/c$,

$$\lambda = \log\frac{8a}{c} + \frac{1}{12} - \frac{\pi x}{3} - \frac{1}{2}\log(1+x^2) + \frac{1}{12x^2}\log(1+x^2)$$
$$+ \frac{1}{12}x^2\log\left(1+\frac{1}{x^2}\right) + \frac{2}{3}\left(x-\frac{1}{x}\right)\tan^{-1}x,$$

$$\mu = \frac{c^2}{96a}\left[\left(\log\frac{8a}{c} - \frac{1}{2}\log(1+x^2)\right)(1+3x^2) + 3{\cdot}45\,x^2\right.$$
$$+ \frac{221}{60} - 1{\cdot}6\,\pi x^3 + 3{\cdot}2\,x^3\tan^{-1}x$$
$$\left. - \frac{1}{10}\frac{1}{x^2}\log(1+x^2) + \frac{1}{2}x^4\log\left(1+\frac{1}{x^2}\right)\right].\}$$

# CHAPTER XV.

## ELECTROMAGNETIC INSTRUMENTS.

### *Galvanometers.*

707.] A GALVANOMETER is an instrument by means of which an electric current is indicated or measured by its magnetic action.

When the instrument is intended to indicate the existence of a feeble current, it is called a Sensitive Galvanometer.

When it is intended to measure a current with the greatest accuracy in terms of standard units, it is called a Standard Galvanometer.

All galvanometers are founded on the principle of Schweigger's Multiplier, in which the current is made to pass through a wire, which is coiled so as to pass many times round an open space, within which a magnet is suspended, so as to produce within this space an electromagnetic force, the intensity of which is indicated by the magnet.

In sensitive galvanometers the coil is so arranged that its windings occupy the positions in which their influence on the magnet is greatest. They are therefore packed closely together in order to be near the magnet.

Standard galvanometers are constructed so that the dimensions and relative positions of all their fixed parts may be accurately known, and that any small uncertainty about the position of the moveable parts may introduce the smallest possible error into the calculations.

In constructing a sensitive galvanometer we aim at making the field of electromagnetic force in which the magnet is suspended as intense as possible. In designing a standard galvanometer we wish to make the field of electromagnetic force near the magnet as uniform as possible, and to know its exact intensity in terms of the strength of the current.

## *On Standard Galvanometers.*

708.] In a standard galvanometer the strength of the current has to be determined from the force which it exerts on the suspended magnet. Now the distribution of the magnetism within the magnet, and the position of its centre when suspended, are not capable of being determined with any great degree of accuracy. Hence it is necessary that the coil should be arranged so as to produce a field of force which is very nearly uniform throughout the whole space occupied by the magnet during its possible motion. The dimensions of the coil must therefore in general be much larger than those of the magnet.

By a proper arrangement of several coils the field of force within them may be made much more uniform than when one coil only

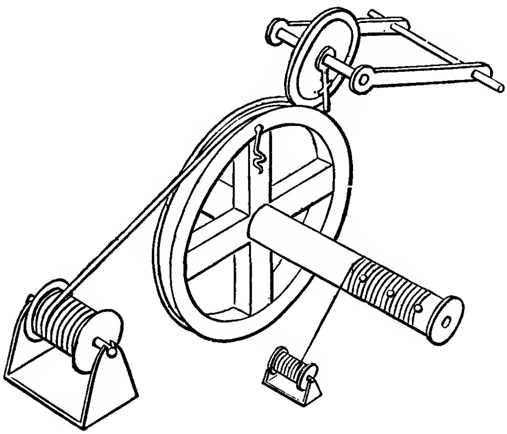


Fig. 49.

is used, and the dimensions of the instrument may be thus reduced and its sensibility increased. The errors of the linear measurements, however, introduce greater uncertainties into the values of the electrical constants for small instruments than for large ones. It is therefore best to determine the electrical constants of small instruments, not by direct measurement of their dimensions, but by an electrical comparison with a large standard instrument, of which the dimensions are more accurately known; see Art. 752.

In all standard galvanometers the coils are circular. The channel in which the coil is to be wound is carefully turned.

Its breadth is made equal to some multiple, $n$, of the diameter of the covered wire. A hole is bored in the side of the channel where the wire is to enter, and one end of the covered wire is pushed out through this hole to form the inner connexion of the coil. The channel is placed on a lathe, and a wooden axis is fastened to it; see Fig. 49. The end of a long string is nailed to the wooden axis at the same part of the circumference as the entrance of the wire. The whole is then turned round, and the wire is smoothly and regularly laid on the bottom of the channel till it is completely covered by $n$ windings. During this process the string has been wound $n$ times round the wooden axis, and a nail is driven into the string at the $n$th turn. The windings of the string should be kept exposed so that they can easily be counted. The external circumference of the first layer of windings is then measured and a new layer is begun, and so on till the proper number of layers has been wound on. The use of the string is to count the number of windings. If for any reason we have to unwind part of the coil, the string is also unwound, so that we do not lose our reckoning of the actual number of windings of the coil. The nails serve to distinguish the number of windings in each layer.

The measure of the circumference of each layer furnishes a test of the regularity of the winding, and enables us to calculate the electrical constants of the coil. For if we take the arithmetic mean of the circumferences of the channel and of the outer layer, and then add to this the circumferences of all the intermediate layers, and divide the sum by the number of layers, we shall obtain the mean circumference, and from this we can deduce the mean radius of the coil. The circumference of each layer may be measured by means of a steel tape, or better by means of a graduated wheel which rolls on the coil as the coil revolves in the process of winding. The value of the divisions of the tape or wheel must be ascertained by comparison with a straight scale.

709.] The moment of the force with which a unit current in the coil acts upon the suspended apparatus may be expressed by the series

$$G_1 g_1 \sin\theta + G_2 g_2 \sin\theta\, P_2'(\theta) + \&c.,$$

where the coefficients $G$ refer to the coil, and the coefficients $g$ to the suspended apparatus, $\theta$ being the angle between the axis of the coil and that of the suspended apparatus; see Art. 700.

When the suspended apparatus is a thin uniformly and longitudinally magnetized bar magnet of length $2l$ and strength unity, suspended by its middle,

$$g_1 = 2l, \quad g_2 = 0, \quad g_3 = 2l^3, \text{ \&c.}$$

The values of the coefficients for a bar magnet of length $2l$ magnetized in any other way are smaller than when it is magnetized uniformly.

710.] When the apparatus is used as a tangent galvanometer, the coil is fixed with its plane vertical and parallel to the direction of the earth's magnetic force. The equation of equilibrium of the magnet is in this case

$$mg_1 H\cos\theta = m\gamma\sin\theta\,\{G_1 g_1 + G_2 g_2 P_2'(\theta) + \text{\&c.}\},$$

where $mg_1$ is the magnetic moment of the magnet, $H$ the horizontal component of the terrestrial magnetic force, and $\gamma$ the strength of the current in the coil. When the length of the magnet is small compared with the radius of the coil the terms after the first in $G$ and $g$ may be neglected, and we find

$$\gamma = \frac{H}{G_1}\cot\theta.$$

The angle usually measured is the deflexion, $\delta$, of the magnet which is the complement of $\theta$, so that $\cot\theta = \tan\delta$.

The current is thus proportional to the tangent of the deflexion, and the instrument is therefore called a Tangent Galvanometer.

Another method is to make the whole apparatus moveable about a vertical axis, and to turn it till the magnet is in equilibrium with its axis parallel to the plane of the coil. If the angle between the plane of the coil and the magnetic meridian is $\delta$, the equation of equilibrium is

$$mg_1 H\sin\delta = m\gamma\,\{G_1 g_1 - \tfrac{3}{2} G_3 g_3 + \text{\&c.}\},$$

whence
$$\gamma = \frac{H}{(G_1 - \text{\&c.})}\sin\delta.$$

Since the current is measured by the sine of the deflexion, the instrument when used in this way is called a Sine Galvanometer.

The method of sines can be applied only when the current is so steady that we can regard it as constant during the time of adjusting the instrument and bringing the magnet to equilibrium.

711.] We have next to consider the arrangement of the coils of a standard galvanometer.

The simplest form is that in which there is a single coil, and the magnet is suspended at its centre.

Let $A$ be the mean radius of the coil, $\xi$ its depth, $\eta$ its breadth, and $n$ the number of windings, the values of the coefficients are

$$G_1 = \frac{2\pi n}{A}\left\{1 + \tfrac{1}{12}\frac{\xi^2}{A^2} - \tfrac{1}{8}\frac{\eta^2}{A^2}\right\},$$

$$G_2 = 0,$$

$$G_3 = -\frac{\pi n}{A^3}\left\{1 + \tfrac{1}{2}\frac{\xi^2}{A^2} - \tfrac{5}{8}\frac{\eta^2}{A^2}\right\},$$

$$G_4 = 0, \text{ \&c.}$$

The principal correction is that arising from $G_3$. The series

$$G_1 g_1 + G_3 g_3 P_3'(\theta)$$

becomes approximately

$$G_1 g_1\left(1 - 3\frac{1}{A^2}\frac{g_3}{g_1}(\cos^2\theta - \tfrac{1}{4}\sin^2\theta)\right).$$

The factor of correction will differ most from unity when the magnet is uniformly magnetized and when $\theta = 0$. In this case it becomes $1 - 3\dfrac{l^2}{A^2}$. It vanishes when $\tan\theta = 2$, or when the deflexion is $\tan^{-1}\frac{1}{2}$, or $26^\circ\ 34'$. Some observers, therefore, arrange their experiments so as to make the observed deflexion as near this angle as possible. The best method, however, is to use a magnet so short compared with the radius of the coil that the correction may be altogether neglected.

The suspended magnet is carefully adjusted so that its centre shall coincide as nearly as possible with the centre of the coil. If, however, this adjustment is not perfect, and if the coordinates of the centre of the magnet relative to the centre of the coil are $x$, $y$, $z$, $z$ being measured parallel to the axis of the coil, the factor of correction is

$$\left(1 + \tfrac{3}{2}\frac{x^2 + y^2 - 2z^2}{A^2}\right).^*$$

When the radius of the coil is large, and the adjustment of the magnet carefully made, we may assume that this correction is insensible.

* {The couple on the bar magnet when its axis makes an angle $\theta$ with that of the coil is

$$ml\left[\sin\theta\left\{G_1 + G_3\tfrac{3}{2}(2z^2 - (x^2+y^2))\right\} + 3\cos\theta\, G_3 z\sqrt{x^2+y^2}\right].$$

Since $G_1 + G_3\frac{3}{2}(2z^2 - (x^2+y^2))$ is the force at $x$, $y$, $z$ parallel to the axis of the coil and

$$3G_3 z\sqrt{x^2+y^2}$$

is the force at right angles to the axis. Thus when the arrangement is used as a sine galvanometer the factor of correction is

$$1 + \frac{G_3}{G_1}\tfrac{3}{2}(2z^2 - (x^2+y^2)) \text{ which is equal to } 1 - \frac{3}{4}\frac{1}{A^2}\left\{2z^2 - (x^2+y^2)\right\}\Big\}.$$

*Gaugain's Arrangement.*

712.] In order to get rid of the correction depending on $G_3$ Gaugain constructed a galvanometer in which this term was rendered zero by suspending the magnet, not at the centre of the coil, but at a point on the axis at a distance from the centre equal to half the radius of the coil. The form of $G_3$ is

$$G_3 = 4\pi \frac{A^2(B^2 - \frac{1}{4}A^2)}{C^7},$$

and, since in this arrangement $B = \frac{1}{2}A$, $G_3 = 0$.

This arrangement would be an improvement on the first form if we could be sure that the centre of the suspended magnet is exactly at the point thus defined. The position of the centre of the magnet, however, is always uncertain, and this uncertainty introduces a factor of correction of unknown amount depending on $G_2$ and of the form $\left(1 - \frac{6}{5}\frac{z}{A}\right)$, where $z$ is the unknown excess of distance of the centre of the magnet from the plane of the coil. This correction depends on the first power of $\frac{z}{A}$. Hence Gaugain's coil with eccentrically suspended magnet is subject to far greater uncertainty than the old form.

*Helmholtz's Arrangement.*

713.] Helmholtz converted Gaugain's galvanometer into a trustworthy instrument by placing a second coil, equal to the first, at an equal distance on the other side of the magnet.

By placing the coils symmetrically on both sides of the magnet we get rid at once of all terms of even order.

Let $A$ be the mean radius of either coil, the distance between their mean planes is made equal to $A$, and the magnet is suspended at the middle point of their common axis. The coefficients are

$$G_1 = \frac{16\pi n}{5\sqrt{5}}\frac{1}{A}\left(1 - \frac{1}{60}\frac{\xi^2}{A^2}\right),$$

$$G_2 = 0,$$

$$G_3 = 0{\cdot}0512\frac{\pi n}{3\sqrt{5}A^5}(31\xi^2 - 36\eta^2),$$

$$G_4 = 0,$$

$$G_5 = -0{\cdot}73728\frac{\pi n}{\sqrt{5}A^5},$$

where $n$ denotes the number of windings in both coils together.

It appears from these results that if the section of the channel of the $\phi$ coils be rectangular, the depth being $\xi$ and the breadth $\eta$, the value of $G_3$, as corrected for the finite size of the section, will be small, and will vanish, if $\xi^2$ is to $\eta^2$ as 36 to 31.

It is therefore quite unnecessary to attempt to wind the coils upon a conical surface, as has been done by some instrument makers, for the conditions may be satisfied by coils of rectangular section, which can be constructed with far greater accuracy than coils wound upon an obtuse cone.

The arrangement of the coils in Helmholtz's double galvanometer is represented in Fig. 53, Art. 725.

The field of force due to the double coil is represented in section in Fig. XIX at the end of this volume.

## *Galvanometer of Four Coils.*

714.] By combining four coils we may get rid of the coefficients $G_2$, $G_3$, $G_4$, $G_5$, and $G_6$. For by any symmetrical combination we get rid of the coefficients of even orders. Let the four coils be parallel circles belonging to the same sphere, corresponding to angles $\theta$, $\phi$, $\pi-\phi$, and $\pi-\theta$.

Let the number of windings on the first and fourth coils be $n$, and the number on the second and third $pn$. Then the condition that $G_3 = 0$ for the combination gives

$$n \sin^2\theta P_3'(\theta) + pn \sin^2\phi P_3'(\phi) = 0, \tag{1}$$

and the condition that $G_5 = 0$ gives

$$n \sin^2\theta P_5'(\theta) + pn \sin^2\phi P_5'(\phi) = 0. \tag{2}$$

Putting $\qquad \sin^2\theta = x \quad \text{and} \quad \sin^2\phi = y, \qquad (3)$

and expressing $P_3'$ and $P_5'$ (Art. 698) in terms of these quantities, the equations (1) and (2) become

$$4x - 5x^2 + 4py - 5py^2 = 0, \tag{4}$$

$$8x - 28x^2 + 21x^3 + 8py - 28py^2 + 21py^3 = 0. \tag{5}$$

Taking twice (4) from (5), and dividing by 3, we get

$$6x^2 - 7x^3 + 6py^2 - 7py^3 = 0.$$

Hence, from (4) and (6),

$$p = \frac{x}{y}\frac{5x-4}{4-5y} = \frac{x^2}{y^2}\frac{7x-6}{6-7y},$$

and we obtain

$$y = \tfrac{4}{7}\frac{7x-6}{5x-4}, \qquad \frac{1}{p} = \frac{32}{49x}\frac{7x-6}{(5x-4)^3}.$$

Both $x$ and $y$ are the squares of the sines of angles and must therefore lie between 0 and 1. Hence, either $x$ is between 0 and $\frac{4}{7}$, in which case $y$ is between $\frac{6}{7}$ and 1, and $1/p$ between $\infty$ and $\frac{49}{32}$, or else $x$ is between $\frac{6}{7}$ and 1, in which case $y$ is between 0 and $\frac{4}{7}$, and $1/p$ between 0 and $\frac{32}{49}$.

### *Galvanometer of Three Coils.*

715.] The most convenient arrangement is that in which $x = 1$. Two of the coils then coincide and form a great circle of the sphere whose radius is $C$. The number of windings in this compound coil is 64. The other two coils form small circles of the sphere. The radius of each of them is $\sqrt{\frac{4}{7}}\,C$. The distance of either of them from the plane of the first is $\sqrt{\frac{3}{7}}\,C$. The number of windings on each of these coils is 49.

The value of $G_1$ is $\dfrac{240\,\pi}{C}$.

This arrangement of coils is represented in Fig. 50.

Since in this three-coiled galvanometer the first term after $G_1$ which has a finite value is $G_7$, a large portion of the sphere on whose surface the coils lie forms a field of force sensibly uniform.

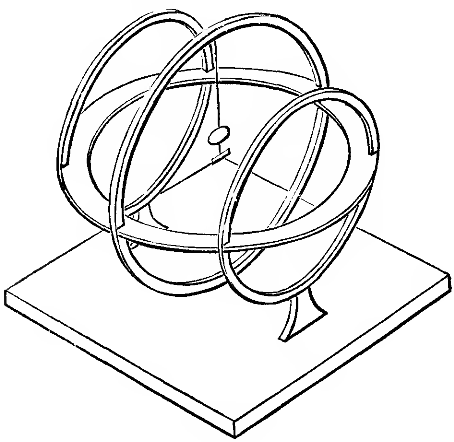


Fig. 50.

If we could wind the wire over the whole of a spherical surface, as described in Art. 672, we should obtain a field of perfectly uniform force. It is practically impossible, however, to distribute the windings on a spherical surface with sufficient accuracy, even if such a coil were not liable to the objection that it forms a closed surface, so that its interior is inaccessible.

By putting the middle coil out of the circuit, and making the current flow in opposite directions through the two side coils, we obtain a field of force which exerts a nearly uniform action in the direction of the axis on a magnet or coil suspended within it, with its axis coinciding with that of the coils; see Art. 673. For in this case all the coefficients of odd orders disappear, and since

$$\mu = \sqrt{\tfrac{3}{7}}, \qquad P_4' = \tfrac{5}{2}\mu(7\mu^2-3) = 0.$$

Hence the expression (6), Art. 695, for the magnetic potential near the centre of the coil becomes, there being $n$ windings in each of the coils,

$$\omega = \tfrac{8}{7}\sqrt{\tfrac{3}{7}}\,\pi n\left\{-3\frac{r^2}{C^2}P_2(\theta)+\tfrac{11}{7}\frac{r^6}{C^6}P_6(\theta)+\&c.\right\}.$$

*On the Proper Thickness of the Wire of a Galvanometer, the External Resistance being given.*

716.] Let the form of the channel in which the galvanometer coil is to be wound be given, and let it be required to determine whether it ought to be filled with a long thin wire or with a shorter thick wire.

Let $l$ be the length of the wire, $y$ its radius, $y+b$ the radius of the wire when covered, $\rho$ its specific resistance, $g$ the value of $G$ for unit of length of the wire, and $r$ the part of the resistance which is independent of the galvanometer.

The resistance of the galvanometer wire is

$$R = \frac{\rho}{\pi}\frac{l}{y^2}.$$

The volume of the coil is

$$V = \pi l(y+b)^2.$$

The electromagnetic force is $\gamma G$, where $\gamma$ is the strength of the current and

$$G = gl.$$

If $E$ is the electromotive force acting in the circuit whose resistance is $R+r$,

$$E = \gamma(R+r).$$

The electromagnetic force due to this electromotive force is

$$E\frac{G}{R+r},$$

which we have to make a maximum by the variation of $y$ and $l$.

Inverting the fraction, we find that

$$\frac{\rho}{\pi g}\frac{1}{y^2}+\frac{r}{gl}$$

is to be made a minimum. Hence

$$2\frac{\rho}{\pi}\frac{dy}{y^3}+\frac{r\,dl}{l^2} = 0.$$

If the volume of the coil remains constant

$$\frac{dl}{l} + 2\frac{dy}{y+b} = 0.$$

Eliminating $dl$ and $dy$, we obtain

$$\frac{\rho}{\pi}\frac{y+b}{y^3} = \frac{r}{l},$$

or

$$\frac{r}{R} = \frac{y+b}{y}.$$

Hence the thickness of the wire of the galvanometer should be such that the external resistance is to the resistance of the galvanometer coil as the diameter of the covered wire to the diameter of the wire itself.

### *On Sensitive Galvanometers.*

717.] In the construction of a sensitive galvanometer the aim of every part of the arrangement is to produce the greatest possible deflexion of the magnet by means of a given small electromotive force acting between the electrodes of the coil.

The current through the wire produces the greatest effect when it is placed as near as possible to the suspended magnet. The magnet, however, must be left free to oscillate, and therefore there is a certain space which must be left empty within the coil. This defines the internal boundary of the coil.

Outside of this space each winding must be placed so as to have the greatest possible effect on the magnet. As the number of windings increases, the most advantageous positions become filled up, so that at last the increased resistance of a new winding diminishes the effect of the current in the former windings more than the new winding itself adds to it. By making the outer windings of thicker wire than the inner ones we obtain the greatest magnetic effect from a given electromotive force.

718.] We shall suppose that the windings of the galvanometer are circles, the axis of the galvanometer passing through the centres of these circles at right angles to their planes.

Let $r \sin\theta$ be the radius of one of these circles, and $r\cos\theta$ the distance of its centre from the centre of the galvanometer, then, if $l$ is the length of a portion of wire coinciding with this circle, and $\gamma$ the current which flows in it, the magnetic force at the centre of the galvanometer resolved in the direction of the axis is

$$\gamma l \frac{\sin\theta}{r^2}.$$

If we write $\qquad r^2 = x^2 \sin\theta, \qquad (1)$

this expression becomes $\gamma \dfrac{l}{x^2}$.

Hence, if a surface be constructed, similar to those represented in section in Fig. 51, whose polar equation is

$$r^2 = x_1{}^2 \sin\theta, \qquad (2)$$

where $x_1$ is any constant, a given length of wire bent into the form of a circular arc will produce a greater magnetic effect when it lies within this surface than when it lies outside it.

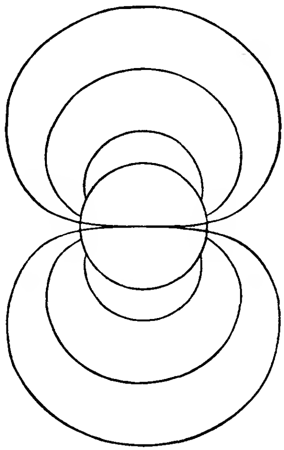


Fig. 51.

It follows from this that the outer surface of any layer of wire ought to have a constant value of $x$, for if $x$ is greater at one place than another a portion of wire might be transferred from the first place to the second, so as to increase the force at the centre of the galvanometer.

The whole force due to the coil is $\gamma G$, where

$$G = \int \frac{dl}{x}, \qquad (3)$$

the integration being extended over the whole length of the wire, $x$ being considered as a function of $l$.

719.] Let $y$ be the radius of the wire, its transverse section will be $\pi y^2$. Let $\rho$ be the specific resistance of the material of which the wire is made referred to unit of volume, then the resistance of a length $l$ is $\dfrac{l\rho}{\pi y^2}$, and the whole resistance of the coil is

$$R = \frac{\rho}{\pi}\int \frac{dl}{y^2}, \qquad (4)$$

where $y$ is considered a function of $l$.

Let $Y^2$ be the area of the quadrilateral whose angles are the sections of the axes of four neighbouring wires of the coil by a plane through the axis, then $Y^2 l$ is the volume occupied in the coil by a length $l$ of wire together with its insulating covering, and including any vacant space necessarily left between the windings of the coil. Hence the whole volume of the coil is

$$V = \int Y^2 dl, \tag{5}$$

where $Y$ is considered a function of $l$.

But since the coil is a figure of revolution

$$V = 2\pi \iint r^2 \sin\theta \, dr \, d\theta, \tag{6}$$

or, expressing $r$ in terms of $x$, by equation (1),

$$V = 2\pi \iint x^2 (\sin\theta)^{\frac{5}{2}} dx \, d\theta. \tag{7}$$

Now $2\pi \int_0^{\pi} (\sin\theta)^{\frac{5}{2}} d\theta$ is a numerical quantity, call it $N$, then

$$V = \tfrac{1}{3} N x^3 - V_0, \tag{8}$$

where $V_0$ is the volume of the interior space left for the magnet.

Let us now consider a layer of the coil contained between the surfaces $x$ and $x + dx$.

The volume of this layer is

$$dV = Nx^2 dx = Y^2 dl, \tag{9}$$

where $dl$ is the length of wire in this layer.

This gives us $dl$ in terms of $dx$. Substituting this in equations (3) and (4), we find

$$dG = N \frac{dx}{Y^2}, \tag{10}$$

$$dR = N \frac{\rho}{\pi} \frac{x^2 dx}{Y^2 y^2}, \tag{11}$$

where $dG$ and $dR$ represent the portions of the values of $G$ and of $R$ due to this layer of the coil.

Now if $E$ be the given electromotive force,

$$E = \gamma (R + r),$$

where $r$ is the resistance of the external part of the circuit, independent of the galvanometer, and the force at the centre is

$$\gamma G = E \frac{G}{R + r}.$$

We have therefore to make $\frac{G}{R + r}$ a maximum, by properly adjusting the section of the wire in each layer. This also necessarily involves a variation of $Y$ because $Y$ depends on $y$.

Let $G_0$ and $R_0$ be the values of $G$ and of $R+r$ when the given layer is excluded from the calculation. We have then

$$\frac{G}{R+r}=\frac{G_0+dG}{R_0+dR}, \tag{12}$$

and to make this a maximum by the variation of the value of $y$ for the given layer we must have

$$\frac{\dfrac{d}{dy}\cdot dG}{\dfrac{d}{dy}\cdot dR}=\frac{G_0+dG}{R_0+dR}=\frac{G}{R+r}. \tag{13}$$

Since $dx$ is very small and ultimately vanishes, $\frac{G_0}{R_0}$ will be sensibly, and ultimately exactly, the same whichever layer is excluded, and we may therefore regard it as constant. We have therefore, by (10) and (11),

$$\frac{\rho}{\pi}\frac{x^2}{y^2}\left(1+\frac{Y}{y}\frac{dy}{dY}\right)=\frac{R+r}{G}=\text{constant}. \tag{14}$$

If the method of covering the wire and of winding it is such that the space occupied by the metal of the wire bears the same proportion to the space between the wires whether the wire is thick or thin, then

$$\frac{Y}{y}\frac{dy}{dY}=1,$$

and we must make both $y$ and $Y$ proportional to $x$, that is to say, the diameter of the wire in any layer must be proportional to the linear dimension of that layer.

If the thickness of the insulating covering is constant and equal to $b$, and if the wires are arranged in square order,

$$Y=2(y+b), \tag{15}$$

and the condition is

$$\frac{x^2(2y+b)}{y^3}=\text{constant}. \tag{16}$$

In this case the diameter of the wire increases with the diameter of the layer of which it forms part, but not at so great a rate.

If we adopt the first of these two hypotheses, which will be nearly true if the wire itself nearly fills up the whole space, then we may put

$$y=ax, \qquad Y=\beta y,$$

where $a$ and $\beta$ are constant numerical quantities, and {by (10) and (11)}

$$G = N\frac{1}{a^2\beta^2}\left(\frac{1}{a}-\frac{1}{x}\right),$$

$$R = N\frac{\rho}{\pi}\frac{1}{a^4\beta^2}\left(\frac{1}{a}-\frac{1}{x}\right),$$

where $a$ is a constant depending upon the size and form of the free space left inside the coil.

Hence, if we make the thickness of the wire vary in the same ratio as $x$, we obtain very little advantage by increasing the external size of the coil after the external dimensions have become a large multiple of the internal dimensions.

720.] If increase of resistance is not regarded as a defect, as when the external resistance is far greater than that of the galvanometer, or when our only object is to produce a field of intense force, we may make $y$ and $Y$ constant. We have then

$$G = \frac{N}{Y^2}(x-a),$$

$$R = \tfrac{1}{3}\frac{N}{Y^2 y^2}\frac{\rho}{\pi}(x^3-a^3),$$

where $a$ is a constant depending on the vacant space inside the coil. In this case the value of $G$ increases uniformly as the dimensions of the coil are increased, so that there is no limit to the value of $G$ except the labour and expense of making the coil.

### *On Suspended Coils.*

721.] In the ordinary galvanometer a suspended magnet is acted on by a fixed coil. But if the coil can be suspended with sufficient delicacy, we may determine the action of the magnet, or of another coil on the suspended coil, by its deflexion from the position of equilibrium.

We cannot, however, introduce the electric current into the coil unless there is metallic connexion between the electrodes of the battery and those of the wire of the coil. This connexion may be made in two different ways, by the Bifilar Suspension, and by wires in opposite directions.

The bifilar suspension has already been described in Art. 459 as applied to magnets. The arrangement of the upper part of the suspension is shewn in Fig. 54. When applied to coils, the two fibres are no longer of silk but of metal, and since the torsion of a metal wire capable of supporting the coil and transmitting the current is much greater than that of a silk fibre, it must be taken specially into account. This suspension has been brought to great perfection in the instruments constructed by M. Weber.

The other method of suspension is by means of a single wire which is connected to one extremity of the coil. The other extremity of the coil is connected to another wire which is made to hang down, in the same vertical straight line with the first wire, into a cup of mercury, as is shewn in Fig. 56, Art. 726. In certain cases it is convenient to fasten the extremities of the two wires to pieces by which they may be tightly stretched, care being taken that the line of these wires passes through the centre of gravity of the coil. The apparatus in this form may be used when the axis is not vertical; see Fig. 52.

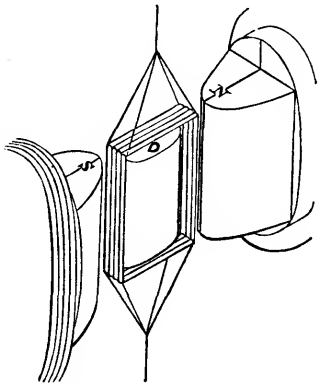


Fig. 52.

722.] The suspended coil may be used as an exceedingly sensitive galvanometer, for, by increasing the intensity of the magnetic force in the field in which it hangs, the force due to a feeble current in the coil may be greatly increased without adding to the mass of the coil. The magnetic force for this purpose may be produced by means of permanent magnets, or by electromagnets excited by an auxiliary current, and it may be powerfully concentrated on the suspended coil by means of soft iron armatures. Thus, in Sir W. Thomson's recording apparatus, Fig. 52, the coil is suspended between the opposite poles of the electromagnets $N$ and $S$, and in order to concentrate the lines of magnetic force on the vertical sides of the coil, a piece of soft iron, $D$, is fixed between the poles of the magnets. This iron becoming magnetized by induction, produces a very powerful field of force, in the intervals between it and the two magnets, through which the vertical sides of the coil are free to move, so that the coil, even when the current through it is very feeble, is acted on by a considerable force tending to turn it about its vertical axis.

723.] Another application of the suspended coil is to determine, by comparison with a tangent galvanometer, the horizontal component of terrestrial magnetism.

The coil is suspended so that it is in stable equilibrium when its plane is parallel to the magnetic meridian. A current $\gamma$ is passed through the coil and causes it to be deflected into a new position of equilibrium, making an angle $\theta$ with the magnetic meridian. If the suspension is bifilar, the moment of the couple which produces this deflexion is $F \sin\theta$, and this must be equal to $H\gamma g \cos\theta$, where $H$ is the horizontal component of terrestrial magnetism, $\gamma$ is the current in the coil, and $g$ is the sum of the areas of all the windings of the coil. Hence

$$H\gamma = \frac{F}{g}\tan\theta.$$

If $A$ is the moment of inertia of the coil about its axis of suspension, and $T$ the time of a half vibration, when no current is passing,

$$FT^2 = \pi^2 A,$$

and we obtain $$H\gamma = \frac{\pi^2 A}{T^2 g}\tan\theta.$$

If the same current passes through the coil of a tangent galvanometer, and deflects the magnet through an angle $\phi$,

$$\frac{\gamma}{H} = \frac{1}{G}\tan\phi,$$

where $G$ is the principal constant of the tangent galvanometer, Art. 710.

From these two equations we obtain

$$H = \frac{\pi}{T}\sqrt{\frac{AG\tan\theta}{g\tan\phi}}, \qquad \gamma = \frac{\pi}{T}\sqrt{\frac{A\tan\theta\tan\phi}{Gg}}.$$

This method was given by F. Kohlrausch *.

724.] Sir William Thomson has constructed a single instrument by means of which the observations required to determine $H$ and $\gamma$ may be made simultaneously by the same observer.

The coil is suspended so as to be in equilibrium with its plane in the magnetic meridian, and is deflected from this position when the current flows through it. A very small magnet is suspended at the centre of the coil, and is deflected by the current in the direction opposite to that of the deflexion of the coil. Let

* Pogg., *Ann.* cxxxviii, pp. 1–10, Aug. 1869.

the deflexion of the coil be $\theta$, and that of the magnet $\phi$, then the variable part of the energy of the system is

$$-H\gamma g \sin\theta - m\gamma G \sin(\theta-\phi) - Hm\cos\phi - F\cos\theta.$$

Differentiating with respect to $\theta$ and $\phi$, we obtain the equations of equilibrium of the coil and of the magnet respectively,

$$-H\gamma g\cos\theta - m\gamma G\cos(\theta-\phi) + F\sin\theta = 0,$$
$$m\gamma G\cos(\theta-\phi) + Hm\sin\phi = 0.$$

From these equations we find, by eliminating $H$ or $\gamma$, a quadratic equation from which $\gamma$ or $H$ may be found. If $m$, the magnetic moment of the suspended magnet, is very small, we obtain the following approximate values,

$$H = \frac{\pi}{T}\sqrt{\frac{-AG\sin\theta\cos(\theta-\phi)}{g\cos\theta\sin\phi}} - \tfrac{1}{2}\frac{mG}{g}\frac{\cos(\theta-\phi)}{\cos\theta},$$
$$\gamma = -\frac{\pi}{T}\sqrt{\frac{-A\sin\theta\sin\phi}{Gg\cos\theta\cos(\theta-\phi)}} + \tfrac{1}{2}\frac{m}{g}\frac{\sin\phi}{\cos\theta}.$$

In these expressions $G$ and $g$ are the principal electric constants of the coil, $A$ its moment of inertia, $T$ its half-time of vibration, $m$ the magnetic moment of the magnet, $H$ the intensity of the horizontal magnetic force, $\gamma$ the strength of the current, $\theta$ the deflexion of the coil, and $\phi$ that of the magnet.

Since the deflexion of the coil is in the opposite direction to the deflexion of the magnet, these values of $H$ and $g$ will always be real.

### *Weber's Electrodynamometer.*

725.] In this instrument a small coil is suspended by two wires within a larger coil which is fixed. When a current is made to flow through both coils, the suspended coil tends to place itself parallel to the fixed coil. This tendency is counteracted by the moment of the forces arising from the bifilar suspension, and it is also affected by the action of terrestrial magnetism on the suspended coil.

In the ordinary use of the instrument the planes of the two coils are nearly at right angles to each other, so that the mutual action of the currents in the coils may be as great as possible, and the plane of the suspended coil is nearly at right angles to the magnetic meridian, so that the action of terrestrial magnetism may be as small as possible.

Let the magnetic azimuth of the plane of the fixed coil be $\alpha$, and let the angle which the axis of the suspended coil makes with the plane of the fixed coil be $\theta+\beta$, where $\beta$ is the value of this angle when the coil is in equilibrium and no current is flowing, and $\theta$ is the deflexion due to the current. The equation of equilibrium is, $\gamma_1$ being the current in the fixed, $\gamma_2$ that in the moveable coil,

$$Gg\gamma_1\gamma_2\cos(\theta+\beta)-Hg\gamma_2\sin(\theta+\beta+\alpha)-F\sin\theta = 0.$$

Let us suppose that the instrument is adjusted so that $\alpha$ and $\beta$ are both very small, and that $Hg\gamma_2$ is small compared with $F$. We have in this case, approximately,

$$\tan\theta = \frac{Gg\gamma_1\gamma_2\cos\beta}{F} - \frac{Hg\gamma_2\sin(\alpha+\beta)}{F} - \frac{HGg^2\gamma_1\gamma_2^2}{F^2} - \frac{G^2g^2\gamma_1^2\gamma_2^2\sin\beta}{F^2}.$$

If the deflexions when the signs of $\gamma_1$ and $\gamma_2$ are changed are as follows,

$\theta_1$ when $\gamma_1$ is $+$ and $\gamma_2$ $+$,
$\theta_2$ ,, $-$ ,, $-$,
$\theta_3$ ,, $+$ ,, $-$,
$\theta_4$ ,, $-$ ,, $+$,

then we find

$$\gamma_1\gamma_2 = \tfrac{1}{4}\frac{F}{Gg\cos\beta}(\tan\theta_1+\tan\theta_2-\tan\theta_3-\tan\theta_4).$$

If it is the same current which flows through both coils we may put $\gamma_1\gamma_2=\gamma^2$, and thus obtain the value of $\gamma$.

When the currents are not very constant it is best to adopt this method, which is called the Method of Tangents.

If the currents are so constant that we can adjust $\beta$, the angle of the torsion-head of the instrument, we may get rid of the correction for terrestrial magnetism at once by the method of sines.

In this method $\beta$ is adjusted till the deflexion is zero, so that

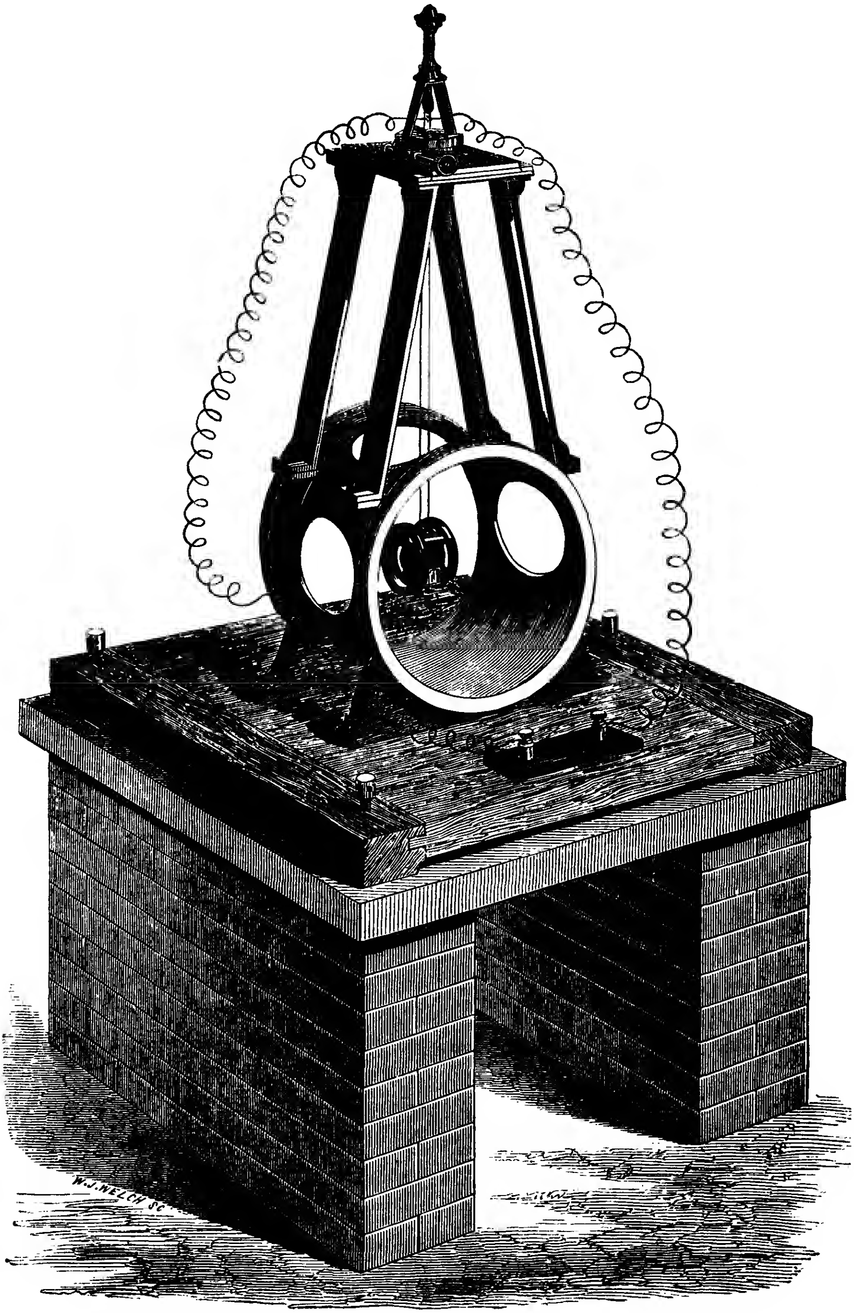
$$\theta = -\beta.$$

If the signs of $\gamma_1$ and $\gamma_2$ are indicated by the suffixes of $\beta$ as before,

$$F\sin\beta_1 = -F\sin\beta_3 = -Gg\gamma_1\gamma_2+Hg\gamma_2\sin\alpha,$$
$$F\sin\beta_2 = -F\sin\beta_4 = -Gg\gamma_1\gamma_2-Hg\gamma_2\sin\alpha,$$

and
$$\gamma_1\gamma_2 = -\frac{F}{4\,Gg}(\sin\beta_1+\sin\beta_2-\sin\beta_3-\sin\beta_4).$$

This is the method adopted by Mr. Latimer Clark in his use of the instrument constructed by the Electrical Committee of


W. J. WELCH SC.

the British Association. We are indebted to Mr. Clark for the drawing of the electrodynamometer in Fig. 53, in which Helmholtz's arrangement of two coils is adopted both for the fixed and for the suspended coil *. The torsion-head of the instrument, by which the bifilar suspension is adjusted, is represented in Fig. 54. The equality of the tensions of the suspension wires is ensured by their being attached to the extremities of a silk

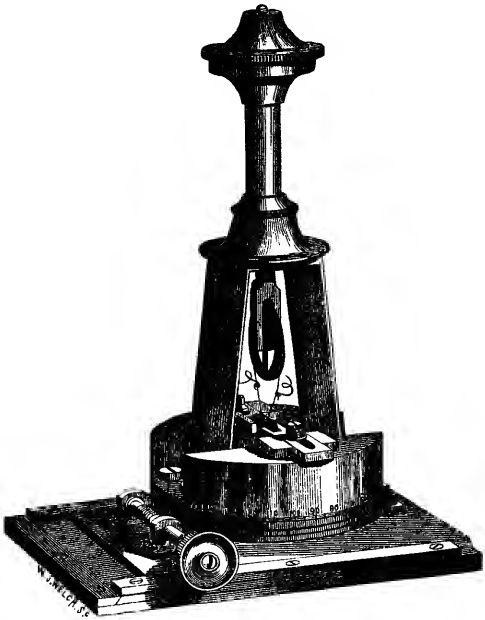


Fig. 54.

thread which passes over a wheel, and their distance is regulated by two guide-wheels, which can be set at the proper distance. The suspended coil can be moved vertically by means of a screw acting on the suspension-wheel, and horizontally in two directions by the sliding pieces shewn at the bottom of Fig. 54. It is adjusted in azimuth by means of the torsion-screw, which turns the torsion-head round a vertical axis (see Art. 459). The azimuth of the suspended coil is ascertained by observing the

* In the actual instrument, the wires conveying the current to and from the coils are not spread out as displayed in the figure, but are kept as close together as possible, so as to neutralize each other's electromagnetic action.

reflexion of a scale in the mirror, shewn just beneath the axis of the suspended coil.

The instrument originally constructed by Weber is described in his *Elektrodynamische Maasbestimmungen*. It was intended for the measurement of small currents, and therefore both the fixed and the suspended coils consisted of many windings, and the suspended coil occupied a larger part of the space within the fixed coil than in the instrument of the British Association, which was primarily intended as a standard instrument, with which more sensitive instruments might be compared. The experiments which he made with it furnish the most complete experimental proof of the accuracy of Ampère's formula as applied to closed currents, and form an important part of the researches by which Weber has raised the numerical determination of electrical quantities to a very high rank as regards precision.

Weber's form of the electrodynamometer, in which one coil is suspended within another, and is acted on by a couple tending to turn it about a vertical axis, is probably the best fitted for absolute measurements. A method of calculating the constants of such an arrangement is given in Art. 700.

726.] If, however, we wish, by means of a feeble current, to produce a considerable electromagnetic force, it is better to place the suspended coil parallel to the fixed coil, and to make it capable of motion to or from it.

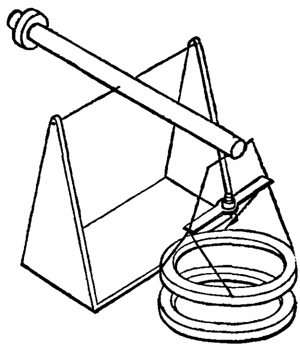


Fig. 55.

The suspended coil in Dr. Joule's current-weigher, Fig. 55, is horizontal, and capable of vertical motion, and the force between it and the fixed coil is estimated by the weight which must be added to or removed from the coil in order to bring it to the same relative position with respect to the fixed coil that it has when no current passes.

The suspended coil may also be fastened to the extremity of the horizontal arm of a torsion-balance, and may be placed between two fixed coils, one of which attracts it, while the other repels it, as in Fig. 56.

By arranging the coils as described in Art. 729, the force acting on the suspended coil may be made nearly uniform within a small distance of the position of equilibrium.

Another coil may be fixed to the other extremity of the arm of the torsion-balance and placed between two fixed coils. If the two suspended coils are similar, but with the current flowing

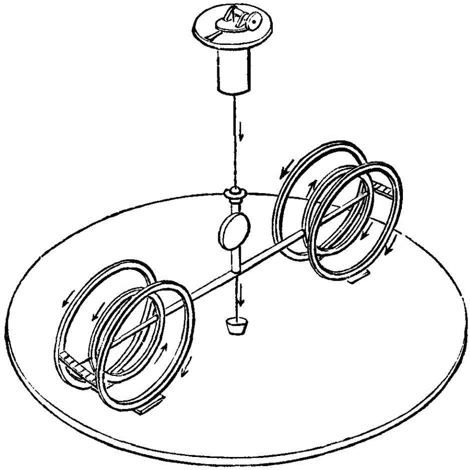


Fig. 56.

in opposite directions, the effect of terrestrial magnetism on the position of the arm of the torsion-balance will be completely eliminated.

727.] If the suspended coil is in the shape of a long solenoid, and is capable of moving parallel to its axis, so as to pass into the interior of a larger fixed solenoid having the same axis, then, if the current is in the same direction in both solenoids, the suspended solenoid will be sucked into the fixed one by a force which will be nearly uniform as long as none of the extremities of the solenoids are near one another.

728.] To produce a uniform longitudinal force on a small coil placed between two equal coils of much larger dimensions, we should make the ratio of the diameter of the large coils to the distance between their planes that of 2 to $\sqrt{3}$. If we send the same current through these coils in opposite directions, then, in the expression for $\omega$, the terms involving odd powers of $r$ disappear, and since $\sin^2\alpha = \frac{4}{7}$ and $\cos^2\alpha = \frac{3}{7}$, the term involving $r^4$ disappears also, and we have, by Art. 715, as the variable part of $\omega$,

$$\tfrac{8}{7}\sqrt{\tfrac{3}{7}}\,\pi n\gamma\left\{3\frac{r^2}{c^2}P_2(\theta)-\tfrac{11}{7}\frac{r^6}{c^6}P_6(\theta)+\&c.\right\},$$

which indicates a nearly uniform force on a small suspended coil. The arrangement of the coils in this case is that of the two outer coils in the galvanometer with three coils, described at Art. 715. See Fig. 50.

729.] If we wish to suspend a coil between two coils placed so near it that the distance between the mutually acting wires is small compared with the radii of the coils, the most uniform force is obtained by making the radius of either of the outer coils exceed that of the middle one by $\frac{1}{\sqrt{3}}$ of the distance between the planes of the middle and outer coils. This follows from the expression proved in Art. 705 for the mutual induction between two circular currents *.

* {In this case, if $M$ is the mutual potential energy of the inside and one of the outside coils, then, using the notation of Art. 705, the variation in the force for a displacement $y$ will, since the coils are symmetrically placed, be proportional to $d^3M/dy^3$. The most important term in this expression is $d^3 \log r/dy^3$, which vanishes when $3x^2 = y^2$.}

# CHAPTER XVI.

## ELECTROMAGNETIC OBSERVATIONS.

730.] So many of the measurements of electrical quantities depend on observations of the motion of a vibrating body that we shall devote some attention to the nature of this motion, and the best methods of observing it.

The small oscillations of a body about a position of stable equilibrium are, in general, similar to those of a point acted on by a force varying directly as the distance from a fixed point. In the case of the vibrating bodies in our experiments there is also a resistance to the motion, depending on a variety of causes, such as the viscosity of the air, and that of the suspension fibre. In many electrical instruments there is another cause of resistance, namely, the reflex action of currents induced in conducting circuits placed near vibrating magnets. These currents are induced by the motion of the magnet, and their action on the magnet is, by the law of Lenz, invariably opposed to its motion. This is in many cases the principal part of the resistance.

A metallic circuit, called a Damper, is sometimes placed near a magnet for the express purpose of damping or deadening its vibrations. We shall therefore speak of this kind of resistance as Damping.

In the case of slow vibrations, such as can be easily observed, the whole resistance, from whatever causes it may arise, appears to be proportional to the velocity. It is only when the velocity is much greater than in the ordinary vibrations of electromagnetic instruments that we have evidence of a resistance proportional to the square of the velocity.

We have therefore to investigate the motion of a body subject to an attraction varying as the distance, and to a resistance varying as the velocity.

731.] The following application, by Professor Tait *, of the principle of the Hodograph, enables us to investigate this kind of motion in a very simple manner by means of the equiangular spiral.

Let it be required to find the acceleration of a particle which describes a logarithmic or equiangular spiral with uniform angular velocity $\omega$ about the pole.

The property of this spiral is, that the tangent $PT$ makes with the radius vector $PS$ a constant angle $\alpha$.

If $v$ is the velocity at the point $P$, then

$$v.\sin\alpha = \omega\,.\,SP.$$

Hence, if we draw $SP'$ parallel to $PT$ and equal to $SP$, the velocity at $P$ will be given both in magnitude and direction by

$$v = \frac{\omega}{\sin\alpha}SP'.$$

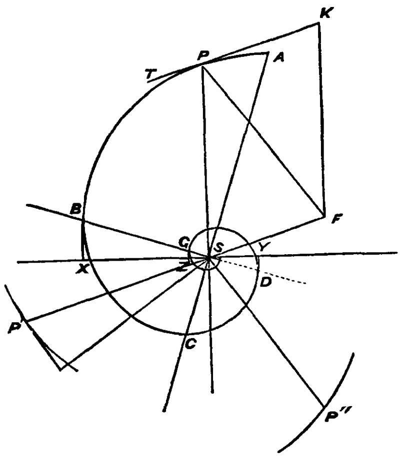



Fig. 57.

Hence $P'$ will be a point in the hodograph. But $SP'$ is $SP$ turned through a constant angle $\pi-\alpha$, so that the hodograph described by $P'$ is the same as the original spiral turned about its pole through an angle $\pi-\alpha$.

The acceleration of $P$ is represented in magnitude and direction by the velocity of $P'$ multiplied by the same factor, $\dfrac{\omega}{\sin\alpha}$.

* *Proc. R. S. Edin.*, Dec. 16, 1867.

Hence, if we perform on $SP'$ the same operation of turning it through an angle $\pi - a$ into the position $SP''$, the acceleration of $P$ will be equal in magnitude and direction to

$$\frac{\omega^2}{\sin^2 a} SP'',$$

where $SP''$ is equal to $SP$ turned through an angle $2\pi - 2a$.

If we draw $PF$ equal and parallel to $SP''$, the acceleration will be $\frac{\omega^2}{\sin^2 a} PF$, which we may resolve into

$$\frac{\omega^2}{\sin^2 a} PS \text{ and } \frac{\omega^2}{\sin^2 a} PK.$$

The first of these components is a central acceleration towards $S$ proportional to the distance.

The second is in a direction opposite to the velocity, and since

$$PK = 2\cos a P'S = -2\frac{\sin a \cos a}{\omega} v,$$

this acceleration may be written

$$-2\frac{\omega \cos a}{\sin a} v.$$

The acceleration of the particle is therefore compounded of two parts, the first of which is due to an attractive force $\mu r$, directed towards $S$, and proportional to the distance, and the second is $-2kv$, a resistance to the motion proportional to the velocity, where

$$\mu = \frac{\omega^2}{\sin^2 a}, \text{ and } k = \omega\frac{\cos a}{\sin a}.$$

If in these expressions we make $a = \frac{\pi}{2}$, the orbit becomes a circle, and we have $\mu_0 = \omega_0^2$, and $k = 0$.

Hence, if the force at unit distance remains the same, $\mu = \mu_0$, and

$$\omega = \omega_0 \sin a,$$

or the angular velocity in different spirals with the same law of attraction is proportional to the sine of the angle of the spiral.

732.] If we now consider the motion of a point which is the projection of the moving point $P$ on the horizontal line $XY$, we shall find that its distance from $S$ and its velocity are the horizontal components of those of $P$. Hence the acceleration of this point is also an attraction towards $S$, equal to $\mu$ times its distance from $S$, together with a retardation equal to $2k$ times its velocity.

We have therefore a complete construction for the rectilinear motion of a point, subject to an attraction proportional to the distance from a fixed point, and to a resistance proportional to the velocity. The motion of such a point is simply the horizontal part of the motion of another point which moves with uniform angular velocity in a logarithmic spiral.

733.] The equation of the spiral is

$$r = Ce^{-\phi \cot \alpha}.$$

To determine the horizontal motion, we put

$$\phi = \omega t, \qquad x = a + r \sin \phi,$$

where $a$ is the value of $x$ for the point of equilibrium.

If we draw $BSD$ making an angle $\alpha$ with the vertical, then the tangents $BX$, $DY$, $GZ$, &c. will be vertical, and $X$, $Y$, $Z$, &c. will be the extremities of successive oscillations.

734.] The observations which are made on vibrating bodies are—

(1) The scale-reading at the stationary points. These are called Elongations.

(2) The time of passing a definite division of the scale in the positive or negative direction.

(3) The scale-reading at certain definite times. Observations of this kind are not often made except in the case of vibrations of long period *.

The quantities which we have to determine are—

(1) The scale-reading at the position of equilibrium.

(2) The logarithmic decrement of the vibrations.

(3) The time of vibration.

### *To determine the Reading at the Position of Equilibrium from Three Consecutive Elongations.*

735.] Let $x_1$, $x_2$, $x_3$ be the observed scale-readings, corresponding to the elongations $X$, $Y$, $Z$, and let $a$ be the reading at the position of equilibrium, $S$, and let $r_1$ be the value of $SB$,

$$\begin{aligned} x_1 - a &= \quad r_1 \sin \alpha, \\ x_2 - a &= -r_1 \sin \alpha e^{-\pi \cot \alpha}, \\ x_3 - a &= \quad r_1 \sin \alpha e^{-2\pi \cot \alpha}. \end{aligned}$$

* See Gauss and W. Weber, *Resultate des magnetischen Vereins*, 1836. Chap. II. pp. 34–50.

From these values we find

$$(x_1-a)(x_3-a) = (x_2-a)^2,$$

whence $$a = \frac{x_1 x_3 - x_2^{\,2}}{x_1 + x_3 - 2x_2}.$$

When $x_3$ does not differ much from $x_1$ we may use as an approximate formula

$$a = \tfrac{1}{4}(x_1 + 2x_2 + x_3).$$

*To determine the Logarithmic Decrement.*

736.] The logarithm of the ratio of the amplitude of a vibration to that of the next following is called the Logarithmic Decrement. If we write $\rho$ for this ratio,

$$\rho = \frac{x_1 - x_2}{x_3 - x_2}, \qquad L = \log_{10}\rho, \qquad \lambda = \log_e \rho.$$

$L$ is called the common logarithmic decrement, and $\lambda$ the Napierian logarithmic decrement. It is manifest that

$$\lambda = L \log_e 10 = \pi \cot \alpha.$$

Hence $$\alpha = \cot^{-1}\frac{\lambda}{\pi},$$

which determines the angle of the logarithmic spiral.

In making a special determination of $\lambda$ we allow the body to perform a considerable number of vibrations. If $c_1$ is the amplitude of the first, and $c_n$ that of the $n^{\text{th}}$ vibration,

$$\lambda = \frac{1}{n-1}\log_e\left(\frac{c_1}{c_n}\right).$$

If we suppose the accuracy of observation to be the same for small vibrations as for large ones, then, to obtain the best value of $\lambda$, we should allow the vibrations to subside till the ratio of $c_1$ to $c_n$ becomes most nearly equal to $e$, the base of the Napierian logarithms. This gives for $n$ the nearest whole number to $\frac{1}{\lambda} + 1$.

Since, however, in most cases time is valuable, it is best to take the second set of observations before the diminution of amplitude has proceeded so far.

737.] In certain cases we may have to determine the position of equilibrium from two consecutive elongations, the logarithmic decrement being known from a special experiment. We have then

$$a = \frac{x_1 + e^{\lambda} x_2}{1 + e^{\lambda}}.$$

## *Time of Vibration.*

738.] Having determined the scale-reading of the point of equilibrium, a conspicuous mark is placed at that point of the scale, or as near it as possible, and the times of the passage of this mark are noted for several successive vibrations.

Let us suppose that the mark is at an unknown but very small distance $x$ on the positive side of the point of equilibrium, and that $t_1$ is the observed time of the first transit of the mark in the positive direction, and $t_2$, $t_3$, &c. the times of the following transits.

If $T$ be the time of vibration {i. e. the time between two consecutive passages through the position of equilibrium}, and $P_1$, $P_2$, $P_3$, &c. the times of transit of the true point of equilibrium,

$$t_1 = P_1 + \frac{x}{v_1}, \qquad t_2 = P_2 + \frac{x}{v_2}, \qquad P_2 - P_1 = P_3 - P_2 = T,$$

where $v_1$, $v_2$, &c. are the successive velocities of transit, which we may suppose uniform for the very small distance $x$.

If $\rho$ is the ratio of the amplitude of a vibration to that of the next in succession,

$$v_2 = -\frac{1}{\rho} v_1, \quad \text{and} \quad \frac{x}{v_2} = -\rho \frac{x}{v_1}.$$

If three transits are observed at times $t_1$, $t_2$, $t_3$, we find

$$\frac{x}{v_1} = \frac{t_1 - 2t_2 + t_3}{(\rho+1)^2}.$$

The time of vibration is therefore

$$T = \tfrac{1}{2}(t_3 - t_1) - \tfrac{1}{2}\frac{\rho - 1}{\rho + 1}(t_1 - 2t_2 + t_3).$$

The time of the second passage of the true point of equilibrium is

$$P_2 = \tfrac{1}{4}(t_1 + 2t_2 + t_3) - \tfrac{1}{4}\frac{(\rho-1)^2}{(\rho+1)^2}(t_1 - 2t_2 + t_3).$$

Three transits are sufficient to determine these three quantities, but any greater number may be combined by the method of least squares. Thus, for five transits,

$$T = \tfrac{1}{10}(2t_5 + t_4 - t_2 - 2t_1) - \tfrac{1}{10}(t_1 - 2t_2 + 2t_3 - 2t_4 + t_5)\frac{\rho - 1}{\rho + 1}\left(2 - \frac{\rho}{1+\rho^2}\right).$$

The time of the third transit is,

$$P_3 = \tfrac{1}{8}(t_1 + 2t_2 + 2t_3 + 2t_4 + t_5) - \tfrac{1}{8}(t_1 - 2t_2 + 2t_3 - 2t_4 + t_5)\frac{(\rho-1)^2}{(\rho+1)^2}.$$

739.] The same method may be extended to a series of any number of vibrations. If the vibrations are so rapid that the time of every transit cannot be recorded, we may record the time of every third or every fifth transit, taking care that the directions of successive transits are opposite. If the vibrations continue regular for a long time, we need not observe during the whole time. We may begin by observing a sufficient number of transits to determine approximately the time of vibration, $T$, and the time of the middle transit, $P$, noting whether this transit is in the positive or the negative direction. We may then either go on counting the vibrations without recording the times of transit, or we may leave the apparatus unwatched. We then observe a second series of transits, and deduce the time of vibration $T'$ and the time of middle transit $P'$, noting the direction of this transit.

If $T$ and $T'$, the times of vibration as deduced from the two sets of observations, are nearly equal, we may proceed to a more accurate determination of the period by combining the two series of observations.

Dividing $P'-P$ by $T$, the quotient ought to be very nearly an integer, even or odd according as the transits $P$ and $P'$ are in the same or in opposite directions. If this is not the case, the series of observations is worthless, but if the result is very nearly a whole number $n$, we divide $P'-P$ by $n$, and thus find the mean value of $T$ for the whole time of swinging.

740.] The time of vibration $T$ thus found is the actual mean time of vibration, and is subject to corrections if we wish to deduce from it the time of vibration in infinitely small arcs and without damping.

To reduce the observed time to the time in infinitely small arcs, we observe that the time of a vibration from rest to rest of amplitude $c$ is in general of the form

$$T = T_1(1+\kappa c^2),$$

where $\kappa$ is a coefficient, which, in the case of the ordinary pendulum, is $\frac{1}{64}$. Now the amplitudes of the successive vibrations are $c$, $c\rho^{-1}$, $c\rho^{-2}$,...$c\rho^{1-n}$, so that the whole time of $n$ vibrations is

$$nT = T_1\left(n+\kappa\,\frac{c_1{}^2\rho^2-c_n{}^2}{\rho^2-1}\right),$$

whe e $T$ is the time deduced from the observations.

Hence, to find the time $T_1$ in infinitely small arcs, we have approximately,

$$T_1 = T\left\{1-\frac{\kappa}{n}\,\frac{c_1{}^2\rho^2-c_n{}^2}{\rho^2-1}\right\}.$$

To find the time $T_0$ when there is no damping, we have Art. 731

$$T_0 = T_1 \sin \alpha$$

$$= T_1 \frac{\pi}{\sqrt{\pi^2 + \lambda^2}}.$$

741.] The equation of the rectilinear motion of a body, attracted to a fixed point {by a force proportional to the distance} and resisted by a force varying as the velocity, is

$$\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + \omega^2(x-a) = 0, \tag{1}$$

where $x$ is the coordinate of the body at the time $t$, and $a$ is the coordinate of the point of equilibrium.

To solve this equation, let

$$x - a = e^{-kt}y; \tag{2}$$

then

$$\frac{d^2y}{dt^2} + (\omega^2 - k^2)y = 0; \tag{3}$$

the solution of which is

$$y = C\cos(\sqrt{\omega^2-k^2}\,t + \alpha), \text{ when } k \text{ is less than } \omega; \tag{4}$$

$$y = A + Bt, \text{ when } k \text{ is equal to } \omega; \tag{5}$$

and $$y = C'\cosh(\sqrt{k^2-\omega^2}\,t + \alpha'), \text{ when } k \text{ is greater than } \omega. \tag{6}$$

The value of $x$ may be obtained from that of $y$ by equation (2). When $k$ is less than $\omega$, the motion consists of an infinite series of oscillations, of constant periodic time, but of continually decreasing amplitude. As $k$ increases, the periodic time becomes longer, and the diminution of amplitude becomes more rapid.

When $k$ (half the coefficient of resistance) becomes equal to or greater than $\omega$, (the square root of the acceleration at unit distance from the point of equilibrium,) the motion ceases to be oscillatory, and during the whole motion the body can only once pass through the point of equilibrium, after which it reaches a position of greatest elongation, and then returns towards the point of equilibrium, continually approaching, but never reaching it.

Galvanometers in which the resistance is so great that the motion is of this kind are called *dead beat* galvanometers. They are useful in many experiments, but especially in telegraphic signalling, in which the existence of free vibrations would quite disguise the movements which are meant to be observed.

Whatever be the values of $k$ and $\omega$, the value of $a$, the scale-reading at the point of equilibrium, may be deduced from five scale-readings, $p, q, r, s, t$, taken at equal intervals of time, by the formula

$$a = \frac{q(rs-qt)+r(pt-r^2)+s(qr-ps)}{(p-2q+r)(r-2s+t)-(q-2r+s)^2}.$$

*On the Observation of the Galvanometer.*

742.] To measure a constant current with the tangent galvanometer, the instrument is adjusted with the plane of its coils parallel to the magnetic meridian, and the zero reading is taken. The current is then made to pass through the coils, and the deflexion of the magnet corresponding to its new position of equilibrium is observed. Let this be denoted by $\phi$.

Then, if $H$ is the horizontal magnetic force, $G$ the coefficient of the galvanometer, and $\gamma$ the strength of the current,

$$\gamma = \frac{H}{G}\tan\phi. \tag{1}$$

If the coefficient of torsion of the suspension fibre is $\tau MH$ (see Art. 452), we must use the corrected formula

$$\gamma = \frac{H}{G}(\tan\phi + \tau\phi\sec\phi). \tag{2}$$

*Best value of the Deflexion.*

743.] In some galvanometers the number of windings of the coil through which the current flows can be altered at pleasure. In others a known fraction of the current can be diverted from the galvanometer by a conductor called a Shunt. In either case the value of $G$, the effect of a unit-current on the magnet, is made to vary.

Let us determine the value of $G$, for which a given error in the observation of the deflexion corresponds to the smallest error of the deduced value of the strength of the current.

Differentiating equation (1), we find

$$\frac{d\gamma}{d\phi} = \frac{H}{G}\sec^2\phi. \tag{3}$$

Eliminating $G$,
$$\frac{d\phi}{d\gamma} = \frac{1}{2\gamma}\sin 2\phi. \tag{4}$$

This is a maximum for a given value of $\gamma$ when the deflexion is 45°. The value of $G$ should therefore be adjusted till $G\gamma$ is as nearly equal to $H$ as is possible; so that for strong currents it is better not to use too sensitive a galvanometer.

*On the Best Method of applying the Current.*

744.] When the observer is able, by means of a key, to make or break the connexions of the circuit at any instant, it is advisable to operate with the key in such a way as to make the magnet arrive at its position of equilibrium with the least possible velocity. The following method was devised by Gauss for this purpose.

Suppose that the magnet is in its position of equilibrium, and that there is no current. The observer now makes contact for a short time, so that the magnet is set in motion towards its new position of equilibrium. He then breaks contact. The force is now towards the original position of equilibrium, and the motion is retarded. If this is so managed that the magnet comes to rest exactly at the new position of equilibrium, and if the observer again makes contact at that instant and maintains the contact, the magnet will remain at rest in its new position.

If we neglect the effect of the resistances and also the inequality of the total force acting in the new and the old positions, then, since we wish the new force to generate as much kinetic energy during the time of its first action as the original force destroys while the circuit is broken, we must prolong the first action of the current till the magnet has moved over half the distance from the first position to the second. Then if the original force acts while the magnet moves over the other half of its course, it will exactly stop it. Now the time required to pass from a point of greatest elongation to a point half way to the position of equilibrium is one-third of the period, from rest to rest.

The operator, therefore, having previously ascertained the time of a vibration from rest to rest, makes contact for one-third of that time, breaks contact for another third of the same time, and then makes contact again during the continuance of the experiment. The magnet is then either at rest, or its vibrations are so small that observations may be taken at once, without waiting for the motion to die away. For this purpose a metronome may be adjusted so as to beat three times for each vibration of the magnet.

The rule is somewhat more complicated when the resistance is of sufficient magnitude to be taken into account, but in this case the vibrations die away so fast that it is unnecessary to apply any corrections to the rule.

When the magnet is to be restored to its original position, the circuit is broken for one-third of a vibration, made again for an equal time, and finally broken. This leaves the magnet at rest in its former position.

If the reversed reading is to be taken immediately after the direct one, the circuit is broken for the time of a single vibration and then reversed. This brings the magnet to rest in the reversed position.

*Measurement by the First Swing.*

745.] When there is no time to make more than one observation, the current may be measured by the extreme elongation observed in the first swing of the magnet. If there is no resistance, the permanent deflexion $\phi$ is half the extreme elongation. If the resistance is such that the ratio of one vibration to the next is $\rho$, and if $\theta_0$ is the zero reading, and $\theta_1$ the extreme elongation in the first swing, the deflexion, $\phi$, corresponding to the point of equilibrium is

$$\phi = \frac{\theta_0 + \rho\,\theta_1}{1+\rho}.$$

In this way the deflexion may be calculated without waiting for the magnet to come to rest in its position of equilibrium.

*To make a Series of Observations.*

746.] The best way of making a considerable number of measures of a constant current is by observing three elongations while the current is in the positive direction, then breaking contact for about the time of a single vibration, so as to let the magnet swing into the position of negative deflexion, then reversing the current and observing three successive elongations on the negative side, then breaking contact for the time of a single vibration and repeating the observations on the positive side, and so on till a sufficient number of observations have been obtained. In this way the errors which may arise from a change in the direction of the earth's magnetic force during the time of observation are eliminated. The operator, by carefully timing the making and breaking of contact, can easily regulate the extent of the vibrations, so as to make them sufficiently small without being indistinct. The motion of the magnet is graphically represented in Fig. 58, where the abscissa represents the time, and the ordinate the deflexion of the magnet. If $\theta_1 \ldots \theta_6$ be the observed algebraical values of the elongations, the deflexion is given by the equation

$$8\phi = \theta_1 + 2\theta_2 + \theta_3 - \theta_4 - 2\theta_5 - \theta_6.$$

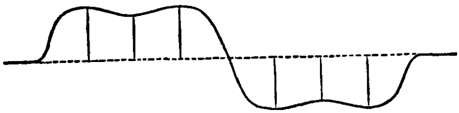


Fig. 58.

### *Method of Multiplication.*

747.] In certain cases, in which the deflexion of the galvanometer magnet is very small, it may be advisable to increase the visible effect by reversing the current at proper intervals, so, as to set up a swinging motion of the magnet. For thispurpose after ascertaining the time, $T$, of a single vibration {i. e. one from rest to rest} of the magnet, the current is sent in the positive direction for a time $T$, then in the reverse direction for an equal time, and so on. When the motion of the magnet has become visible, we may make the reversal of the current at the observed times of greatest elongation.

Let the magnet be at the positive elongation $\theta_0$, and let the current be sent through the coil in the negative direction. The point of equilibrium is then $-\phi$, and the magnet will swing to a negative elongation $\theta_1$, such that

$$-\rho(\phi + \theta_1) = (\theta_0 + \phi),$$

or $$-\rho\theta_1 = \theta_0 + (\rho + 1)\phi.$$

Similarly, if the current is now made positive while the magnet swings to $\theta_2$,

$$\rho\theta_2 = -\theta_1 + (\rho + 1)\phi,$$

or $$\rho^2\theta_2 = \theta_0 + (\rho + 1)^2\phi;$$

and if the current is reversed $n$ times in succession, we find

$$(-1)^n\theta_n = \rho^{-n}\theta_0 + \frac{\rho+1}{\rho-1}(1 - \rho^{-n})\phi,$$

whence we may find $\phi$ in the form

$$\phi = (\theta_n - \rho^{-n}\theta_0)\frac{\rho-1}{\rho+1}\frac{1}{1-\rho^{-n}}.$$

If $n$ is a number so great that $\rho^{-n}$ may be neglected, the expression becomes

$$\phi = \theta_n \frac{\rho-1}{\rho+1}.$$

The application of this method to exact measurement requires an accurate knowledge of $\rho$, the ratio of one vibration of the magnet to the next under the influence of the resistances which it experiences. The uncertainties arising from the difficulty of avoiding irregularities in the value of $\rho$ generally outweigh the advantages of the large angular elongation. It is only where we wish to establish the existence of a very small current by causing it to produce a visible movement of the needle that this method is really valuable.

*On the Measurement of Transient Currents.*

748.] When a current lasts only during a very small fraction of the time of vibration of the galvanometer-magnet, the whole quantity of electricity transmitted by the current may be measured by the angular velocity communicated to the magnet during the passage of the current, and this may be determined from the elongation of the first vibration of the magnet.

If we neglect the resistance which damps the vibrations of the magnet, the investigation becomes very simple.

Let $\gamma$ be the intensity of the current at any instant, and $Q$ the quantity of electricity which it transmits, then

$$Q = \int \gamma\, dt. \tag{1}$$

Let $M$ be the magnetic moment, $A$ the moment of inertia of the magnet and suspended apparatus, and $\theta$ the angle the magnet makes with the plane of the coil,

$$A\frac{d^2\theta}{dt^2} + MH\sin\theta = MG\gamma\cos\theta. \tag{2}$$

If the time of the passage of the current is very small, we may integrate with respect to $t$ during this short time without regarding the change of $\theta$, and we find

$$A\frac{d\theta}{dt} = MG\cos\theta_0 \int \gamma\, dt + C = MGQ\cos\theta_0 + C. \tag{3}$$

This shews that the passage of the quantity $Q$ produces an angular momentum $MGQ\cos\theta_0$ in the magnet, where $\theta_0$ is the value of $\theta$ at the instant of passage of the current. If the magnet is initially in equilibrium, we may put $\theta_0 = 0$, $C = 0$.

The magnet then swings freely and reaches an elongation $\theta_1$. If there is no resistance, the work done against the magnetic force during this swing is $MH(1-\cos\theta_1)$.

The energy communicated to the magnet by the current is

$$\tfrac{1}{2}A\left|\frac{d\theta}{dt}\right|^2.$$

Equating these quantities, we find

$$\left|\frac{d\theta}{dt}\right|^2 = 2\frac{MH}{A}(1-\cos\theta_1), \tag{4}$$

whence
$$\frac{d\theta}{dt} = 2\sqrt{\frac{MH}{A}}\sin\tfrac{1}{2}\theta_1$$

$$= \frac{MG}{A}Q \text{ by (3)}. \tag{5}$$

But if $T$ be the time of a single vibration of the magnet from rest to rest,

$$T = \pi\sqrt{\frac{A}{MH}}, \tag{6}$$

and we find
$$Q = \frac{H}{G}\frac{T}{\pi}2\sin\tfrac{1}{2}\theta_1, \tag{7}$$

where $H$ is the horizontal magnetic force, $G$ the coefficient of the galvanometer, $T$ the time of a single vibration, and $\theta_1$ the first elongation of the magnet.

749.] In many actual experiments the elongation is a small angle, and it is then easy to take into account the effect of resistance, for we may treat the equation of motion as a linear equation.

Let the magnet be at rest at its position of equilibrium, let an angular velocity $v$ be communicated to it instantaneously, and let its first elongation be $\theta_1$.

The equation of motion is

$$\theta = Ce^{-\omega_1 t\tan\beta}\sin\omega_1 t, \tag{8}$$

$$\frac{d\theta}{dt} = C\omega_1\sec\beta\, e^{-\omega_1 t\tan\beta}\cos(\omega_1 t+\beta). \tag{9}$$

When $t = 0$, $\theta = 0$, and $\dfrac{d\theta}{dt} = C\omega_1 = v$.

When $\omega_1 t + \beta = \frac{\pi}{2}$,

$$\theta = C e^{-\left(\frac{\pi}{2} - \beta\right)\tan\beta} \cos\beta = \theta_1. \tag{10}$$

Hence $$\theta_1 = \frac{v}{\omega_1} e^{-\left(\frac{\pi}{2} - \beta\right)\tan\beta} \cos\beta. \tag{11}$$

Now by Art. (741) $$\frac{MH}{A} = \omega^2 = \omega_1^2 \sec^2\beta, \tag{12}$$

$$\tan\beta = \frac{\lambda}{\pi}, \qquad \omega_1 = \frac{\pi}{T_1}, \tag{13}$$

and by equation (5) $$v = \frac{MG}{A} Q. \tag{14}$$

Hence $$\theta_1 = \frac{QG}{H} \frac{\sqrt{\pi^2 + \lambda^2}}{T_1} e^{-\frac{\lambda}{\pi}\tan^{-1}\frac{\pi}{\lambda}}, \tag{15}$$

and $$Q = \frac{H}{G} \frac{T_1 \theta_1}{\sqrt{\pi^2 + \lambda^2}} e^{\frac{\lambda}{\pi}\tan^{-1}\frac{\pi}{\lambda}}, \tag{16}$$

which gives the first elongation in terms of the quantity of electricity in the transient current, and conversely, where $T_1$ is the observed time of a single vibration as affected by the actual resistance of damping. When $\lambda$ is small we may use the approximate formula

$$Q = \frac{H}{G} \frac{T}{\pi} (1 + \tfrac{1}{2}\lambda)\, \theta_1. \tag{17}$$

*Method of Recoil.*

750.] The method given above supposes the magnet to be at rest in its position of equilibrium when the transient current is passed through the coil. If we wish to repeat the experiment we must wait till the magnet is again at rest. In certain cases, however, in which we are able to produce transient currents of equal intensity, and to do so at any desired instant, the following method, described by Weber*, is the most convenient for making a continued series of observations.

Suppose that we set the magnet swinging by means of a transient current whose value is $Q_0$. If, for brevity, we write

$$\frac{G}{H} \frac{\sqrt{\pi^2 + \lambda^2}}{T_1} e^{-\frac{\lambda}{\pi}\tan^{-1}\frac{\pi}{\lambda}} = K, \tag{18}$$

then the first elongation

$$\theta_1 = KQ_0 = a_1 \text{ (say)}. \tag{19}$$

* Gauss & Weber, *Resultate des Magnetischen Vereins*, 1838, p. 98.

The velocity instantaneously communicated to the magnet at starting is

$$v_0 = \frac{MG}{A} Q_0. \tag{20}$$

When it returns through the point of equilibrium in a negative direction its velocity will be

$$v_1 = -ve^{-\lambda}. \tag{21}$$

The next negative elongation will be

$$\theta_2 = -\theta_1 e^{-\lambda} = b_1. \tag{22}$$

When the magnet returns to the point of equilibrium, its velocity will be

$$v_2 = v_0 e^{-2\lambda}. \tag{23}$$

Now let an instantaneous current, whose total quantity is $-Q$, be transmitted through the coil at the instant when the magnet is at the zero point. It will change the velocity $v_2$ into $v_2 - v$, where

$$v = \frac{MG}{A} Q. \tag{24}$$

If $Q$ is greater than $Q_0 e^{-2\lambda}$, the new velocity will be negative and equal to

$$-\frac{MG}{A}(Q - Q_0 e^{-2\lambda}).$$

The motion of the magnet will thus be reversed, and the next elongation will be negative,

$$\theta_3 = -K(Q - Q_0 e^{-2\lambda}) = c_1 = -KQ + \theta_1 e^{-2\lambda}. \tag{25}$$

The magnet is then allowed to come to its positive elongation

$$\theta_4 = -\theta_3 e^{-\lambda} = d_1 = e^{-\lambda}(KQ - a_1 e^{-2\lambda}), \tag{26}$$

and when it again reaches the point of equilibrium a positive current whose quantity is $Q$ is transmitted. This throws the magnet back in the positive direction to the positive elongation

$$\theta_5 = KQ + \theta_3 e^{-2\lambda}; \tag{27}$$

or, calling this the first elongation of a second series of four,

$$a_2 = KQ(1 - e^{-2\lambda}) + a_1 e^{-4\lambda}. \tag{28}$$

Proceeding in this way, by observing two elongations + and −, then sending a negative current and observing two elongations − and +, then sending a positive current, and so on, we obtain a series consisting of sets of four elongations, in each of which

$$\frac{d-b}{a-c} = e^{-\lambda}, \tag{29}$$

and

$$KQ = \frac{(a-b)e^{-2\lambda} + d - c}{1 + e^{-\lambda}}; \tag{30}$$

If $n$ series of elongations have been observed, then we find the logarithmic decrement from the equation

$$\frac{\Sigma(d)-\Sigma(b)}{\Sigma(a)-\Sigma(c)} = e^{-\lambda}, \tag{31}$$

and $Q$ from the equation

$$KQ(1+e^{-\lambda})(2n-1)$$
$$= \Sigma_n(a-b-c+d)(1+e^{-2\lambda})-(a_1-b_1)-(d_n-c_n)e^{-2\lambda}. \tag{32}$$

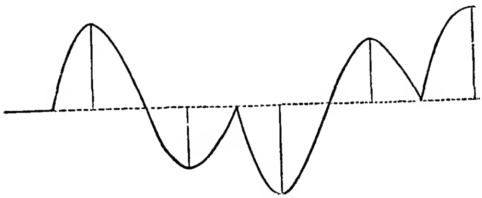


Fig. 59.

The motion of the magnet in the method of recoil is graphically represented in Fig. 59, where the abscissa represents the time, and the ordinate the deflexion of the magnet at that time. See Art. 760.

*Method of Multiplication.*

751.] If we make the transient current pass every time that the magnet passes through the zero point, and always so as to increase the velocity of the magnet, then, if $\theta_1$, $\theta_2$, &c. are the successive elongations,

$$\theta_2 = -KQ - e^{-\lambda}\theta_1, \tag{33}$$
$$\theta_3 = +KQ - e^{-\lambda}\theta_2. \tag{34}$$

The ultimate value to which the elongation tends after a great many vibrations is found by putting $\theta_n = -\theta_{n-1}$, whence we find

$$\theta = \pm\frac{1}{1-e^{-\lambda}}KQ. \tag{35}$$

If $\lambda$ is small, the value of the ultimate elongation may be large, but since this involves a long continued experiment, and a careful determination of $\lambda$, and since a small error in $\lambda$ introduces a large error in the determination of $Q$, this method is rarely useful for numerical determination, and should be reserved for obtaining evidence of the existence or non-existence of currents too small to be observed directly.

In all experiments in which transient currents are made to act on the moving magnet of the galvanometer, it is essential that the whole current should pass while the distance of the magnet from the zero point remains a small fraction of the total elongation. The time of vibration should therefore be large compared with the time required to produce the current, and the operator should have his eye on the motion of the magnet, so as to regulate the instant of passage of the current by the instant of passage of the magnet through its point of equilibrium.

To estimate the error introduced by a failure of the operator to produce the current at the proper instant, we observe that the effect of an impulse in increasing the elongation varies as

$$e^{\phi \tan\beta} \cos(\phi+\beta),*$$

and that this is a maximum when $\phi = 0$. Hence the error arising from a mistiming of the current will always lead to an under-estimation of its value, and the amount of the error may be estimated by comparing the cosine of the phase of the vibration at the time of the passage of the current with unity.

* {I have not succeeded in verifying this expression; using the notation of Art. 748. I find that the elongation when the impulse is applied at $\phi$ bears to the elongation produced by the same impulse when $\phi = 0$ the ratio

$$e^{\frac{A\omega_1}{MGQ}\phi\tan\beta}\left\{1+\frac{A\omega_1\phi\tan\beta}{MGQ}\right\},$$

where $\phi$ has been assumed to be so small that its squares and higher powers may be neglected.}

# CHAPTER XVII.

## COMPARISON OF COILS.

### *Experimental Determination of the Electrical Constants of a Coil.*

752.] WE have seen in Art. 717 that in a sensitive galvanometer the coils should be of small radius, and should contain many windings of the wire. It would be extremely difficult to determine the electrical constants of such a coil by direct measurement of its form and dimensions, even if we could obtain access to every winding of the wire in order to measure it. But in fact the greater number of the windings are not only completely hidden by the outer windings, but we are uncertain whether the pressure of the outer windings may not have altered the form of the inner ones after the coiling of the wire.

It is better therefore to determine the electrical constants of the coil by direct electrical comparison with a standard coil whose constants are known.

Since the dimensions of the standard coil must be determined by actual measurement, it must be made of considerable size, so that the unavoidable error of measurement of its diameter or circumference may be as small as possible compared with the quantity measured. The channel in which the coil is wound should be of rectangular section, and the dimensions of the section should be small compared with the radius of the coil. This is necessary, not so much in order to diminish the correction for the size of the section, as to prevent any uncertainty about the position of those windings of the coil which are hidden by the external windings*.

* Large tangent galvanometers are sometimes made with a single circular conducting ring of considerable thickness, which is sufficiently stiff to maintain its form without any support. This is not a good plan for a standard instrument. The distribution of the current within the conductor depends on the relative conductivity

The principal constants which we wish to determine are—

(1) The magnetic force at the centre of the coil due to a unit-current. This is the quantity denoted by $G_1$ in Art. 700.

(2) The magnetic moment of the coil due to a unit-current. This is the quantity $g_1$.

753.] *To determine* $G_1$. Since the coils of the working galvanometer are much smaller than the standard coil, we place the galvanometer within the standard coil, so that their centres coincide, the planes of both coils being vertical and parallel to the earth's magnetic force. We have thus obtained a differential galvanometer one of whose coils is the standard coil, for which the value of $G_1$ is known, while the constant of the other coil is $G_1'$, the value of which we have to determine.

The magnet suspended in the centre of the galvanometer coil is acted on by the currents in both coils. If the strength of the current in the standard coil is $\gamma$, and that in the galvanometer coil $\gamma'$, then, if these currents flowing in opposite directions produce a deflexion $\delta$ of the magnet,

$$H \tan \delta = G_1'\gamma' - G_1\gamma, \tag{1}$$

where $H$ is the horizontal magnetic force of the earth.

If the currents are so arranged as to produce no deflexion, we may find $G_1'$ by the equation

$$G_1' = \frac{\gamma}{\gamma'} G_1. \tag{2}$$

We may determine the ratio of $\gamma$ to $\gamma'$ in several ways. Since the value of $G_1$ is in general greater for the galvanometer than for the standard coil, we may arrange the circuit so that the whole current $\gamma$ flows through the standard coil, and is then divided so that $\gamma'$ flows through the galvanometer and resistance coils, the combined resistance of which is $R_1$, while the remainder $\gamma - \gamma'$ flows through another set of resistance coils whose combined resistance is $R_2$.

of its various parts. Hence any concealed flaw in the continuity of the metal may cause the main stream of electricity to flow either close to the outside or close to the inside of the circular ring. Thus the true path of the current becomes uncertain. Besides this, when the current flows only once round the circle, especial care is necessary to avoid any action on the suspended magnet due to the current on its way to or from the circle, because the current in the electrodes is equal to that in the circle. In the construction of many instruments the action of this part of the current seems to have been altogether lost sight of.

The most perfect method is to make one of the electrodes in the form of a metal tube, and the other a wire covered with insulating material, and placed inside the tube and concentric with it. The external action of the electrodes when thus arranged is zero, by Art. 683.

We have then, by Art. 276,

$$\gamma' R_1 = (\gamma - \gamma') R_2, \tag{3}$$

or
$$\frac{\gamma}{\gamma'} = \frac{R_1 + R_2}{R_2}, \tag{4}$$

and
$$G_1' = \frac{R_1 + R_2}{R_2} G_1. \tag{5}$$

If there is any uncertainty about the actual resistance of the galvanometer coil (on account, say, of an uncertainty as to its temperature) we may add resistance coils to it, so that the resistance of the galvanometer itself forms but a small part of $R_1$, and thus introduces but little uncertainty into the final result.

754.] *To determine* $g_1$, the magnetic moment of a small coil due to a unit current flowing through it, the magnet is still suspended at the centre of the standard coil, but the small coil is moved parallel to itself along the common axis of both coils, till the same current, flowing in opposite directions round the coils, no longer deflects the magnet. If the distance between the centres of the coils is $r$, we have now (Art. 700)

$$G_1 = 2\frac{g_1}{r^3} + 3\frac{g_2}{r^4} + 4\frac{g_3}{r^5} + \&c. \tag{6}$$

By repeating the experiment with the small coil on the opposite side of the standard coil, and measuring the distance between the positions of the small coil, we eliminate the uncertain error in the determination of the position of the centres of the magnet and of the small coil, and we get rid of the terms in $g_2$, $g_4$, &c.

If the standard coil is so arranged that we can send the current through half the number of windings, so as to give a different value to $G_1$, we may determine a new value of $r$, and thus, as in Art. 454, we may eliminate the term involving $g_3$.

It is often possible, however, to determine $g_3$ by direct measurement of the small coil with sufficient accuracy to make it available in calculating the value of the correction to be applied to $g_1$ in the equation

$$g_1 = \frac{1}{2} G_1 r^3 - 2\frac{g_3}{r^2}, \tag{7}$$

where $g_3 = -\frac{1}{8}\pi a^2 (6a^2 + 3\xi^2 - 2\eta^2)$, by Art. 700.

*Comparison of Coefficients of Induction.*

755.] It is only in a small number of cases that the direct calculation of the coefficients of induction from the form and position of the circuits can be easily performed. In order to attain a sufficient degree of accuracy, it is necessary that the distance between the circuits should be capable of exact measurement. But when the distance between the circuits is sufficient to prevent errors of measurement from introducing large errors into the result, the coefficient of induction itself is necessarily very much reduced in magnitude. Now for many experiments it is necessary to make the coefficient of induction large, and we can only do so by bringing the circuits close together, so that the method of direct measurement becomes impossible, and, in order to determine the coefficient of induction, we must compare it with that of a pair of coils arranged so that their coefficient may be obtained by direct measurement and calculation.

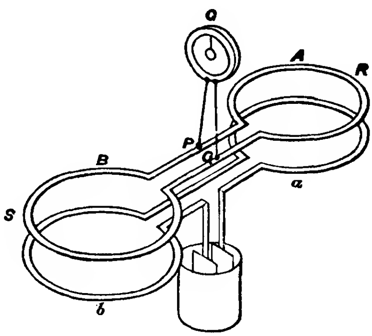


Fig. 60.

This may be done as follows:

Let $A$ and $a$ be the standard pair of coils, $B$ and $b$ the coils to be compared with them. Connect $A$ and $B$ in one circuit, and place the electrodes of the galvanometer, $G$, at $P$ and $Q$, so that the resistance of $PAQ$ is $R$, and that of $QBP$ is $S$, $K$ being the resistance of the galvanometer. Connect $a$ and $b$ in one circuit with the battery.

Let the current in $A$ be $\dot{x}$, that in $B$, $\dot{y}$, and that in the galvanometer, $\dot{x}-\dot{y}$, that in the battery circuit being $\gamma$.

Then, if $M_1$ is the coefficient of induction between $A$ and $a$, and $M_2$ that between $B$ and $b$, the integral induction current through the galvanometer at breaking the battery circuit is

$$x-y=\gamma\frac{\dfrac{M_2}{S}-\dfrac{M_1}{R}}{1+\dfrac{K}{R}+\dfrac{K}{S}}. \qquad (8)$$

By adjusting the resistances $R$ and $S$ till there is no current through the galvanometer at making or breaking the battery circuit, the ratio of $M_2$ to $M_1$ may be determined by measuring that of $S$ to $R$.

* [The expression (8) may be proved as follows: Let $L_1$, $L_2$, $N$ and $\Gamma$ be the coefficients of self-induction of the coils $A$, $B$, $ab$ and the galvanometer respectively. The kinetic energy $T$ of the system is then approximately,

$$\tfrac{1}{2}L_1\dot{x}^2+\tfrac{1}{2}L_2\dot{y}^2+\tfrac{1}{2}\Gamma(\dot{x}-\dot{y})^2+\tfrac{1}{2}N\gamma^2+M_1\dot{x}\gamma+M_2\dot{y}\gamma.$$

The dissipation function $F$, i.e. half the rate at which the energy of the currents is wasted in heating the coils, is (see Lord Rayleigh's *Theory of Sound*, vol. i. p. 78)

$$\tfrac{1}{2}\dot{x}^2R+\tfrac{1}{2}\dot{y}^2S+\tfrac{1}{2}(\dot{x}-\dot{y})^2K+\tfrac{1}{2}\gamma^2Q,$$

where $Q$ is the resistance of the battery and battery coils.

The equation of currents corresponding to any variable $x$ is then of the form
$$\frac{d}{dt}\frac{dT}{d\dot{x}}-\frac{dT}{dx}+\frac{dF}{d\dot{x}}=\xi,$$

where $\xi$ is the corresponding electromotive force.

Hence we have

$$L_1\ddot{x}+\Gamma(\ddot{x}-\ddot{y})+M_1\dot{\gamma}+R\dot{x}+K(\dot{x}-\dot{y})=0,$$
$$L_2\ddot{y}-\Gamma(\ddot{x}-\ddot{y})+M_2\dot{\gamma}+S\dot{y}-K(\dot{x}-\dot{y})=0.$$

These equations can be at once integrated in regard to $t$. Observing that $x$, $\dot{x}$, $y$, $\dot{y}$, $\gamma$ are zero initially, if we write $x-y=z$ we find, on eliminating $y$, an equation of the form

$$A\ddot{z}+B\dot{z}+Cz=D\dot{\gamma}+E\gamma. \qquad (8')$$

A short time after battery contact the current $\gamma$ will have become steady and the current $\dot{z}$ will have died away. Hence

$$Cz=E\gamma.$$

This gives the expression (8) above, and it shews that when the total quantity of electricity passing through the galvanometer is zero we must have $E=0$, or $M_2R-M_1S=0$. The equation (8′) further shews that if there is no current whatever in the galvanometer we must also have $D=0$, or $M_2L_1-M_1L_2=0$.]†

* [The investigation in square brackets, taken from Mr. Fleming's notes of Professor Clerk Maxwell's Lectures, possesses a melancholy interest as being part of the last lecture delivered by the Professor. In Mr. Fleming's notes the plan of the experiment differs from that given in the text in having the battery and galvanometer interchanged.]

† {Unless the condition $M_2L_1-M_1L_2=0$ is approximately fulfilled the unsteadiness caused in the zero of the galvanometer by the transient currents prevents our determining with accuracy whether there is or is not a 'kick' of the galvanometer on closing the battery circuit.}

*Comparison of a Coefficient of Self-Induction with a Coefficient of Mutual Induction.*

756.] In the branch $AF$ of Wheatstone's Bridge let a coil be inserted, the coefficient of self-induction of which we wish to find. Let us call it $L$.

In the connecting wire between $A$ and the battery another coil is inserted. The coefficient of mutual induction between this coil and the coil in $AF$ is $M$. It may be measured by the method described in Art. 755.

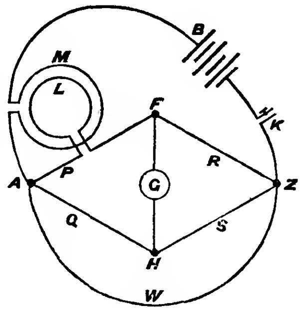


Fig. 61.

If the current from $A$ to $F$ is $x$, and that from $A$ to $H$ is $y$, that from $Z$ to $A$, through $B$, will be $x+y$. The external electromotive force from $A$ to $F$ is

$$A-F=Px+L\frac{dx}{dt}+M\left(\frac{dx}{dt}+\frac{dy}{dt}\right). \quad (9)$$

The external electromotive force along $AH$ is

$$A-H=Qy. \quad (10)$$

If the galvanometer placed between $F$ and $H$ indicates no current, either transient or permanent, then by (9) and (10), since $H-F=0$,

$$Px=Qy; \quad (11)$$

and

$$L\frac{dx}{dt}+M\left(\frac{dx}{dt}+\frac{dy}{dt}\right)=0, \quad (12)$$

whence

$$L=-\left(1+\frac{P}{Q}\right)M. \quad (13)$$

Since $L$ is always positive, $M$ must be negative, and therefore the current must flow in opposite directions through the coils placed in $P$ and in $B$. In making the experiment we may either begin by adjusting the resistances so that

$$PS=QR, \quad (14)$$

which is the condition that there may be no permanent current, and then adjust the distance between the coils till the galvanometer ceases to indicate a transient current on making and breaking the battery connexion; or, if this distance is not capable of adjustment, we may get rid of the transient current by altering the resistances $Q$ and $S$ in such a way that the ratio of $Q$ to $S$ remains constant.

If this double adjustment is found too troublesome, we may adopt a third method. Beginning with an arrangement in which the transient current due to self-induction is slightly in excess of that due to mutual induction, we may get rid of the inequality by inserting a conductor whose resistance is $W$ between $A$ and $Z$. The condition of no permanent current through the galvanometer is not affected by the introduction of $W$. We may therefore get rid of the transient current by adjusting the resistance of $W$ alone. When this is done the value of $L$ is

$$L = -\left(1 + \frac{P}{Q} + \frac{P+R}{W}\right) M. \tag{15}$$

*Comparison of the Coefficients of Self-Induction of Two Coils.*

757.] Insert the coils in two adjacent branches of Wheatstone's Bridge. Let $L$ and $N$ be the coefficients of self-induction of the coils inserted in $P$ and in $R$ respectively, then the condition of no galvanometer current is Fig. 61,

$$\left(Px + L\frac{dx}{dt}\right)Sy = Qy\left(Rx + N\frac{dx}{dt}\right), \tag{16}$$

whence $\quad PS = QR$, for no permanent current, $\quad$ (17)

and $\quad \dfrac{L}{P} = \dfrac{N}{R}$, for no transient current. $\quad$ (18)

Hence, by a proper adjustment of the resistances, both the permanent and the transient currents can be got rid of, and then the ratio of $L$ to $N$ can be determined by a comparison of the resistances.

# APPENDIX TO CHAPTER XVII.

{THE method of measuring the coefficient of self-induction of a coil is described in the following extract from Maxwell's paper on a Dynamical Theory of the Electromagnetic Field, *Phil. Trans.* 155, pp. 475–477.

'*On the Determination of Coefficients of Induction by the Electric Balance.*

The electric balance consists of six conductors joining four points $A$, $C$, $D$, $E$, two and two.

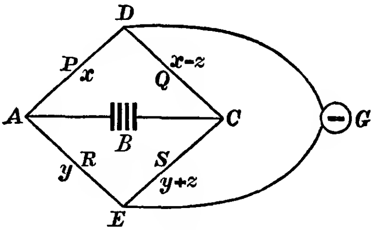



Fig. 62.

One pair, $AC$, of these points is connected through the battery $B$. The opposite pair, $DE$, is connected through the galvanometer $G$. Then if the resistances of the four remaining conductors are represented by $P$, $Q$, $R$, $S$, and the currents in them by $x$, $x-z$, $y$, and $y+z$, the current through $G$ will be $z$. Let the potentials at the four points be $A$, $C$, $D$, $E$. Then the conditions of steady currents may be found from the equations

$$\left.\begin{array}{ll} Px = A-D, & Q(x-z) = D-C, \\ Ry = A-E, & S(y+z) = E-C, \\ Gz = D-E. & B(x+y) = -A+C+F. \end{array}\right\} \qquad (21)$$

Solving these equations for $z$, we find

$$z\left\{\frac{1}{P}+\frac{1}{Q}+\frac{1}{R}+\frac{1}{S}+B\left(\frac{1}{P}+\frac{1}{R}\right)\left(\frac{1}{Q}+\frac{1}{S}\right)+G\left(\frac{1}{P}+\frac{1}{Q}\right)\left(\frac{1}{R}+\frac{1}{S}\right)\right.$$
$$\left.+\frac{BG}{PQRS}(P+Q+R+S)\right\} = F\left(\frac{1}{PS}-\frac{1}{QR}\right). \qquad (22)$$

In this expression $F$ is the electromotive force of the battery; $z$ the current through the galvanometer when it has become steady; $P$, $Q$, $R$, $S$, the resistances in the four arms; $B$ that of the battery and electrodes, and $G$ that of the galvanometer.

(44) If $PS = QR$, then $z = 0$, and there will be no steady current, but a transient current through the galvanometer may be produced on making or breaking circuit on account of induction, and the indications of the galvanometer may be used to determine the coefficients of induction, provided we understand the actions which take place.

We shall suppose $PS = QR$, so that the current $z$ vanishes when sufficient time is allowed, and

$$x(P+Q) = y(R+S) = \frac{F(P+Q)(R+S)}{(P+Q)(R+S)+B(P+Q+R+S)}. \tag{23}$$

Let the induction coefficients between $P$, $Q$, $R$, $S$ be given by the following Table, the coefficient of induction of $P$ on itself being $p$, between $P$ and $Q$, $h$, and so on.

| | $P$ | $Q$ | $R$ | $S$ |
|---|---|---|---|---|
| $P$ | $p$ | $h$ | $k$ | $l$ |
| $Q$ | $h$ | $q$ | $m$ | $n$ |
| $R$ | $k$ | $m$ | $r$ | $o$ |
| $S$ | $l$ | $n$ | $o$ | $s$ |

Let $g$ be the coefficient of induction of the galvanometer on itself, and let it be out of reach of the induction influence of $P$, $Q$, $R$, $S$ (as it must be in order to avoid direct action of $P$, $Q$, $R$, $S$ on the needle). Let $X$, $Y$, $Z$ be the integrals of $x$, $y$, $z$ with respect to $t$. At making contact $x$, $y$, $z$ are zero. After a time $z$ disappears, and $x$ and $y$ reach constant values. The equations for each conductor will therefore be

$$\left.\begin{aligned} PX+(p+h)\,x+(k+l)\,y &= \textstyle\int A\,dt - \int D\,dt, \\ Q(X-Z)+(h+q)\,x+(m+n)\,y &= \textstyle\int D\,dt - \int C\,dt, \\ RY+(k+m)\,x+(r+o)\,y &= \textstyle\int A\,dt - \int E\,dt, \\ S(Y+Z)+(l+n)\,x+(o+s)\,y &= \textstyle\int E\,dt - \int C\,dt, \end{aligned}\right\} \tag{24}$$

$$GZ = \textstyle\int D\,dt - \int E\,dt.$$

Solving these equations for $Z$ we find,

$$\begin{aligned} Z\Big\{\frac{1}{P}+\frac{1}{Q}+\frac{1}{R}+\frac{1}{S}+B\Big(\frac{1}{P}+\frac{1}{R}\Big)\Big(\frac{1}{Q}+\frac{1}{S}\Big)+G\Big(\frac{1}{P}+\frac{1}{Q}\Big)\Big(\frac{1}{R}+\frac{1}{S}\Big) \\ + \frac{BG}{PQRS}(P+Q+R+S)\Big\} = -F\frac{1}{PS}\Big\{\frac{p}{P}-\frac{q}{Q}-\frac{r}{R}+\frac{s}{S} \\ + h\Big(\frac{1}{P}-\frac{1}{Q}\Big)+k\Big(\frac{1}{R}-\frac{1}{P}\Big)+l\Big(\frac{1}{R}+\frac{1}{Q}\Big)-m\Big(\frac{1}{P}+\frac{1}{S}\Big) \\ +n\Big(\frac{1}{Q}-\frac{1}{S}\Big)+o\Big(\frac{1}{S}-\frac{1}{R}\Big)\Big\}. \end{aligned} \tag{25}$$

Now let the deflexion of the galvanometer by the instantaneous current whose intensity {total quantity} is $Z$ be $a$.

Let the permanent deflexion produced by making the ratio of $PS$ to $QR$, $\rho$ instead of unity, be $\theta$.

Also let the time of vibration of the galvanometer needle from rest to rest be $T$. Then calling the quantity

$$\frac{p}{P}-\frac{q}{Q}-\frac{r}{R}+\frac{s}{S}+h\left(\frac{1}{P}-\frac{1}{Q}\right)+k\left(\frac{1}{R}-\frac{1}{P}\right)+l\left(\frac{1}{R}+\frac{1}{Q}\right)$$
$$-m\left(\frac{1}{P}+\frac{1}{S}\right)+n\left(\frac{1}{Q}-\frac{1}{S}\right)+o\left(\frac{1}{S}-\frac{1}{R}\right)=\tau, \quad (26)$$

we find
$$\frac{Z}{z}=\frac{2\sin\frac{1}{2}a}{\tan\theta}\frac{T}{\pi}=\frac{\tau}{\rho-1}. \quad (27)$$

In determining $\tau$ by experiment it is best to make the alteration in the resistance in one of the arms by means of the arrangement described by Mr. Jenkin in the Report of the British Association for 1863, by which any value of $\rho$ from 1 to 1·01 can be accurately measured.

We observe $\{a\}$, the greatest deflexion {throw} due to the impulse of induction when the galvanometer is in circuit, when the connexions are made, and when the resistances are so adjusted as to give no permanent current.

We then observe $\{\beta\}$, the greatest deflexion {throw} produced by the permanent current when the resistance of one of the arms is increased in the ratio of $\rho$ to 1, the galvanometer not being in circuit till a little while after the connexion is made with the battery.

In order to eliminate the effects of resistance of the air, it is best to vary $\rho$ till $\beta = 2a$ nearly: then

$$\tau = T\frac{1}{\pi}(\rho-1)\frac{2\sin\frac{1}{2}a}{\tan\frac{1}{2}\beta}.$$

If all the arms of the balance except $P$ consist of resistance coils of very fine wire of no great length and doubled before being coiled, the induction coefficients belonging to these coils will be insensible, and $\tau$ will be reduced to $p/P$. The electric balance therefore affords the means of measuring the self-induction of any circuit whose resistance is known.'

# CHAPTER XVIII.

## ELECTROMAGNETIC UNIT OF RESISTANCE.

### *On the Determination of the Resistance of a Coil in Electromagnetic Measure.*

758.] THE resistance of a conductor is defined as the ratio of the numerical value of the electromotive force to that of the current which it produces in the conductor. The determination of the value of the current in electromagnetic measure can be made by means of a standard galvanometer, when we know the value of the earth's magnetic force. The determination of the value of the electromotive force is more difficult, as the only case in which we can directly calculate its value is when it arises from the relative motion of the circuit with respect to a known magnetic system.

759.] The first determination of the resistance of a wire in electromagnetic measure was made by Kirchhoff*. He employed two coils of known form, $A_1$ and $A_2$, and calculated their coefficient of mutual induction from the geometrical data of their form and position. These coils were placed in circuit with a galvanometer, $G$, and a battery, $B$, and two points of the circuit, $P$, between the coils, and $Q$, between the battery and galvanometer, were joined by the wire whose resistance, $R$, was to be measured.

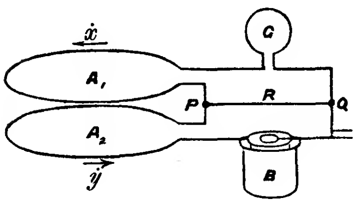



Fig. 63.

When the current is steady it is divided between the wire and the galvanometer circuit, and produces a certain permanent

* 'Bestimmung der Constanten, von welcher die Intensität inducirter elektrischer Ströme abhängt.' Pogg., *Ann.*, lxxvi (April 1849).

deflexion of the galvanometer. If the coil $A_1$ is now removed quickly from $A_2$ and placed in a position in which the coefficient of mutual induction between $A_1$ and $A_2$ is zero (Art. 538), a current of induction is produced in both circuits, and the galvanometer needle receives an impulse which produces a certain transient deflexion *.

The resistance of the wire, $R$, is deduced from a comparison between the permanent deflexion, due to the steady current, and the transient deflexion, due to the current of induction.

Let the resistance of $QGA_1P$ be $K$, of $PA_2BQ$, $B$, and of $PQ$, $R$.

Let $L$, $M$ and $N$ be the coefficients of induction of $A_1$ and $A_2$.

Let $\dot{x}$ be the current in $G$, and $\dot{y}$ that in $B$, then the current from $P$ to $Q$ is $\dot{x}-\dot{y}$.

Let $E$ be the electromotive force of the battery, then

$$(K+R)\dot{x}-R\dot{y}+\frac{d}{dt}(L\dot{x}+M\dot{y}) = 0, \tag{1}$$

$$-R\dot{x}+(B+R)\dot{y}+\frac{d}{dt}(M\dot{x}+N\dot{y}) = E. \tag{2}$$

When the currents are constant, and everything at rest,

$$(K+R)\dot{x}-R\dot{y} = 0. \tag{3}$$

If $M$ now suddenly becomes zero on account of the separation of $A_1$ from $A_2$, then, integrating with respect to $t$,

$$(K+R)x-Ry-M\dot{y} = 0, \tag{4}$$

$$-Rx+(B+R)y-M\dot{x} = \int E\,dt = 0\,; \tag{5}$$

whence
$$x = M\frac{(B+R)\dot{y}+R\dot{x}}{(B+R)(K+R)-R^2}. \tag{6}$$

Substituting the value of $\dot{y}$ in terms of $\dot{x}$ from (3), we find

$$\frac{x}{\dot{x}} = \frac{M}{R}\,\frac{(B+R)(K+R)+R^2}{(B+R)(K+R)-R^2} \tag{7}$$

$$= \frac{M}{R}\left\{1+\frac{2R^2}{(B+R)(K+R)}+\&c.\right\}. \tag{8}$$

* {Instead of removing the coil $A_1$, it is more convenient to reverse the current through $A_2$; in this case the quantity of electricity passing through the ballistic galvanometer is twice that in the text. Kirchhoff's method has been used by Messrs. Glazebrook, Sargant and Dodds to determine a resistance in absolute measure. *Phil. Trans.* 1883, pp. 223–268.}

When, as in Kirchhoff's experiment, both $B$ and $K$ are large compared with $R$, this equation is reduced to

$$\frac{x}{\dot{x}} = \frac{M}{R}. \tag{9}$$

Of these quantities, $x$ is found from the throw of the galvanometer due to the induction current. See Art. 748. The permanent current, $\dot{x}$, is found from the permanent deflexion due to the steady current; see Art. 746. $M$ is found either by direct calculation from the geometrical data, or by a comparison with a pair of coils, for which this calculation has been made; see Art. 755. From these three quantities $R$ can be determined in electromagnetic measure.

These methods involve the determination of the period of vibration of the galvanometer magnet, and of the logarithmic decrement of its oscillations.

### *Weber's Method by Transient Currents*.*

760.] A coil of considerable size is mounted on an axle, so as to be capable of revolving about a vertical diameter. The wire of this coil is connected with that of a tangent galvanometer so as to form a single circuit. Let the resistance of this circuit be $R$. Let the large coil be placed with its positive face perpendicular to the magnetic meridian, and let it be quickly turned round half a revolution. There will be an induced current due to the earth's magnetic force, and the total quantity of electricity in this current in electromagnetic measure will be

$$Q = \frac{2g_1 H}{R}, \tag{1}$$

where $g_1$ is the magnetic moment of the coil for unit current, which in the case of a large coil may be determined directly, by measuring the dimensions of the coil, and calculating the sum of the areas of its windings. $H$ is the horizontal component of terrestrial magnetism, and $R$ is the resistance of the circuit formed by the coil and galvanometer together. This current sets the magnet of the galvanometer in motion.

If the magnet is originally at rest, and if the motion of the coil occupies but a small fraction of the time of a vibration of

* *Elekt. Maasb.*; or Pogg., *Ann.*, lxxxii. pp. 337–369 (1851).

the magnet, then, if we neglect the resistance to the motion of the magnet, we have, by Art. 748,

$$Q = \frac{H}{G}\frac{T}{\pi} 2 \sin \tfrac{1}{2}\theta, \tag{2}$$

where $G$ is the constant of the galvanometer, $T$ is the time of vibration of the magnet, and $\theta$ is the observed elongation. From these equations we obtain

$$R = \pi G g_1 \frac{1}{T \sin \frac{1}{2}\theta}. \tag{3}$$

The value of $H$ does not appear in this result, provided it is the same at the position of the coil and at that of the galvanometer. This should not be assumed to be the case, but should be tested by comparing the time of vibration of the same magnet, first at one of these places and then at the other.

761.] To make a series of observations Weber began with the coil parallel to the magnetic meridian. He then turned it with its positive face north, and observed the first elongation due to the negative current. He then observed the second elongation of the freely swinging magnet, and on the return of the magnet through the point of equilibrium he turned the coil with its positive face south. This caused the magnet to recoil to the positive side. The series was continued as in Art. 750, and the result corrected for resistance. In this way the value of the resistance of the combined circuit of the coil and galvanometer was ascertained.

In all such experiments it is necessary, in order to obtain sufficiently large deflexions, to make the wire of copper, a metal which, though it is the best conductor, has the disadvantage of altering considerably in resistance with alterations of temperature. It is also very difficult to ascertain the temperature of every part of the apparatus. Hence, in order to obtain a result of permanent value from such an experiment, the resistance of the experimental circuit should be compared with that of a carefully constructed resistance-coil, both before and after each experiment.

*Weber's Method by observing the Decrement of the Oscillations of a Magnet.*

762.] A magnet of considerable magnetic moment is suspended at the centre of a galvanometer coil. The period of vibration and the logarithmic decrement of the oscillations is observed, first with the circuit of the galvanometer open, and then with the circuit closed, and the conductivity of the galvanometer coil is deduced from the effect which the currents induced in it by the motion of the magnet have in resisting that motion.

If $T$ is the observed time of a single vibration, and $\lambda$ the Napierian logarithmic decrement for each single vibration, then, if we write

$$\omega = \frac{\pi}{T}, \tag{1}$$

and

$$a = \frac{\lambda}{T}, \tag{2}$$

the equation of motion of the magnet is of the form

$$\phi = Ce^{-at}\cos(\omega t + \beta). \tag{3}$$

This expresses the nature of the motion as determined by observation. We must compare this with the dynamical equation of motion.

Let $M$ be the coefficient of induction between the galvanometer coil and the suspended magnet. It is of the form

$$M = G_1 g_1 P_1(\theta) + G_2 g_2 P_2(\theta) + \&\text{c.}, \tag{4}$$

where $G_1$, $G_2$, &c. are coefficients belonging to the coil, $g_1$, $g_2$, &c. to the magnet, and $P_1(\theta)$, $P_2(\theta)$, &c. are zonal harmonics of the angle between the axes of the coil and the magnet. See Art. 700. By a proper arrangement of the coils of the galvanometer, and by building up the suspended magnet of several magnets placed side by side at proper distances, we may cause all the terms of $M$ after the first to become insensible compared with the first. If we also put $\phi = \frac{\pi}{2} - \theta$, we may write

$$M = Gm\sin\phi, \tag{5}$$

where $G$ $\{= G_1\}$ is the principal coefficient of the galvanometer, $m$ is the magnetic moment of the magnet, and $\phi$ is the angle between the axis of the magnet and the plane of the coil, which, in this experiment, is always a small angle.

If $L$ is the coefficient of self-induction of the coil, and $R$ its resistance, and $\gamma$ the current in the coil,

$$\frac{d}{dt}(L\gamma + M) + R\gamma = 0, \tag{6}$$

or

$$L\frac{d\gamma}{dt} + R\gamma + Gm\cos\phi\frac{d\phi}{dt} = 0. \tag{7}$$

The moment of the force with which the current $\gamma$ acts on the magnet is $\gamma\dfrac{dM}{d\phi}$, or $Gm\gamma\cos\phi$. The angle $\phi$ is in this experiment so small, that we may suppose $\cos\phi = 1$.

Let us suppose that the equation of motion of the magnet when the circuit is broken is

$$A\frac{d^2\phi}{dt^2} + B\frac{d\phi}{dt} + C\phi = 0, \tag{8}$$

where $A$ is the moment of inertia of the suspended apparatus, $B\dfrac{d\phi}{dt}$ expresses the resistance arising from the viscosity of the air and of the suspension fibre, &c., and $C\phi$ expresses the moment of the force arising from the earth's magnetism, the torsion of the suspension apparatus, &c. tending to bring the magnet to its position of equilibrium.

The equation of motion, as affected by the current, will be

$$A\frac{d^2\phi}{dt^2} + B\frac{d\phi}{dt} + C\phi = Gm\gamma. \tag{9}$$

To determine the motion of the magnet, we have to combine this equation with (7) and eliminate $\gamma$. The result is

$$\left(L\frac{d}{dt} + R\right)\left(A\frac{d^2}{dt^2} + B\frac{d}{dt} + C\right)\phi + G^2m^2\frac{d\phi}{dt} = 0, \tag{10}$$

a linear differential equation of the third order.

We have no occasion, however, to solve this equation, because the data of the problem are the observed elements of the motion of the magnet, and from these we have to determine the value of $R$.

Let $a_0$ and $\omega_0$ be the values of $a$ and $\omega$ in equation (3) when the circuit is broken. In this case $R$ is infinite, and the equation (10) is reduced to the form (8). We thus find

$$B = 2Aa_0, \qquad C = A(a_0^2 + \omega_0^2). \tag{11}$$

Solving equation (10) for $R$, and writing

$$\frac{d}{dt} = -(a + i\omega), \quad \text{where } i = \sqrt{-1}, \tag{12}$$

we find

$$R = \frac{G^2m^2}{A}\,\frac{a + i\omega}{a^2 - \omega^2 + 2ia\omega - 2a_0(a + i\omega) + a_0^2 + \omega_0^2} + L(a + i\omega). \tag{13}$$

Since the value of $\omega$ is in general much greater than that of $a$, the best value of $R$ is found by equating the terms in $i\omega$,

$$R = \frac{G^2 m^2}{2A(a-a_0)} + \tfrac{1}{2}L\left(3a - a_0 - \frac{\omega^2-\omega_0^2}{a-a_0}\right). \tag{14}$$

We may also obtain a value of $R$ by equating the terms not involving $i$, but as these terms are small, the equation is useful only as a means of testing the accuracy of the observations. From these equations we find the following testing equation,

$$G^2 m^2 \{a^2+\omega^2-a_0^2-\omega_0^2\}$$
$$= LA\{(a-a_0)^4 + 2(a-a_0)^2(\omega^2+\omega_0^2) + (\omega^2-\omega_0^2)^2\}. \tag{15}$$

Since $LA\omega^2$ is very small compared with $G^2m^2$, this equation gives

$$\omega^2 - \omega_0^2 = a_0^2 - a^2; \tag{16}$$

and equation (14) may be written

$$R = \frac{G^2 m^2}{2A(a-a_0)} + 2La. \tag{17}$$

In this expression $G$ may be determined either from the linear measurement of the galvanometer coil, or better, by comparison with a standard coil, according to the method of Art. 753. $A$ is the moment of inertia of the magnet and its suspended apparatus, which is to be found by the proper dynamical method. $\omega$, $\omega_0$, $a$ and $a_0$, are given by observation.

The determination of the value of $m$, the magnetic moment of the suspended magnet, is the most difficult part of the investigation, because it is affected by temperature, by the earth's magnetic force, and by mechanical violence, so that great care must be taken to measure this quantity when the magnet is in the very same circumstances as when it is vibrating.

The second term of $R$, that which involves $L$, is of less importance, as it is generally small compared with the first term. The value of $L$ may be determined either by calculation from the known form of the coil, or by an experiment on the extra-current of induction. See Art. 756.

*Thomson's Method by a Revolving Coil.*

763.] This method was suggested by Thomson to the Committee of the British Association on Electrical Standards, and the experiment was made by MM. Balfour Stewart, Fleeming Jenkin, and the author in 1863 *.

* See *Report of the British Association for* 1863, pp. 111–176.

A circular coil is made to revolve with uniform velocity about a vertical axis. A small magnet is suspended by a silk fibre at the centre of the coil. An electric current is induced in the coil by the earth's magnetism, and also by the suspended magnet. This current is periodic, flowing in opposite directions through the wire of the coil during different parts of each revolution, but the effect of the current on the suspended magnet is to produce a deflexion from the magnetic meridian in the direction of the rotation of the coil.

764.] Let $H$ be the horizontal component of the earth's magnetism.

Let $\gamma$ be the strength of the current in the coil.

$g$ the total area inclosed by all the windings of the wire.

$G$ the magnetic force at the centre of the coil due to unit-current.

$L$ the coefficient of self-induction of the coil.

$M$ the magnetic moment of the suspended magnet.

$\theta$ the angle between the plane of the coil and the magnetic meridian.

$\phi$ the angle between the axis of the suspended magnet and the magnetic meridian.

$A$ the moment of inertia of the suspended magnet.

$MH\tau$ the coefficient of torsion of the suspension fibre.

$a$ the azimuth of the magnet when there is no torsion.

$R$ the resistance of the coil.

The kinetic energy of the system is

$$T = \tfrac{1}{2}L\gamma^2 - Hg\gamma\sin\theta - MG\gamma\sin(\theta-\phi) + MH\cos\phi + \tfrac{1}{2}A\dot{\phi}^2. \quad (1)$$

The first term, $\frac{1}{2}L\gamma^2$, expresses the energy of the current as depending on the coil itself. The second term depends on the mutual action of the current and terrestrial magnetism, the third on that of the current and the magnetism of the suspended magnet, the fourth on that of the magnetism of the suspended magnet and terrestrial magnetism, and the last expresses the kinetic energy of the matter composing the magnet and the suspended apparatus which moves with it.

The {variable part of the} potential energy of the suspended apparatus arising from the torsion of the fibre is

$$V = \frac{MH}{2}\tau(\phi^2 - 2\phi a). \quad (2)$$

The electromagnetic momentum of the current is

$$p = \frac{dT}{d\gamma} = L\gamma - Hg\sin\theta - MG\sin(\theta-\phi), \tag{3}$$

and if $R$ is the resistance of the coil, the equation of the current is

$$R\gamma + \frac{d^2T}{dt\,d\gamma} = 0, \tag{4}$$

or, since

$$\theta = \omega t, \tag{5}$$

$$\left(R + L\frac{d}{dt}\right)\gamma = Hg\omega\cos\theta + MG(\omega - \dot{\phi})\cos(\theta-\phi). \tag{6}$$

765.] It is the result alike of theory and observation that $\phi$, the azimuth of the magnet, is subject to two kinds of periodic variations. One of these is a free oscillation, whose periodic time depends on the intensity of terrestrial magnetism, and is, in the experiment, several seconds. The other is a forced vibration whose period is half that of the revolving coil, and whose amplitude is, as we shall see, insensible. Hence, in determining $\gamma$, we may treat $\phi$ as sensibly constant.

We thus find

$$\gamma = \frac{Hg\omega}{R^2 + L^2\omega^2}(R\cos\theta + L\omega\sin\theta) \tag{7}$$

$$+ \frac{MG\omega}{R^2 + L^2\omega^2}\{R\cos(\theta-\phi) + L\omega\sin(\theta-\phi)\}, \tag{8}$$

$$+ Ce^{-\frac{R}{L}t}. \tag{9}$$

The last term of this expression soon dies away when the rotation is continued uniform.

The equation of motion of the suspended magnet is

$$\frac{d^2T}{dt\,d\dot{\phi}} - \frac{dT}{d\phi} + \frac{dV}{d\phi} = 0, \tag{10}$$

whence $\quad A\ddot{\phi} - MG\gamma\cos(\theta-\phi) + MH(\sin\phi + \tau(\phi - a)) = 0. \quad (11)$

Substituting the value of $\gamma$, and arranging the terms according to the functions of multiples of $\theta$, then we know from observation that

$$\phi = \phi_0 + be^{-lt}\cos nt + c\cos 2(\theta - \beta), \tag{12}$$

where $\phi_0$ is the mean value of $\phi$, and the second term expresses the free vibrations gradually decaying, and the third the forced vibrations arising from the variation of the deflecting current.

Beginning with the terms in (11) which do not involve $\theta$, and which must collectively vanish, we find approximately

$$\frac{MG\omega}{R^2+L^2\omega^2}\left\{Hg\,(R\cos\phi_0+L\omega\sin\phi_0)+GMR\right\}$$
$$= 2\,MH\,(\sin\phi_0+\tau(\phi_0-a)). \quad (13)$$

Since $L\tan\phi_0$ is generally small compared with $Gg$, {and $GM\sec\phi$ with $gH$,} the solution of the quadratic (13) gives approximately

$$R = \frac{Gg\omega}{2\tan\phi_0\left(1+\tau\dfrac{\phi_0-a}{\sin\phi_0}\right)}\left\{1+\frac{GM}{gH}\sec\phi_0-\frac{2L}{Gg}\left(\frac{2L}{Gg}-1\right)\tan^2\phi_0\right.$$
$$\left.-\left(\frac{2L}{Gg}\right)^2\left(\frac{2L}{Gg}-1\right)^2\tan^4\phi_0\right\}. \quad (14)$$

If we now employ the leading term in this expression in equations (7), (8), and (11)*, we shall find that the value of $n$ in equation (12) is $\sqrt{\dfrac{HM}{A}\sec\phi_0}$. That of $c$, the amplitude of the forced vibrations, is $\frac{1}{4}\dfrac{n^2}{\omega^2}\sin\phi_0$. Hence, when the coil makes many revolutions during one free vibration of the magnet, the amplitude of the forced vibrations of the magnet is very small, and we may neglect the terms in (11) which involve $c$.

766.] The resistance is thus determined in electromagnetic measure in terms of the velocity $\omega$ and the deviation $\phi$. It is not necessary to determine $H$, the horizontal terrestrial magnetic force, provided it remains constant during the experiment.

To determine $\dfrac{M}{H}$ we must make use of the suspended magnet to deflect the magnet of the magnetometer, as described in Art. 454. In this experiment $M$ should be small, so that this correction becomes of secondary importance.

For the other corrections required in this experiment see the *Report of the British Association for* 1863, p. 168.

### *Joule's Calorimetric Method.*

767.] The heat generated by a current $\gamma$ in passing through a conductor whose resistance is $R$ is, by Joule's law, Art. 242,

$$h = \frac{1}{J}\int R\gamma^2\,dt, \quad (1)$$

* {It is shorter and as accurate to put $L=0$ in equation (6) and substitute the corresponding value of $\gamma$ in (11).}

where $J$ is the equivalent in dynamical measure of the unit of heat employed.

Hence, if $R$ is constant during the experiment, its value is

$$R = \frac{Jh}{\int \gamma^2 dt}. \tag{2}$$

This method of determining $R$ involves the determination of $h$, the heat generated by the current in a given time, and of $\gamma^2$, the square of the strength of the current.

In Joule's experiments*, $h$ was determined by the rise of temperature of the water in a vessel in which the conducting wire was immersed. It was corrected for the effects of radiation, &c. by alternate experiments in which no current was passed through the wire.

The strength of the current was measured by means of a tangent galvanometer. This method involves the determination of the intensity of terrestrial magnetism, which was done by the method described in Art. 457. These measurements were also tested by the current weigher, described in Art. 726, which measures $\gamma^2$ directly. The most direct method of measuring $\int \gamma^2 dt$, however, is to pass the current through a self-acting electrodynamometer (Art. 725) with a scale which gives readings proportional to $\gamma^2$, and to make the observations at equal intervals of time, which may be done approximately by taking the reading at the extremities of every vibration of the instrument during the whole course of the experiment †.

* *Report on Standards of Electrical Resistance of the British Association for* 1867, pp. 474–522.

† {For the relative merits of the various methods of finding the absolute measure of a resistance the reader is referred to a paper by Lord Rayleigh, *Phil. Mag.* Nov. 1882. An excellent method not given in the text, due to Lorentz, is fully described by Lord Rayleigh and Mrs. Sidgwick in the *Phil. Trans.* 1883, Part I, pp. 295–322. The reader should also consult the paper by the same authors entitled 'Experiments to determine the value of the British Association Unit of Resistance in Absolute Measure,' *Phil. Trans.* 1882, Part II, pp. 661–697.}

# CHAPTER XIX.

## COMPARISON OF THE ELECTROSTATIC WITH THE ELECTROMAGNETIC UNITS.

### *Determination of the Number of Electrostatic Units of Electricity in one Electromagnetic Unit.*

768.] THE absolute magnitudes of the electrical units in both systems depend on the units of length, time, and mass which we adopt, and the mode in which they depend on these units is different in the two systems, so that the ratio of the electrical units will be expressed by a different number, according to the different units of length and time.

It appears from the table of dimensions, Art. 628, that the number of electrostatic units of electricity in one electromagnetic unit varies inversely as the magnitude of the unit of length, and directly as the magnitude of the unit of time which we adopt.

If, therefore, we determine a velocity which is represented numerically by this number, then, even if we adopt new units of length and of time, the number representing this velocity will still be the number of electrostatic units of electricity in one electromagnetic unit, according to the new system of measurement.

This velocity, therefore, which indicates the relation between electrostatic and electromagnetic phenomena, is a natural quantity of definite magnitude, and the measurement of this quantity is one of the most important researches in electricity.

To shew that the quantity we are in search of is really a velocity, we may observe that in the case of two parallel currents the attraction experienced by a length $a$ of one of them is, by Art. 686,

$$F = 2CC'\frac{a}{b},$$

where $C$, $C'$ are the numerical values of the currents in electromagnetic measure, and $b$ the distance between them. If we make $b = 2a$, then

$$F = CC'.$$

Now the quantity of electricity transmitted by the current $C$ in the time $t$ is $Ct$ in electromagnetic measure, or $nCt$ in electrostatic measure, if $n$ is the number of electrostatic units in one electromagnetic unit.

Let two small conductors be charged with the quantities of electricity transmitted by the two currents in the time $t$, and placed at a distance $r$ from each other. The repulsion between them will be

$$F' = \frac{CC'n^2t^2}{r^2}.$$

Let the distance $r$ be so chosen that this repulsion is equal to the attraction of the currents, then

$$\frac{CC'n^2t^2}{r^2} = CC'.$$

Hence

$$r = nt;$$

or the distance $r$ must increase with the time $t$ at the rate $n$. Hence $n$ is a velocity, the absolute magnitude of which is the same, whatever units we assume.

769.] To obtain a physical conception of this velocity, let us imagine a plane surface charged with electricity to the electrostatic surface-density $\sigma$, and moving in its own plane with a velocity $v$. This moving electrified surface will be equivalent to an electric current-sheet, the strength of the current flowing through unit of breadth of the surface being $\sigma v$ in electrostatic measure, or $\frac{1}{n}\sigma v$ in electromagnetic measure, if $n$ is the number of electrostatic units in one electromagnetic unit. If another plane surface parallel to the first is electrified to the surface-density $\sigma'$, and moves in the same direction with the velocity $v'$, it will be equivalent to a second current-sheet.

The electrostatic repulsion between the two electrified surfaces is, by Art. 124, $2\pi\sigma\sigma'$ for every unit of area of the opposed surfaces.

The electromagnetic attraction between the two current-sheets is, by Art. 653, $2\pi uu'$ for every unit of area, $u$ and $u'$ being the surface-densities of the currents in electromagnetic measure.

But $u = \frac{1}{n}\sigma v$, and $u' = \frac{1}{n}\sigma' v'$, so that the attraction is

$$2\pi\sigma\sigma'\frac{vv'}{n^2}.$$

The ratio of the attraction to the repulsion is equal to that of $vv'$ to $n^2$. Hence, since the attraction and the repulsion are quantities of the same kind, $n$ must be a quantity of the same kind as $v$, that is, a velocity. If we now suppose the velocity of each of the moving planes to be *equal* to $n$, the attraction will be equal to the repulsion, and there will be no mechanical action between them. Hence we may define the ratio of the electric units to be a velocity, such that two electrified surfaces, moving in the same direction with this velocity, have no mutual action. Since this velocity is about 300000 kilometres per second, it is impossible to make the experiment above described.

770.] If the electric surface-density and the velocity can be made so great that the magnetic force is a measurable quantity, we may at least verify our supposition that a moving electrified body is equivalent to an electric current.

We may assume* that an electrified surface in air would begin to discharge itself by sparks when the electric force $2\pi\sigma$ reaches the value 130. The magnetic force due to the current-sheet is $2\pi\sigma\frac{v}{n}$. The horizontal magnetic force in Britain is about 0·175. Hence a surface electrified to the highest degree, and moving with a velocity of 100 metres per second, would act on a magnet with a force equal to about one-four-thousandth part of the earth's horizontal force, a quantity which can be measured. The electrified surface may be that of a non-conducting disk revolving in the plane of the magnetic meridian, and the magnet may be placed close to the ascending or descending portion of the disk, and protected from its electrostatic action by a screen of metal. I am not aware that this experiment has been hitherto attempted †.

* Sir W. Thomson, *R. S. Proc.* or Reprint, Art. xix. pp. 247–259.

† {This effect was discovered by Prof. Rowland in 1876. For subsequent experiments on this subject see Rowland and Hutchinson, *Phil. Mag.* 27. 445 (1887); Röntgen, *Wied. Ann.* 40. 93; Himstedt, *Wied. Ann.* 40. 720.}

## I. *Comparison of Units of Electricity.*

771.] Since the ratio of the electromagnetic to the electrostatic unit of electricity is represented by a velocity, we shall in future denote it by the symbol $v$. The first numerical determination of this velocity was made by Weber and Kohlrausch *.

Their method was founded on the measurement of the same quantity of electricity, first in electrostatic and then in electromagnetic measure.

The quantity of electricity measured was the charge of a Leyden jar. It was measured in electrostatic measure as the product of the capacity of the jar into the difference of potential of its coatings. The capacity of the jar was determined by comparison with that of a sphere suspended in an open space at a distance from other bodies. The capacity of such a sphere is expressed in electrostatic measure by its radius. Thus the capacity of the jar may be found and expressed as a certain length. See Art. 227.

The difference of the potentials of the coatings of the jar was measured by connecting the coatings with the electrodes of an electrometer, the constants of which were carefully determined, so that the difference of the potentials, $E$, became known in electrostatic measure.

By multiplying this by $c$, the capacity of the jar, the charge of the jar was expressed in electrostatic measure.

To determine the value of the charge in electromagnetic measure, the jar was discharged through the coil of a galvanometer. The effect of the transient current on the magnet of the galvanometer communicated to the magnet a certain angular velocity. The magnet then swung round to a certain deviation, at which its velocity was entirely destroyed by the opposing action of the earth's magnetism.

By observing the extreme deviation of the magnet the quantity of electricity in the discharge may be determined in electromagnetic measure, as in Art. 748, by the formula

$$Q = \frac{H}{G}\frac{T}{\pi}\, 2 \sin \tfrac{1}{2}\theta,$$

* *Elektrodynamische Maasbestimmungen;* and Pogg., *Ann.*, xcix (Aug. pp. 10–25, 1856).

where $Q$ is the quantity of electricity in electromagnetic measure. We have therefore to determine the following quantities:—

$H$, the intensity of the horizontal component of terrestrial magnetism; see Art. 456.

$G$, the principal constant of the galvanometer; see Art. 700.

$T$, the time of a single vibration of the magnet; and

$\theta$, the deviation due to the transient current.

The value of $v$ obtained by MM. Weber and Kohlrausch was

$$v = 310740000 \text{ metres per second.}$$

The property of solid dielectrics, to which the name of Electric Absorption has been given, renders it difficult to estimate correctly the capacity of a Leyden jar. The apparent capacity varies according to the time which elapses between the charging or discharging of the jar and the measurement of the potential, and the longer the time the greater is the value obtained for the capacity of the jar.

Hence, since the time occupied in obtaining a reading of the electrometer is large in comparison with the time during which the discharge through the galvanometer takes place, it is probable that the estimate of the discharge in electrostatic measure is too high, and the value of $v$, derived from it, is probably also too high.

## II. '$v$' *expressed as a Resistance.*

772.] Two other methods for the determination of $v$ lead to an expression of its value in terms of the resistance of a given conductor, which, in the electromagnetic system, is also expressed as a velocity.

In Sir William Thomson's form of the experiment, a constant current is made to flow through a wire of great resistance. The electromotive force which urges the current through the wire is measured electrostatically by connecting the extremities of the wire with the electrodes of an absolute electrometer, Arts. 217, 218. The strength of the current in the wire is measured in electromagnetic measure by the deflexion of the suspended coil of an electrodynamometer through which it passes, Art. 725. The resistance of the circuit is known in electromagnetic measure by comparison with a standard coil or Ohm. By multiplying the strength of the current by this resistance we obtain the electromotive force in electromagnetic measure, and from a comparison of this with the electrostatic measure the value of $v$ is obtained.

This method requires the simultaneous determination of two forces, by means of the electrometer and electrodynamometer respectively, and it is only the ratio of these forces which appears in the result.

773.] Another method, in which these forces, instead of being separately measured, are directly opposed to each other, was employed by the present writer. The ends of the great resistance coil are connected with two parallel disks, one of which is moveable. The same difference of potentials which sends the current through the great resistance, also causes an attraction between these disks. At the same time, an electric current which, in the actual experiment, was distinct from the primary current, is sent through two coils, fastened, one to the back of the fixed disk, and the other to the back of the moveable disk. The current flows in opposite directions through these coils, so that they repel one another. By adjusting the distance of the two disks the attraction is exactly balanced by the repulsion, while at the same time another observer, by means of a differential galvanometer with shunts, determines the ratio of the primary to the secondary current.

In this experiment the only measurement which must be referred to a material standard is that of the great resistance, which must be determined in absolute measure by comparison with the Ohm. The other measurements are required only for the determination of ratios, and may therefore be determined in terms of any arbitrary unit.

Thus the ratio of the two forces is a ratio of equality.

The ratio of the two currents is found by a comparison of resistances when there is no deflexion of the differential galvanometer.

The attractive force depends on the square of the ratio of the diameter of the disks to their distance.

The repulsive force depends on the ratio of the diameter of the coils to their distance.

The value of $v$ is therefore expressed directly in terms of the resistance of the great coil, which is itself compared with the Ohm.

The value of $v$, as found by Thomson's method, was 28·2 Ohms*; by Maxwell's, 28·8 Ohms †.

III. *Electrostatic Capacity in Electromagnetic Measure.*

774.] The capacity of a condenser may be ascertained in electromagnetic measure by a comparison of the electromotive force which produces the charge, and the quantity of electricity in the current of discharge. By means of a voltaic battery a current is maintained through a circuit containing a coil of great resistance. The condenser is charged by putting its electrodes in contact with those of the resistance coil. The current through the coil is measured by the deflexion which it produces in a galvanometer. Let $\phi$ be this deflexion, then the current is, by Art. 742,

$$\gamma = \frac{H}{G} \tan \phi,$$

where $H$ is the horizontal component of terrestrial magnetism, and $G$ is the principal constant of the galvanometer.

If $R$ is the resistance of the coil through which this current is made to flow, the difference of the potentials at the ends of the coil is

$$E = R\gamma,$$

and the charge of electricity produced in the condenser, whose capacity in electromagnetic measure is $C$, will be

$$Q = EC.$$

Now let the electrodes of the condenser, and then those of the galvanometer, be disconnected from the circuit, and let the magnet of the galvanometer be brought to rest at its position of equilibrium. Then let the electrodes of the condenser be connected with those of the galvanometer. A transient current will flow through the galvanometer, and will cause the magnet to swing to an extreme deflexion $\theta$. Then, by Art. 748, if the discharge is equal to the charge,

$$Q = \frac{H}{G}\frac{T}{\pi} 2 \sin \tfrac{1}{2} \theta.$$

We thus obtain as the value of the capacity of the condenser in electromagnetic measure

$$C = \frac{T}{\pi}\frac{1}{R}\frac{2 \sin \frac{1}{2} \theta}{\tan \phi}.$$

* *Report of British Association*, 1869, p. 434.

† *Phil. Trans.*, 1868. p. 643; and *Report of British Association*, 1869, p. 436.

The capacity of the condenser is thus determined in terms of the following quantities:—

$T$, the time of vibration of the magnet of the galvanometer from rest to rest.

$R$, the resistance of the coil.

$\theta$, the extreme limit of the swing produced by the discharge.

$\phi$, the constant deflexion due to the current through the coil $R$.

This method was employed by Professor Fleeming Jenkin in determining the capacity of condensers in electromagnetic measure *.

If $c$ be the capacity of the same condenser in electrostatic measure, as determined by comparison with a condenser whose capacity can be calculated from its geometrical data,

$$c = v^2 C.$$

Hence $$v^2 = \pi R \frac{c}{T} \frac{\tan \phi}{2 \sin \frac{1}{2} \theta}.$$

The quantity $v$ may therefore be found in this way. It depends on the determination of $R$ in electromagnetic measure, but as it involves only the square root of $R$, an error in this determination will not affect the value of $v$ so much as in the methods of Arts. 772, 773.

*Intermittent Current.*

775.] If the wire of a battery-circuit be broken at any point, and the broken ends connected with the electrodes of a condenser, the current will flow into the condenser with a strength which diminishes as the difference of the potentials of the plates of the condenser increases, so that when the condenser has received the full charge corresponding to the electromotive force acting on the wire the current ceases entirely.

If the electrodes of the condenser are now disconnected from the ends of the wire, and then again connected with them in the reverse order, the condenser will discharge itself through the wire, and will then become recharged in the opposite way, so that a transient current will flow through the wire, the total quantity of which is equal to two charges of the condenser.

By means of a piece of mechanism (commonly called a Commutator, or *wippe*) the operation of reversing the connexions of the condenser can be repeated at regular intervals of time, each interval being equal to $T$. If this interval is sufficiently long to

* *Report of British Association*, 1867, pp. 483–488.

allow of the complete discharge of the condenser, the quantity of electricity transmitted by the wire in each interval will be $2EC$, where $E$ is the electromotive force, and $C$ is the capacity of the condenser.

If the magnet of a galvanometer included in the circuit is loaded, so as to swing so slowly that a great many discharges of the condenser occur in the time of one free vibration of the magnet, the succession of discharges will act on the magnet like a steady current whose strength is $\frac{2EC}{T}$.

If the condenser is now removed, and a resistance coil substituted for it, and adjusted till the steady current through the galvanometer produces the same deflexion as the succession of discharges, and if $R$ is the resistance of the whole circuit when this is the case,

$$\frac{E}{R} = \frac{2EC}{T}; \tag{1}$$

or

$$R = \frac{T}{2C}. \tag{2}$$

We may thus compare the condenser with its commutator in motion to a wire of a certain electrical resistance, and we may make use of the different methods of measuring resistance described in Arts. 345 to 357 in order to determine this resistance.

776.] For this purpose we may substitute for any one of the wires in the method of the Differential Galvanometer, Art. 346, or in that of Wheatstone's Bridge, Art. 347, a condenser with its commutator. Let us suppose that in either case a zero deflexion of the galvanometer has been obtained, first with the condenser and commutator, and then with a coil of resistance $R_1$ in its place, then the quantity $\frac{T}{2C}$ will be measured by the resistance of the circuit of which the coil $R_1$ forms part, and which is completed by the remainder of the conducting system including the battery. Hence the resistance, $R$, which we have to calculate, is equal to $R_1$, that of the resistance coil, together with $R_2$; the resistance of the remainder of the system (including the battery), the extremities of the resistance coil being taken as the electrodes of the system.

In the cases of the differential galvanometer and Wheatstone's Bridge it is not necessary to make a second experiment by substituting a resistance coil for the condenser. The value of the resistance required for this purpose may be found by calculation from the other known resistances in the system.

Using the notation of Art. 347, and supposing the condenser and commutator substituted for the conductor $AC$ in Wheatstone's Bridge, and the galvanometer inserted in $OA$, and that the deflexion of the galvanometer is zero, then we know that the resistance of a coil, which placed in $AC$ would give a zero deflexion, is

$$b = \frac{c\gamma}{\beta} = R_1. \tag{3}$$

The other part of the resistance, $R_2$, is that of the system of conductors $AO$, $OC$, $AB$, $BC$ and $OB$, the points $A$ and $C$ being considered as the electrodes. Hence

$$R_2 = \frac{\beta(c+a)(\gamma+a)+ca(\gamma+a)+\gamma a(c+a)}{(c+a)(\gamma+a)+\beta(c+a+\gamma+a)}. \tag{4}$$

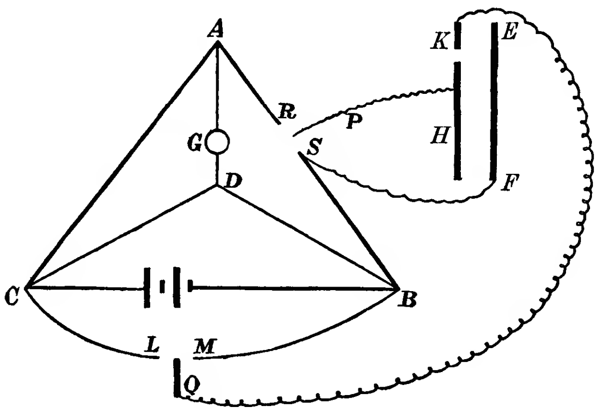
In this expression $a$ denotes the internal resistance of the battery and its connexions, the value of which cannot be determined with certainty; but by making it small compared with the other resistances, this uncertainty will only slightly affect the value of $R_2$.

The value of the capacity of the condenser in electromagnetic measure is

$$C = \frac{T}{2(R_1+R_2)}.* \tag{5}$$

* {As this method is of great importance in measuring the capacity of a condenser in electromagnetic measure, we subjoin a somewhat fuller investigation of it, adapted to the case when the cylinder has a guard-ring.

The arrangement employed in this measurement is represented in the annexed figure.



777.] If the condenser has a large capacity, and the commutator is very rapid in its action, the condenser may not be fully

*ABCD* is a Wheatstone's Bridge with the galvanometer at *G*, and the battery between *B* and *C*. The arm *AB* is broken at *R* and *S*, which are two poles of a commutator, which alternately come into contact with a spring *P*, connected with the middle-plate, *H*, of the condenser. The plate without the guard-ring is connected to *S*. The points *C* and *B* are connected respectively with *L* and *M*, the two poles of a commutator, which alternately come into contact with a spring *Q*, attached to the guard-ring of the condenser. The system is arranged so that when the commutators are working the order of events is as follows :

I. *P* on *S*. Condenser discharged.
*Q* on *M*. Guard-ring discharged.
II. *P* on *R*. Condenser begins to charge.
*Q* on *M*.
III. *P* on *R*. Condenser completely charged to potential $(A)-(B)$.
*Q* on *L*. Guard-ring charged to potential $(C)-(B)$.
IV. *P* on *S*. Condenser begins discharging.
*Q* on *L*.
V. *P* on *S*. Condenser discharged.
*Q* on *M*. Guard-ring discharged.

Thus, when the commutators are working, there will, owing to the flow of electricity to the condenser, be a succession of momentary currents through the galvanometer. The resistances are so adjusted that the effect of these momentary currents on the galvanometer just balances the effect due to the steady current, and there is no deflexion of the galvanometer.

To investigate the relation between the resistances when this is the case, let us suppose that when the guard-ring and condenser are charging

$$\dot{x} = \text{current through } BC,$$
$$\dot{y} = \text{current through } AR,$$
$$\dot{z} = \text{current through } AD,$$
$$\dot{w} = \text{current through } CL.$$

Thus, if $a$, $b$, $\alpha$, $\beta$, $\gamma$ are the resistances in the arms $BC$, $AC$, $AD$, $BD$, $CD$ respectively, $L$ the coefficient of self induction of the galvanometer, and $E$ the electromotive force of the battery, we have from circuits $ADC$ and $BCD$ respectively,

$$L\ddot{z}+(b+\gamma+\alpha)\dot{z}+(b+\gamma)\dot{y}+\gamma\dot{w}-\gamma\dot{x} = 0, \tag{1}$$
$$(a+\gamma+\beta)\dot{x}-(\gamma+\beta)\dot{y}-\gamma\dot{z}-(\gamma+\beta)\dot{w}-E = 0. \tag{2}$$

Now it is evident that the currents are expressed by equations of the following kind,

$$\dot{x} = \dot{x}_1+\dot{x}_2,$$
$$\dot{z} = \dot{z}_1+\dot{z}_2,$$

where $\dot{x}_1$ and $\dot{z}_1$ express the steady currents when no electricity is flowing into the condenser, and $\dot{x}_2$, $\dot{z}_2$ are of the form $Ae^{-\lambda t}$, $Be^{-\lambda t}$, and express the variable parts of the currents due to the charging of the condenser; $\dot{y}$ and $\dot{w}$ will be of the form $C'e^{-\lambda t}$, $De^{-\lambda t}$; $t$ in all these expressions is the time which has elapsed since the condenser commenced to charge.

Equations (1) and (2) will thus contain constant terms, and terms multiplied by $e^{-\lambda t}$, and the latter must separately vanish, hence we have

$$L\ddot{z}_2+(b+\gamma+\alpha)\dot{z}_2+(b+\gamma)\dot{y}+\gamma\dot{w}-\gamma\dot{x}_2 = 0, \tag{3}$$
$$(a+\gamma+\beta)\,\dot{x}_2-(\gamma+\beta)\,\dot{y}-\gamma\dot{z}_2-(\gamma+\beta)\,\dot{w} = 0. \tag{4}$$

Let $Z$, $X$ be the quantities of electricity which have passed through the galvanometer and battery respectively, in consequence of the charging of the condenser, and $Y$ and $W$ the charges in the condenser and guard-ring. Then integrating equations (3) and (4) over a time extending from just before the condenser began to charge until it is fully charged, remembering that at each of these times $\dot{z}_2 = 0$, we get

$$(b+\gamma+\alpha)\,Z+(b+\gamma)\,Y+\gamma\,W-\gamma X = 0,$$
$$(a+\gamma+\beta)X-(\gamma+\beta)\,Y-\gamma Z-(\gamma+\beta)\,W = 0;$$

discharged at each reversal. The equation of the electric current during the discharge is

$$Q + R_2 C \frac{dQ}{dt} + EC = 0, \tag{6}$$

where $Q$ is the charge, $C$ the capacity of the condenser, $R_2$ the resistance of the rest of the system between the electrodes of the condenser, and $E$ the electromotive force due to the connexion with the battery.

Hence $$Q = (Q_0 + EC)\, e^{-\frac{t}{R_2 C}} - EC, \tag{7}$$

where $Q_0$ is the initial value of $Q$.

hence eliminating $X$,

$$Z\left(b+\gamma+a-\frac{\gamma^2}{a+\gamma+\beta}\right)+Y\left(b+\gamma-\frac{\gamma(\gamma+\beta)}{a+\gamma+\beta}\right)+W\gamma\frac{a}{a+\gamma+\beta}=0.$$

In practice the battery resistance is very small indeed compared with $\beta$, $b$ or $\gamma$, so that the third term may be neglected in comparison with the second, and we get, neglecting the battery resistance,

$$Z = -\frac{b}{b+\gamma+a-\dfrac{\gamma^2}{\gamma+\beta}}\,Y.$$

If $\{A\}$, $\{B\}$, $\{D\}$ denote the potentials of $A$, $B$, $D$ when the condenser is fully charged, $C$ the capacity of the condenser, then

$$Y = C[\{A\} - \{B\}].$$

But

$$\frac{\{A\}-\{B\}}{a+\beta\dfrac{(b+a+\gamma)}{\gamma}} = \frac{\{A\}-\{D\}}{a}.$$

The right-hand side of this equation is evidently $\dot{z}_1$, the steady current through the galvanometer, so that

$$Y = C\dot{z}_1\left(a+\beta\frac{(b+a+\gamma)}{\gamma}\right), \tag{5}$$

$$Z = -\dot{z}_1 b C \frac{\left\{a+\beta\dfrac{(b+a+\gamma)}{\gamma}\right\}}{b+\gamma+a-\dfrac{\gamma^2}{\gamma+\beta}}. \tag{6}$$

If the condenser is charged $n$ times per second, the quantity of electricity which passes in consequence through the galvanometer per second is $nZ$. If the galvanometer needle remains undeflected, the quantity of electricity which passes through the galvanometer in unit time must be zero. But this quantity is $nZ + \dot{z}_1$, so that

$$nZ + \dot{z}_1 = 0.$$

Substituting this relation in equation (6), we get

$$C = \frac{1}{n}\frac{\gamma}{b\beta}\frac{\left\{1-\dfrac{\gamma^2}{(\gamma+\beta)(b+a+\gamma)}\right\}}{1+\dfrac{\gamma a}{(b+a+\gamma)\beta}}. \tag{7}$$

From this equation, if we know the resistances and the speed, we can calculate the capacity. See J. J. Thomson and Searle, "A Determination of '$v$,'" *Phil. Trans.* 1890, A, p. 583.}

If $\tau$ is the time during which contact is maintained during each discharge, the quantity in each discharge is

$$Q = 2EC\frac{1-e^{-\frac{\tau}{R_2C}}}{1+e^{-\frac{\tau}{R_2C}}}. \tag{8}$$

By making $c$ and $\gamma$ in equation (4) large compared with $\beta$, $\alpha$, or $a$, the time represented by $R_2C$ may be made so small compared with $\tau$, that in calculating the value of the exponential expression we may use the value of $C$ in equation (5). We thus find

$$\frac{\tau}{R_2C} = 2\frac{R_1+R_2}{R_2}\frac{\tau}{T}, \tag{9}$$

where $R_1$ is the resistance which must be substituted for the condenser to produce an equivalent effect. $R_2$ is the resistance of the rest of the system, $T$ is the interval between the beginning of a discharge and the beginning of the next discharge, and $\tau$ is the duration of contact for each discharge. We thus obtain for the corrected value of $C$ in electromagnetic measure

$$C = \frac{1}{2}\frac{T}{R_1+R_2}\frac{1+e^{-2\frac{R_1+R_2}{R_2}\frac{\tau}{T}}}{1-e^{-2\frac{R_1+R_2}{R_2}\frac{\tau}{T}}}. \tag{10}$$

IV. *Comparison of the Electrostatic Capacity of a Condenser with the Electromagnetic Capacity of Self-induction of a Coil.*

778.] If two points of a conducting circuit, between which the resistance is $R$, are connected with the electrodes of a condenser whose capacity is $C$, then, when an electromotive force acts on the circuit, part of the current, instead of passing through the resistance $R$, will be employed in charging the condenser. The current through $R$ will therefore rise to its final value from zero in a gradual manner. It appears from the mathematical theory that the manner in which the current through $R$ rises from zero to its final value is expressed by a formula of exactly the same kind as that which expresses the value of a current urged by a constant electromotive force through the coil of an electromagnet. Hence we may place

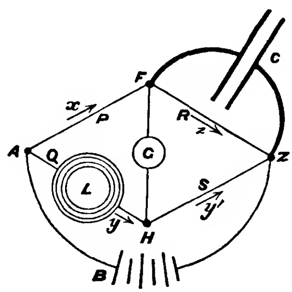



Fig. 65.

a condenser and an electromagnet in two opposite members of Wheatstone's Bridge in such a way that the current through the galvanometer is always zero, even at the instant of making or breaking the battery circuit.

In the figure, let $P$, $Q$, $R$, $S$ be the resistances of the four members of Wheatstone's Bridge respectively. Let a coil, whose coefficient of self-induction is $L$, be made part of the member $AH$, whose resistance is $Q$, and let the electrodes of a condenser, whose capacity is $C$, be connected by pieces of small resistance with the points $F$ and $Z$. For the sake of simplicity, we shall assume that there is no current in the galvanometer $G$, the electrodes of which are connected to $F$ and $H$. We have therefore to determine the condition that the potential at $F$ may be equal to that at $H$. It is only when we wish to estimate the degree of accuracy of the method that we require to calculate the current through the galvanometer when this condition is not fulfilled.

Let $x$ be the total quantity of electricity which has passed through the member $AF$, and $z$ that which has passed through $FZ$ at the time $t$, then $x-z$ will be the charge of the condenser. The electromotive force acting between the electrodes of the condenser is, by Ohm's law, $R\dfrac{dz}{dt}$, so that if the capacity of the condenser is $C$,

$$x-z = RC\frac{dz}{dt}. \tag{1}$$

Let $y$ be the total quantity of electricity which has passed through the member $AH$, the electromotive force from $A$ to $H$ must be equal to that from $A$ to $F$, or

$$Q\frac{dy}{dt}+L\frac{d^2y}{dt^2} = P\frac{dx}{dt}. \tag{2}$$

Since there is no current through the galvanometer, the quantity which has passed through $HZ$ must be also $y$, and we find

$$S\frac{dy}{dt} = R\frac{dz}{dt}. \tag{3}$$

Substituting in (2) the value of $x$, derived from (1), and comparing with (3), we find as the condition of no current through the galvanometer

$$RQ\left(1+\frac{L}{Q}\frac{d}{dt}\right)z = SP\left(1+RC\frac{d}{dt}\right)z. \tag{4}$$

The condition of no final current is, as in the ordinary form of Wheatstone's Bridge,
$$QR = SP. \tag{5}$$

The additional condition of no current at making and breaking the battery connexion is
$$\frac{L}{Q} = RC. \tag{6}$$

Here $\frac{L}{Q}$ and $RC$ are the time-constants of the members $Q$ and $R$ respectively, and if, by varying $Q$ or $R$, we can adjust the members of Wheatstone's Bridge till the galvanometer indicates no current, either at making and breaking the circuit, or when the current is steady, then we know that the time-constant of the coil is equal to that of the condenser.

The coefficient of self-induction, $L$, can be determined in electromagnetic measure from a comparison with the coefficient of mutual induction of two circuits, whose geometrical data are known (Art. 756). It is a quantity of the dimensions of a line.

The capacity of the condenser can be determined in electrostatic measure by comparison with a condenser whose geometrical data are known (Art. 229). This quantity is also a length, $c$. The electromagnetic measure of the capacity is
$$C = \frac{c}{v^2}. \tag{7}$$

Substituting this value in equation (6), we obtain for the value of $v^2$
$$v^2 = \frac{c}{L} QR, \tag{8}$$
where $c$ is the capacity of the condenser in electrostatic measure, $L$ the coefficient of self-induction of the coil in electromagnetic measure, and $Q$ and $R$ the resistances in electromagnetic measure. The value of $v$, as determined by this method, depends on the determination of the unit of resistance, as in the second method, Arts. 772, 773.

V. *Combination of the Electrostatic Capacity of a Condenser with the Electromagnetic Capacity of Self-induction of a Coil.*

779.] Let $C$ be the capacity of the condenser, the surfaces of which are connected by a wire of resistance $R$. In this wire let the coils $L$ and $L'$ be inserted, and let $L$ denote the sum of their capacities of self-induction. The coil $L'$ is hung by a bifilar suspension, and consists of two parallel coils in vertical planes, between which passes a vertical axis which carries the magnet $M$, the axis of which revolves in a horizontal plane between the coils $L'L'$. The coil $L$ has a large coefficient of self-induction, and is fixed. The suspended coil $L'$ is protected from the currents of air caused by the rotation of the magnet by enclosing the rotating parts in a hollow case.

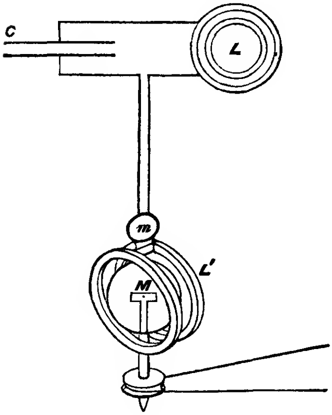


Fig. 66.

The motion of the magnet causes currents of induction in the coil, and these are acted on by the magnet, so that the plane of the suspended coil is deflected in the direction of the rotation of the magnet. Let us determine the strength of the induced currents, and the magnitude of the deflexion of the suspended coil.

Let $x$ be the charge of electricity on the upper surface of the condenser $C$, then, if $E$ is the electromotive force which produces this charge, we have, by the theory of the condenser,

$$x = CE. \tag{1}$$

We have also, by the theory of electric currents,

$$R\dot{x} + \frac{d}{dt}(L\dot{x} + M\cos\theta) + E = 0, \tag{2}$$

where $M$ is the electromagnetic momentum of the circuit $L'$, when the axis of the magnet is normal to the plane of the coil, and $\theta$ is the angle between the axis of the magnet and this normal.

The equation to determine $x$ is therefore

$$CL\frac{d^2x}{dt^2} + CR\frac{dx}{dt} + x = CM\sin\theta\frac{d\theta}{dt}. \tag{3}$$

If the coil is in a position of equilibrium, and if the rotation of the magnet is uniform, the angular velocity being $n$,

$$\theta = nt. \tag{4}$$

The expression for the current consists of two parts, one of which is independent of the term on the right-hand of the equation, and diminishes according to an exponential function of the time. The other, which may be called the forced current, depends entirely on the term in $\theta$, and may be written

$$x = A\sin\theta + B\cos\theta. \tag{5}$$

Finding the values of $A$ and $B$ by substitution in the equation (3), we obtain

$$x = -MCn\frac{RCn\cos\theta - (1-CLn^2)\sin\theta}{R^2C^2n^2 + (1-CLn^2)^2}. \tag{6}$$

The moment of the force with which the magnet acts on the coil $L'$, in which the current $\dot{x}$ is flowing, being the reverse of that acting on the magnet the coil being by supposition fixed, is given by

$$\Theta = -\dot{x}\frac{d}{d\theta}(M\cos\theta) = M\sin\theta\frac{dx}{dt}. \tag{7}$$

Integrating this expression with respect to $t$ for one revolution, and dividing by the time, we find, for the mean value of $\Theta$,

$$\overline{\Theta} = \tfrac{1}{2}\frac{M^2RC^2n^3}{R^2C^2n^2 + (1-CLn^2)^2}. \tag{8}$$

If the coil has a considerable moment of inertia, its forced vibrations will be very small, and its mean deflexion will be proportional to $\overline{\Theta}$.

Let $D_1$, $D_2$, $D_3$ be the observed deflexions corresponding to angular velocities $n_1$, $n_2$, $n_3$ of the magnet, then in general

$$P\frac{n}{D} = \left(\frac{1}{n} - CLn\right)^2 + R^2C^2, \tag{9}$$

where $P$ is a constant.

Eliminating $P$ and $R$ from three equations of this form, we find

$$C^2L^2 = \frac{1}{n_1^2n_2^2n_3^2}\frac{\dfrac{n_1^3}{D_1}(n_2^2-n_3^2) + \dfrac{n_2^3}{D_2}(n_3^2-n_1^2) + \dfrac{n_3^3}{D_3}(n_1^2-n_2^2)}{\dfrac{n_1}{D_1}(n_2^2-n_3^2) + \dfrac{n_2}{D_2}(n_3^2-n_1^2) + \dfrac{n_3}{D_3}(n_1^2-n_2^2)}. \tag{10}$$

If $n_2$ is such that $CLn_2^2 = 1$, the value of $\frac{n}{D}$ will be a minimum for this value of $n$. The other values of $n$ should be taken, one greater, and the other less, than $n_2$.

The value of $CL$, determined from equation (10), is of the dimensions of the square of a time. Let us call it $\tau^2$.

If $C_s$ be the electrostatic measure of the capacity of the condenser, and $L_m$ the electromagnetic measure of the self-induction of the coil, both $C_s$ and $L_m$ are lines, and the product

$$C_s L_m = v^2 C_s L_s = v^2 C_m L_m = v^2 \tau^2; \tag{11}$$

and

$$v^2 = \frac{C_s L_m}{\tau^2}, \tag{12}$$

where $\tau^2$ is the value of $C^2 L^2$, determined by this experiment. The experiment here suggested as a method of determining $v$ is of the same nature as one described by Sir W. R. Grove, *Phil. Mag.*, March 1868, p. 184. See also remarks on that experiment, by the present writer, in the number for May 1868, pp. 360–363.

VI. *Electrostatic Measurement of Resistance.* (See Art. 355.)

780.] Let a condenser of capacity $C$ be discharged through a conductor of resistance $R$, then, if $x$ is the charge at any instant,

$$\frac{x}{C} + R\frac{dx}{dt} = 0. \tag{1}$$

Hence

$$x = x_0 e^{-\frac{t}{RC}}. \tag{2}$$

If, by any method, we can make contact for a short time, which is accurately known, so as to allow the current to flow through the conductor for the time $t$, then, if $E_0$ and $E_1$ are the readings of an electrometer put in connexion with the condenser before and after the operation,

$$RC(\log_e E_0 - \log_e E_1) = t. \tag{3}$$

If $C$ is known in electrostatic measure as a linear quantity, $R$ may be found from this equation in electrostatic measure as the reciprocal of a velocity.

If $R_s$ is the numerical value of the resistance as thus determined, and $R_m$ the numerical value of the resistance in electromagnetic measure,

$$v^2 = \frac{R_m}{R_s}. \tag{4}$$

Since it is necessary for this experiment that $R$ should be very great, and since $R$ must be small in the electromagnetic experiments of Arts. 763, &c., the experiments must be made on separate conductors, and the resistance of these conductors compared by the ordinary methods.

# CHAPTER XX.

## ELECTROMAGNETIC THEORY OF LIGHT.

781.] In several parts of this treatise an attempt has been made to explain electromagnetic phenomena by means of mechanical action transmitted from one body to another by means of a medium occupying the space between them. The undulatory theory of light also assumes the existence of a medium. We have now to shew that the properties of the electromagnetic medium are identical with those of the luminiferous medium.

To fill all space with a new medium whenever any new phenomenon is to be explained is by no means philosophical, but if the study of two different branches of science has independently suggested the idea of a medium, and if the properties which must be attributed to the medium in order to account for electromagnetic phenomena are of the same kind as those which we attribute to the luminiferous medium in order to account for the phenomena of light, the evidence for the physical existence of the medium will be considerably strengthened.

But the properties of bodies are capable of quantitative measurement. We therefore obtain the numerical value of some property of the medium, such as the velocity with which a disturbance is propagated through it, which can be calculated from electromagnetic experiments, and also observed directly in the case of light. If it should be found that the velocity of propagation of electromagnetic disturbances is the same as the velocity of light, and this not only in air, but in other transparent media, we shall have strong reasons for believing that light is an electromagnetic phenomenon, and the combination of the optical with the electrical evidence will produce a conviction of the reality of the medium similar to that which we obtain, in the case of other kinds of matter, from the combined evidence of the senses.

782.] When light is emitted, a certain amount of energy is expended by the luminous body, and if the light is absorbed by another body, this body becomes heated, shewing that it has received energy from without. During the interval of time after the light left the first body and before it reached the second, it must have existed as energy in the intervening space.

According to the theory of emission, the transmission of energy is effected by the actual transference of light-corpuscules from the luminous to the illuminated body, carrying with them their kinetic energy, together with any other kind of energy of which they may be the receptacles.

According to the theory of undulation, there is a material medium which fills the space between the two bodies, and it is by the action of contiguous parts of this medium that the energy is passed on, from one portion to the next, till it reaches the illuminated body.

The luminiferous medium is therefore, during the passage of light through it, a receptacle of energy. In the undulatory theory, as developed by Huygens, Fresnel, Young, Green, &c., this energy is supposed to be partly potential and partly kinetic. The potential energy is supposed to be due to the distortion of the elementary portions of the medium. We must therefore regard the medium as elastic. The kinetic energy is supposed to be due to the vibratory motion of the medium. We must therefore regard the medium as having a finite density.

In the theory of electricity and magnetism adopted in this treatise, two forms of energy are recognised, the electrostatic and the electrokinetic (see Arts. 630 and 636), and these are supposed to have their seat, not merely in the electrified or magnetized bodies, but in every part of the surrounding space, where electric or magnetic force is observed to act. Hence our theory agrees with the undulatory theory in assuming the existence of a medium which is capable of becoming a receptacle of two forms of energy*.

* 'For my own part, considering the relation of a vacuum to the magnetic force and the general character of magnetic phenomena external to the magnet, I am more inclined to the notion that in the transmission of the force there is such an action, external to the magnet, than that the effects are merely attraction and repulsion at a distance. Such an action may be a function of the æther; for it is not at all unlikely that, if there be an æther, it should have other uses than simply the conveyance of radiations.'—Faraday's *Experimental Researches*, 3075.

783.] Let us next determine the conditions of the propagation of an electromagnetic disturbance through a uniform medium, which we shall suppose to be at rest, that is, to have no motion except that which may be involved in electromagnetic disturbances.

Let $C$ be the specific conductivity of the medium, $K$ its specific capacity for electrostatic induction, and $\mu$ its magnetic 'permeability'.

To obtain the general equations of electromagnetic disturbance, we shall express the true current $\mathfrak{C}$ in terms of the vector potential $\mathfrak{A}$ and the electric potential $\Psi$.

The true current $\mathfrak{C}$ is made up of the conduction current $\mathfrak{K}$ and the variation of the electric displacement $\mathfrak{D}$, and since both of these depend on the electromotive intensity $\mathfrak{E}$, we find, as in Art. 611,

$$\mathfrak{C} = \left(C + \frac{1}{4\pi} K \frac{d}{dt}\right) \mathfrak{E}. \tag{1}$$

But since there is no motion of the medium, we may express the electromotive intensity, as in Art. 599,

$$\mathfrak{E} = -\dot{\mathfrak{A}} - \nabla \Psi. \tag{2}$$

Hence
$$\mathfrak{C} = -\left(C + \frac{1}{4\pi} K \frac{d}{dt}\right)\left(\frac{d\mathfrak{A}}{dt} + \nabla \Psi\right). \tag{3}$$

But we may determine a relation between $\mathfrak{C}$ and $\mathfrak{A}$ in a different way, as is shewn in Art. 616, the equations (4) of which may be written

$$4\pi\mu\mathfrak{C} = \nabla^2 \mathfrak{A} + \nabla J, \tag{4}$$

where
$$J = \frac{dF}{dx} + \frac{dG}{dy} + \frac{dH}{dz}. \tag{5}$$

Combining equations (3) and (4), we obtain

$$\mu\left(4\pi C + K\frac{d}{dt}\right)\left(\frac{d\mathfrak{A}}{dt} + \nabla\Psi\right) + \nabla^2\mathfrak{A} + \nabla J = 0, \tag{6}$$

which we may express in the form of three equations as follows—

$$\left.\begin{aligned}
\mu\left(4\pi C + K\frac{d}{dt}\right)\left(\frac{dF}{dt} + \frac{d\Psi}{dx}\right) + \nabla^2 F + \frac{dJ}{dx} &= 0,\\
\mu\left(4\pi C + K\frac{d}{dt}\right)\left(\frac{dG}{dt} + \frac{d\Psi}{dy}\right) + \nabla^2 G + \frac{dJ}{dy} &= 0,\\
\mu\left(4\pi C + K\frac{d}{dt}\right)\left(\frac{dH}{dt} + \frac{d\Psi}{dz}\right) + \nabla^2 H + \frac{dJ}{dz} &= 0.
\end{aligned}\right\} \tag{7}$$

These are the general equations of electromagnetic disturbances.

If we differentiate these equations with respect to $x$, $y$, and $z$ respectively, and add, we obtain

$$\mu\left(4\pi C+K\frac{d}{dt}\right)\left(\frac{dJ}{dt}-\nabla^2\Psi\right)=0. \tag{8}$$

If the medium is a non-conductor, $C=0$, and $\nabla^2\Psi$, which is proportional to the volume-density of free electricity, is independent of $t$. Hence $J$ must be a linear function of $t$, or a constant, or zero, and we may therefore leave $J$ and $\Psi$ out of account in considering periodic disturbances.

### *Propagation of Undulations in a Non-conducting Medium.*

784.] In this case, $C=0$, and the equations become

$$\left.\begin{aligned} K\mu\frac{d^2F}{dt^2}+\nabla^2F=0,\\ K\mu\frac{d^2G}{dt^2}+\nabla^2G=0,\\ K\mu\frac{d^2H}{dt^2}+\nabla^2H=0.\end{aligned}\right\} \tag{9}$$

The equations in this form are similar to those of the motion of an incompressible elastic solid, and when the initial conditions are given, the solution can be expressed in a form given by Poisson*, and applied by Stokes to the Theory of Diffraction †.

Let us write $$V=\frac{1}{\sqrt{K\mu}}. \tag{10}$$

If the values of $F$, $G$, $H$, and of $\frac{dF}{dt}$, $\frac{dG}{dt}$, $\frac{dH}{dt}$ are given at every point of space at the epoch ($t=0$), then we can determine their values at any subsequent time, $t$, as follows.

Let $O$ be the point for which we wish to determine the value of $F$ at the time $t$. With $O$ as centre, and with radius $Vt$, describe a sphere. Find the initial value of $F$ at every point of the spherical surface, and take the *mean*, $\overline{F}$, of all these values. Find also the initial values of $\frac{dF}{dt}$ at every point of the spherical surface, and let the mean of these values be $\frac{d\overline{F}}{dt}$.

* *Mém. de l'Acad.*, tom. iii. p. 130, et seq.
† *Cambridge Transactions*, vol. ix. pp. 1–62 (1849).

Then the value of $F$ at the point $O$, at the time $t$, is

Similarly
$$\left.\begin{aligned} F &= \frac{d}{dt}(\bar{F}t) + t\frac{d\bar{F}}{dt}, \\ G &= \frac{d}{dt}(\bar{G}t) + t\frac{d\bar{G}}{dt}, \\ H &= \frac{d}{dt}(\bar{H}t) + t\frac{d\bar{H}}{dt}. \end{aligned}\right\} \tag{11}$$

785.] It appears, therefore, that the condition of things at the point $O$ at any instant depends on the condition of things at a distance $Vt$ and at an interval of time $t$ previously, so that any disturbance is propagated through the medium with the velocity $V$.

Let us suppose that when $t$ is zero the quantities $\mathfrak{A}$ and $\dot{\mathfrak{A}}$ are zero except within a certain space $S$. Then their values at $O$ at the time $t$ will be zero, unless the spherical surface described about $O$ as centre with radius $Vt$ lies in whole or in part within the space $S$. If $O$ is outside the space $S$ there will be no disturbance at $O$ until $Vt$ becomes equal to the shortest distance from $O$ to the space $S$. The disturbance at $O$ will then begin, and will go on till $Vt$ is equal to the greatest distance from $O$ to any part of $S$. The disturbance at $O$ will then cease for ever.

786.] The quantity $V$, in Art. 784, which expresses the velocity of propagation of electromagnetic disturbances in a non-conducting medium is, by equation (10), equal to $\dfrac{1}{\sqrt{K\mu}}$.

If the medium is air, and if we adopt the electrostatic system of measurement, $K = 1$ and $\mu = \dfrac{1}{v^2}$, so that $V = v$, or the velocity of propagation is numerically equal to the number of electrostatic units of electricity in one electromagnetic unit. If we adopt the electromagnetic system, $K = \dfrac{1}{v^2}$ and $\mu = 1$, so that the equation $V = v$ is still true.

On the theory that light is an electromagnetic disturbance, propagated in the same medium through which other electromagnetic actions are transmitted, $V$ must be the velocity of light, a quantity the value of which has been estimated by several methods. On the other hand, $v$ is the number of electrostatic units of electricity in one electromagnetic unit, and the methods of determining this quantity have been described in the last chapter. They are quite independent of the methods of finding the velocity of light. Hence the agreement or disagreement of the values of $V$ and of $v$ furnishes a test of the electromagnetic theory of light.

787.] In the following table, the principal results of direct observation of the velocity of light, either through the air or through the planetary spaces, are compared with the principal results of the comparison of the electric units :—

| Velocity of Light (mètres per second). | | Ratio of Electric Units (mètres per second). | |
|---|---|---|---|
| Fizeau | 314000000 | Weber | 310740000 |
| Aberration, &c., and Sun's Parallax | 308000000 | Maxwell | 288000000 |
| Foucault | 298360000 | Thomson | 282000000 |

It is manifest that the velocity of light and the ratio of the units are quantities of the same order of magnitude. Neither of them can be said to be determined as yet with such a degree of accuracy as to enable us to assert that the one is greater or less than the other. It is to be hoped that, by further experiment, the relation between the magnitudes of the two quantities may be more accurately determined.

In the meantime our theory, which asserts that these two quantities are equal, and assigns a physical reason for this equality, is certainly not contradicted by the comparison of these results such as they are.

* {In the following table, taken from a paper by E. B. Rosa, *Phil. Mag.* 28, p. 315, 1889, the determinations of '$v$' corrected for the error in the B.A. unit are given :—

| | | |
|---|---|---|
| 1856 | Weber and Kohlrausch | $3{\cdot}107 \times 10^{10}$ (cm. per second) |
| 1868 | Maxwell | $2{\cdot}842 \times 10^{10}$ |
| 1869 | W. Thomson and King | $2{\cdot}808 \times 10^{10}$ |
| 1872 | McKichan | $2{\cdot}896 \times 10^{10}$ |
| 1879 | Ayrton and Perry | $2{\cdot}960 \times 10^{10}$ |
| 1880 | Shida | $2{\cdot}955 \times 10^{10}$ |
| 1883 | J. J. Thomson | $2{\cdot}963 \times 10^{10}$ |
| 1884 | Klemenčič | $3{\cdot}019 \times 10^{10}$ |
| 1888 | Himstedt | $3{\cdot}009 \times 10^{10}$ |
| 1889 | W. Thomson | $3{\cdot}004 \times 10^{10}$ |
| 1889 | E. B. Rosa | $2{\cdot}9993 \times 10^{10}$ |
| 1890 | J. J. Thomson and Searle | $2{\cdot}9955 \times 10^{10}$ |

VELOCITY OF LIGHT IN AIR.

| | |
|---|---|
| Cornu (1878) | $3{\cdot}003 \times 10^{10}$ |
| Michelson (1879) | $2{\cdot}9982 \times 10^{10}$ |
| Michelson (1882) | $2{\cdot}9976 \times 10^{10}$ |
| Newcomb (1885) | $\left\{\begin{matrix} 2{\cdot}99615 \\ 2{\cdot}99682 \\ 2{\cdot}99766 \end{matrix}\right\} \times 10^{10}$ |

}

788.] In other media than air, the velocity $V$ is inversely proportional to the square root of the product of the dielectric and the magnetic inductive capacities. According to the undulatory theory, the velocity of light in different media is inversely proportional to their indices of refraction.

There are no transparent media for which the magnetic capacity differs from that of air more than by a very small fraction. Hence the principal part of the difference between these media must depend on their dielectric capacity. According to our theory, therefore, the dielectric capacity of a transparent medium should be equal to the square of its index of refraction.

But the value of the index of refraction is different for light of different kinds, being greater for light of more rapid vibrations. We must therefore select the index of refraction which corresponds to waves of the longest periods, because these are the only waves whose motion can be compared with the slow processes by which we determine the capacity of the dielectric.

789.] The only dielectric of which the capacity has been hitherto determined with sufficient accuracy is paraffin, for which in the solid form MM. Gibson and Barclay found *

$$K = 1{\cdot}975. \tag{12}$$

Dr. Gladstone has found the following values of the index of refraction of melted paraffin, sp. g. 0·779, for the lines $A$, $D$ and $H$:—

| Temperature | $A$ | $D$ | $H$ |
|---|---|---|---|
| 54°C | 1·4306 | 1·4357 | 1·4499 |
| 57°C | 1·4294 | 1·4343 | 1·4493 ; |

from which I find that the index of refraction for waves of infinite length would be about 1·422.

The square root of $K$ is 1·405.

The difference between these numbers is greater than can be accounted for by errors of observation, and shews that our theories of the structure of bodies must be much improved before we can deduce their optical from their electrical properties. At the same time, I think that the agreement of the numbers is such that if no greater discrepancy were found between the numbers derived from the optical and the electrical properties of a considerable number of substances, we should be warranted in

* *Phil. Trans.* 1871, p. 573.

concluding that the square root of $K$, though it may not be the complete expression for the index of refraction, is at least the most important term in it *.

### *Plane Waves.*

790.] Let us now confine our attention to plane waves, the fronts of which we shall suppose normal to the axis of $z$. All the quantities, the variation of which constitutes such waves, are functions of $z$ and $t$ only, and are independent of $x$ and $y$. Hence the equations of magnetic induction, (A), Art. 591, are reduced to

$$a = -\frac{dG}{dz}, \qquad b = \frac{dF}{dz}, \qquad c = 0, \tag{13}$$

or the magnetic disturbance is in the plane of the wave. This agrees with what we know of that disturbance which constitutes light.

Putting $\mu\alpha$, $\mu\beta$ and $\mu\gamma$ for $a$, $b$ and $c$ respectively, the equations of electric currents, Art. 607, become

$$\left.\begin{aligned} 4\pi\mu u &= -\frac{db}{dz} = -\frac{d^2F}{dz^2}, \\ 4\pi\mu v &= \quad\frac{da}{dz} = -\frac{d^2G}{dz^2}, \\ 4\pi\mu w &= 0. \end{aligned}\right\} \tag{14}$$

Hence the electric disturbance is also in the plane of the wave, and if the magnetic disturbance is confined to one direction, say that of $x$, the electric disturbance is confined to the perpendicular direction, or that of $y$.

But we may calculate the electric disturbance in another way, for if $f$, $g$, $h$ are the components of electric displacement in a non-conducting medium,

$$u = \frac{df}{dt}, \qquad v = \frac{dg}{dt}, \qquad w = \frac{dh}{dt}. \tag{15}$$

* [In a paper read to the Royal Society on June 14, 1877, Dr. J. Hopkinson gives the results of experiments made for the purpose of determining the specific inductive capacities of various kinds of glass. These results do not verify the theoretical conclusions arrived at in the text, the value of $K$ being in each case in excess of that of the square of the refractive index. In a subsequent paper to the Royal Society, read on Jan. 6, 1881, Dr. Hopkinson finds that, if $\mu\infty$ denote the index of refraction for waves of infinite length, then $K = \mu^2\infty$ for hydrocarbons, but for animal and vegetable oils $K > \mu^2\infty$.]

{Under electrical vibrations with a frequency of about twenty-five millions per second $K$ the specific inductive capacity of glass, according to the experiments of J. J. Thomson, *Proc. Roy. Soc.*, June 20, 1889, and Blondlot, *Comptes Rendus*, May 11, 1891, p. 1058, approximates to $\mu^2$. Lecher (*Wied. Ann.* 42, p. 142) came to the opposite conclusion that the divergence under such circumstances was greater than for steady forces.}

If $P$, $Q$, $R$ are the components of the electromotive intensity,

$$f = \frac{K}{4\pi}P, \qquad g = \frac{K}{4\pi}Q, \qquad h = \frac{K}{4\pi}R; \tag{16}$$

and since there is no motion of the medium, equations (B), Art. 598, become

$$P = -\frac{dF}{dt}, \qquad Q = -\frac{dG}{dt}, \qquad R = -\frac{dH}{dt}. \tag{17}$$

Hence $$u = -\frac{K}{4\pi}\frac{d^2F}{dt^2}, \qquad v = -\frac{K}{4\pi}\frac{d^2G}{dt^2}, \qquad w = -\frac{K}{4\pi}\frac{d^2H}{dt^2}. \tag{18}$$

Comparing these values with those given in equation (14), we find

$$\left.\begin{aligned} \frac{d^2F}{dz^2} &= K\mu\frac{d^2F}{dt^2}, \\ \frac{d^2G}{dz^2} &= K\mu\frac{d^2G}{dt^2}, \\ 0 &= K\mu\frac{d^2H}{dt^2}. \end{aligned}\right\} \tag{19}$$

The first and second of these equations are the equations of propagation of a plane wave, and their solution is of the well-known form

$$\left.\begin{aligned} F &= f_1(z-Vt) + f_2(z+Vt), \\ G &= f_3(z-Vt) + f_4(z+Vt). \end{aligned}\right\} \tag{20}$$

The solution of the third equation is

$$H = A + Bt, \tag{21}$$

where $A$ and $B$ are functions of $z$. $H$ is therefore either constant or varies directly with the time. In neither case can it take part in the propagation of waves.

791.] It appears from this that the directions, both of the magnetic and the electric disturbances, lie in the plane of the wave. The mathematical form of the disturbance therefore agrees with that of the disturbance which constitutes light, being transverse to the direction of propagation.

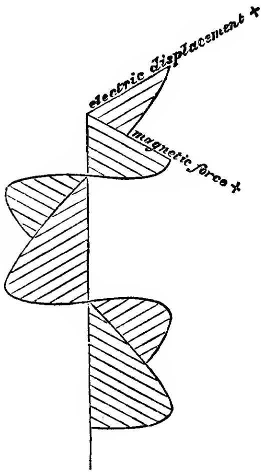


Fig. 67.

If we suppose $G = 0$, the disturbance will correspond to a plane-polarized ray of light.

The magnetic force is in this case parallel to the axis of $y$ and equal to $\frac{1}{\mu}\frac{dF}{dz}$, and the electromotive intensity is parallel to the axis of $x$ and equal to $-\frac{dF}{dt}\cdot$ The magnetic force is therefore in a plane perpendicular to that which contains the electric intensity.

The values of the magnetic force and of the electromotive intensity at a given instant at different points of the ray are represented in Fig. 67, for the case of a simple harmonic disturbance in one plane. This corresponds to a ray of plane-polarized light, but whether the plane of polarization corresponds to the plane of the magnetic disturbance, or to the plane of the electric disturbance, remains to be seen. See Art. 797.

*Energy and Stress of Radiation.*

792.] The electrostatic energy per unit of volume at any point of the wave in a non-conducting medium is

$$\tfrac{1}{2}fP = \frac{K}{8\pi}P^2 = \frac{K}{8\pi}\left|\frac{\overline{dF}}{dt}\right|^2. \tag{22}$$

The electrokinetic energy at the same point is

$$\frac{1}{8\pi}b\beta = \frac{1}{8\pi\mu}b^2 = \frac{1}{8\pi\mu}\left|\frac{\overline{dF}}{dz}\right|^2. \tag{23}$$

In virtue of equation (20) these two expressions are equal for a single wave, so that at every point of the wave the intrinsic energy of the medium is half electrostatic and half electrokinetic.

Let $p$ be the value of either of these quantities, that is, either the electrostatic or the electrokinetic energy per unit of volume, then, in virtue of the electrostatic state of the medium, there is a tension whose magnitude is $p$, in a direction parallel to $x$, combined with a pressure, also equal to $p$, parallel to $y$ and $z$. See Art. 107.

In virtue of the electrokinetic state of the medium there is a tension equal to $p$ in a direction parallel to $y$, combined with a pressure equal to $p$ in directions parallel to $x$ and $z$. See Art. 643.

Hence the combined effect of the electrostatic and the electrokinetic stresses is a *pressure* equal to $2p$ in the direction of the propagation of the wave. Now $2p$ also expresses the whole energy in unit of volume.

Hence in a medium in which waves are propagated there is a pressure in the direction normal to the waves, and numerically equal to the energy in unit of volume.

793.] Thus, if in strong sunlight the energy of the light which falls on one square foot is 83·4 foot pounds per second, the mean energy in one cubic foot of sunlight is about 0·0000000882 of a foot pound, and the mean pressure on a square foot is 0·0000000882 of a pound weight. A flat body exposed to sunlight would experience this pressure on its illuminated side only, and would therefore be repelled from the side on which the light falls. It is probable that a much greater energy of radiation might be obtained by means of the concentrated rays of the electric lamp. Such rays falling on a thin metallic disk, delicately suspended in a vacuum, might perhaps produce an observable mechanical effect. When a disturbance of any kind consists of terms involving sines or cosines of angles which vary with the time, the maximum energy is double of the mean energy. Hence, if $P$ is the maximum electromotive intensity and $\beta$ the maximum magnetic force which are called into play during the propagation of light,

$$\frac{K}{8\pi}P^2 = \frac{\mu}{8\pi}\beta^2 = \text{mean energy in unit of volume.} \tag{24}$$

With Pouillet's data for the energy of sunlight, as quoted by Thomson, *Trans. R. S. E.*, 1854, this gives in electromagnetic measure

$P = 60000000$, or about 600 Daniell's cells per mètre;*

$\beta = 0{\cdot}193$, or rather more than a tenth of the horizontal magnetic force in Britain †.

* {I have not been able to verify these numbers, if we assume $v = 3 \times 10^{10}$, the mean energy in one c. c. of sunlight is, according to Pouillet's data, as quoted by Thomson, $3{\cdot}92 \times 10^{-5}$, ergs, the corresponding values of $P$ and $\beta$ as given by (24) are in C. G. S. units

$P = 9{\cdot}42 \times 10^8$ or 9·42 volts per centimetre,

$\beta = {\cdot}0314$ or rather more than a sixth of the earth's horizontal magnetic force.}

† {We may regard the forces exerted by the incident light on the reflecting surface from a different point of view. Let us suppose that the reflecting surface is metallic, then when the light falls on the surface the variation of the magnetic force induces currents in the metal, and these currents produce opposite inductive effects to the incident light so that the inductive force is screened off from the interior of the metal plate, thus the currents in the plate, and therefore the intensity of the light, rapidly diminish as we recede from the surface of the plate. The currents in the plate are accompanied by magnetic forces at right angles to them, the corresponding mechanical force is at right angles both to the current and the magnetic force, and therefore parallel to the direction of propagation of the light. If the light were passing through a non-absorbent medium this mechanical force would be reversed after half a wave length, and when integrated over a finite time and distance would have no resultant effect. When however the currents rapidly die away as we recede from the surface, the effects due to the currents close to the surface are not counterbalanced by the effects of those at some distance away from it, so that the resultant effect does not vanish.

*Propagation of a Plane Wave in a Crystallized Medium.*

794.] In calculating, from data furnished by ordinary electromagnetic experiments, the electrical phenomena which would result from periodic disturbances, millions of millions of which

We can calculate the magnitude of this effect in the following way. Let us consider the case of light incident normally on a metal plate which we shall take as the plane of $xy$. Let $\sigma$ be the specific resistance of the material. Let the vector potential of the incident ray be given by the equation

$$F = Ae^{i(pt-az)},$$

of the reflected ray by $$F' = A'e^{i(pt+az)},$$

of the refracted ray by $$F'' = A''e^{i(pt-a'z)};$$

then in the air $$\frac{d^2F}{dz^2} = \frac{1}{V^2}\frac{d^2F}{dt^2},$$

where $V$ is the velocity of light in air, hence

$$a = \frac{p}{V};$$

in the metal $$\frac{d^2F}{dz^2} = \frac{4\pi\mu}{\sigma}\frac{dF}{dt},$$

and therefore $$a'^2 = -\frac{4\pi\mu ip}{\sigma} = -2in^2, \text{ say};$$

thus $$a' = n(1-i),$$
$$F'' = A''e^{-nz}e^{i(pt-nz)}.$$

The vector potential at the surface is continuous, hence

$$A + A' = A''.$$

The magnetic force parallel to the surface is also continuous, and hence

$$a(A-A') = \frac{a'A''}{\mu},$$

or $$A'' = \frac{2A}{1+\frac{a}{a\mu}};$$

or, since $a'/a$ is very large, we may write this as

$$A'' = 2A\frac{a\mu}{a'}$$
$$= \frac{2A\mu p}{V\sqrt{2}n}e^{i\frac{\pi}{4}},$$

so that in the metal the real part of the vector potential is

$$F'' = \frac{2A\mu p}{V\sqrt{2}n}e^{-nz}\cos\left(pt-nz+\tfrac{\pi}{4}\right).$$

The intensity of the current is $-\frac{1}{\sigma}\frac{dF''}{dt}$, that is,

$$\frac{2A\mu p^2}{\sigma V\sqrt{2}n}e^{-nz}\sin\left(pt-nz+\tfrac{\pi}{4}\right).$$

The magnetic induction $\frac{dF''}{dz}$ is

$$-\frac{2A\mu p}{V\sqrt{2}}e^{-nz}\left\{\cos\left(pt-nz+\tfrac{\pi}{4}\right)-\sin\left(pt-nz+\tfrac{\pi}{4}\right)\right\}.$$

occur in a second, we have already put our theory to a very severe test, even when the medium is supposed to be air or vacuum. But if we attempt to extend our theory to the case of dense media, we become involved not only in all the ordinary difficulties of molecular theories, but in the deeper mystery of the relation of the molecules to the electromagnetic medium.

To evade these difficulties, we shall assume that in certain media the specific capacity for electrostatic induction is different in different directions, or in other words, the electric displacement, instead of being in the same direction as the electromotive intensity, and proportional to it, is related to it by a system of linear equations similar to those given in Art. 297. It may be shewn, as in Art. 436, that the system of coefficients must be

The mechanical force per unit volume parallel to $z$ is the product of these two quantities,

$$-\frac{2A^2\mu^2p^3}{\sigma V^2 n}e^{-2nz}\left\{\tfrac{1}{2}\sin 2\left(pt-nz+\tfrac{\pi}{4}\right)-\tfrac{1}{2}\left(1-\cos 2\left(pt-nz+\tfrac{\pi}{4}\right)\right)\right\}.$$

The mean value of this is expressed by the non-periodic term and is equal to

$$\frac{A^2\mu^2p^3}{\sigma V^2 n}e^{-2nz}.$$

Integrating this expression with respect to $z$ from $z=0$ to $z=\infty$, we find that the force on the plate per unit area

$$=\tfrac{1}{2}\frac{A^2\mu^2p^3}{\sigma V^2 n^2}=\frac{A^2\mu p^2}{4\pi V^2}.$$

A similar investigation will show that when we have absorption there is a force on the absorbing medium from the places where the light is strong to those where it is faint. In the case of sunlight the effect seems small, if the absorption however were caused by a very rare gas, the pressure-gradient might be large enough to produce very considerable effects, and it has been suggested that this cause is one of the agents at work in causing comets' tails to be repelled by the sun. When the electric vibrations are such as are produced in Hertz's experiments the magnetic forces are very much greater than those in sunlight, and the effect ought to be capable of detection, if the vibrators could be kept at work anything like continuously.

We also get mechanical forces whose mean value at any point is not zero when we have stationary vibrations. We may take as an example of the stationary vibrations the reflected and incident waves in the above example.

In the air the vector potential is, remembering that $a/a'$ is small,

$$Ae^{i(pt-az)}+A'e^{i(pt+az)},$$

or, taking the real part, since $A+A'=0$ approximately,

$$2A\sin pt\sin az.$$

The current is $$\frac{1}{4\pi\mu}\frac{d^2F}{dz^2}=\frac{a^2A}{2\pi\mu}\sin pt\sin az.$$

The magnetic induction is $\quad 2Aa\sin pt\cos az$;

the mechanical force is therefore

$$\frac{A^2a^3}{2\pi\mu}(1-\cos 2pt)\sin az\cos az,$$

and the mean value of this is $$\frac{A^2a^3}{2\pi\mu}\sin az\cos az.\}$$

symmetrical, so that, by a proper choice of axes, the equations become

$$f=\frac{1}{4\pi}K_1P, \qquad g=\frac{1}{4\pi}K_2Q, \qquad h=\frac{1}{4\pi}K_3R, \tag{1}$$

where $K_1$, $K_2$, and $K_3$ are the principal inductive capacities of the medium. The equations of propagation of disturbances are therefore

$$\left.\begin{aligned}
\frac{d^2F}{dy^2}+\frac{d^2F}{dz^2}-\frac{d^2G}{dx\,dy}-\frac{d^2H}{dz\,dx}&=K_1\mu\left(\frac{d^2F}{dt^2}+\frac{d^2\Psi}{dx\,dt}\right),\\
\frac{d^2G}{dz^2}+\frac{d^2G}{dx^2}-\frac{d^2H}{dy\,dz}-\frac{d^2F}{dx\,dy}&=K_2\mu\left(\frac{d^2G}{dt^2}+\frac{d^2\Psi}{dy\,dt}\right),\\
\frac{d^2H}{dx^2}+\frac{d^2H}{dy^2}-\frac{d^2F}{dz\,dx}-\frac{d^2G}{dy\,dz}&=K_3\mu\left(\frac{d^2H}{dt^2}+\frac{d^2\Psi}{dz\,dt}\right).
\end{aligned}\right\} \tag{2}$$

795.] If $l$, $m$, $n$ are the direction-cosines of the normal to the wave-front, and $V$ the velocity of the wave, and if

$$lx+my+nz-Vt=w, \tag{3}$$

and if we write $F''$, $G''$, $H''$, $\Psi''$ for the second differential coefficients of $F$, $G$, $H$, $\Psi$ respectively with respect to $w$, and put

$$K_1\mu=\frac{1}{a^2}, \qquad K_2\mu=\frac{1}{b^2}, \qquad K_3\mu=\frac{1}{c^2}, \tag{4}$$

where $a$, $b$ $c$ are the three principal velocities of propagation, the equations become

$$\left.\begin{aligned}
\left(m^2+n^2-\frac{V^2}{a^2}\right)F''-lmG''-nlH''+V\Psi''\frac{l}{a^2}&=0,\\
-lmF''+\left(n^2+l^2-\frac{V^2}{b^2}\right)G''-mnH''+V\Psi''\frac{m}{b^2}&=0,\\
-nlF''-mnG''+\left(l^2+m^2-\frac{V^2}{c^2}\right)H''+V\Psi''\frac{n}{c^2}&=0.
\end{aligned}\right\} \tag{5}$$

796.] If we write

$$\frac{l^2}{V^2-a^2}+\frac{m^2}{V^2-b^2}+\frac{n^2}{V^2-c^2}=U, \tag{6}$$

we obtain from these equations

$$\left.\begin{aligned}
VU(VF''-l\Psi'')&=0,\\
VU(VG''-m\Psi'')&=0,\\
VU(VH''-n\Psi'')&=0.
\end{aligned}\right\} \tag{7}$$

Hence, either $V=0$, in which case the wave is not propagated at all; or, $U=0$, which leads to the equation for $V$ given by Fresnel; or the quantities within brackets vanish, in which case the vector whose components are $F''$, $G''$, $H''$ is normal to the wave-front and proportional to the electric volume-density.

Since the medium is a non-conductor, the electric density at any given point is constant, and therefore the disturbance indicated by these equations is not periodic, and cannot constitute a wave. We may therefore consider $\Psi'' = 0$ in the investigation of the wave.

797.] The velocity of the propagation of the wave is therefore completely determined from the equation $U = 0$, or

$$\frac{l^2}{V^2-a^2} + \frac{m^2}{V^2-b^2} + \frac{n^2}{V^2-c^2} = 0. \tag{8}$$

There are therefore two, and only two, values of $V^2$ corresponding to a given direction of wave-front.

If $\lambda, \mu, \nu$ are the direction-cosines of the electric current whose components are $u, v, w$,

$$\lambda : \mu : \nu :: \frac{1}{a^2} F'' : \frac{1}{b^2} G'' : \frac{1}{c^2} H'', \tag{9}$$

then
$$l\lambda + m\mu + n\nu = 0; \tag{10}$$
or the current is in the plane of the wave-front, and its direction in the wave-front is determined by the equation

$$\frac{l}{\lambda}(b^2-c^2) + \frac{m}{\mu}(c^2-a^2) + \frac{n}{\nu}(a^2-b^2) = 0. \tag{11}$$

These equations are identical with those given by Fresnel if we define the plane of polarization as a plane through the ray perpendicular to the plane of the electric disturbance.

According to this electromagnetic theory of double refraction the wave of normal disturbance, which constitutes one of the chief difficulties of the ordinary theory, does not exist, and no new assumption is required in order to account for the fact that a ray polarized in a principal plane of the crystal is refracted in the ordinary manner*.

### *Relation between Electric Conductivity and Opacity.*

798.] If the medium, instead of being a perfect insulator, is a conductor whose conductivity per unit of volume is $C$, the disturbance will consist not only of electric displacements but of currents of conduction, in which electric energy is transformed into heat, so that the undulation is absorbed by the medium.

If the disturbance is expressed by a circular function, we may write
$$F = e^{-pz} \cos(nt - qz), \tag{1}$$

* See Stokes' 'Report on Double Refraction,' *Brit. Assoc. Report*, 1862, p. 253.

for this will satisfy the equation

$$\frac{d^2F}{dz^2} = \mu K \frac{d^2F}{dt^2} + 4\pi\mu C \frac{dF}{dt}, \tag{2}$$

provided $$q^2 - p^2 = \mu K n^2, \tag{3}$$
and $$2pq = 4\pi\mu C n. \tag{4}$$

The velocity of propagation is

$$V = \frac{n}{q}, \tag{5}$$

and the coefficient of absorption is

$$p = 2\pi\mu C V. \tag{6}$$

Let $R$ be the resistance {to a current along the length of the plate}, in electromagnetic measure, of a plate whose length is $l$, breadth $b$, and thickness $z$,

$$R = \frac{l}{bzC}. \tag{7}$$

The proportion of the incident light which will be transmitted by this plate will be

$$e^{-2pz} = e^{-4\pi\mu \frac{l}{b}\frac{V}{R}}. \tag{8}$$

799.] Most transparent solid bodies are good insulators, and all good conductors are very opaque. There are, however, many exceptions to the law that the opacity of a body is the greater, the greater its conductivity.

Electrolytes allow an electric current to pass, and yet many of them are transparent. We may suppose, however, that in the case of the rapidly alternating forces which come into play during the propagation of light, the electromotive intensity acts for so short a time in one direction that it is unable to effect a complete separation between the combined molecules. When, during the other half of the vibration, the electromotive intensity acts in the opposite direction it simply reverses what it did during the first half. There is thus no true conduction through the electrolyte, no loss of electric energy, and consequently no absorption of light.

800.] Gold, silver, and platinum are good conductors, and yet, when formed into very thin plates, they allow light to pass through them*. From experiments which I have made on a piece of gold leaf, the resistance of which was determined by Mr. Hockin, it appears that its transparency is very much

* {Wien (*Wied. Ann.* 35, p. 48) has verified the conclusion that the transparency of thin metallic films is much greater than that indicated by the preceding theory.}

greater than is consistent with our theory, unless we suppose that there is less loss of energy when the electromotive forces are reversed for every semivibration of light than when they act for sensible times, as in our ordinary experiments.

801.] Let us next consider the case of a medium in which the conductivity is large in proportion to the inductive capacity.

In this case we may leave out the term involving $K$ in the equations of Art. 783, and they then become

$$\left.\begin{aligned} \nabla^2 F + 4\pi\mu C\frac{dF}{dt} &= 0,\\ \nabla^2 G + 4\pi\mu C\frac{dG}{dt} &= 0,\\ \nabla^2 H + 4\pi\mu C\frac{dH}{dt} &= 0. \end{aligned}\right\} \qquad (1)$$

Each of these equations is of the same form as the equation of the diffusion of heat given in Fourier's *Traité de la Chaleur*.

802.] Taking the first as an example, the component $F$ of the vector-potential will vary according to time and position in the same way as the temperature of a homogeneous solid varies according to time and position, the initial and the surface conditions being made to correspond in the two cases, and the quantity $4\pi\mu C$ being numerically equal to the reciprocal of the thermometric conductivity of the substance, that is to say, the *number of units of volume of the substance which would be heated one degree by the heat which passes through a unit cube of the substance, two opposite faces of which differ by one degree of temperature, while the other faces are impermeable to heat**.

The different problems in thermal conduction, of which Fourier has given the solution, may be transformed into problems in the diffusion of electromagnetic quantities, remembering that $F, G, H$ are the components of a vector, whereas the temperature, in Fourier's problem, is a scalar quantity.

Let us take one of the cases of which Fourier has given a complete solution †, that of an infinite medium, the initial state of which is given.

* See Maxwell's *Theory of Heat*, p. 235 first edition, p. 255 fourth edition.

† *Traité de la Chaleur*, Art. 384. The equation which determines the temperature, $v$, at a point $(x, y, z)$ after a time $t$, in terms of $f(\alpha, \beta, \gamma)$, the initial temperature at the point $(\alpha, \beta, \gamma)$, is

$$v = \iiint \frac{d\alpha\, d\beta\, d\gamma}{2^3\sqrt{k^3\pi^3 t^3}}\, e^{-\left(\frac{(\alpha-x)^2+(\beta-y)^2+(\gamma-z)^2}{4kt}\right)} f(\alpha, \beta, \gamma),$$

where $k$ is the thermometric conductivity.

The state of any point of the medium at the time $t$ is found by taking the average of the state of every part of the medium, the weight assigned to each part in taking the average being

$$e^{-\frac{\pi\mu C r^2}{t}},$$

where $r$ is the distance of that part from the point considered. This average, in the case of vector-quantities, is most conveniently taken by considering each component of the vector separately.

803.] We have to remark in the first place, that in this problem the thermal conductivity of Fourier's medium is to be taken inversely proportional to the electric conductivity of our medium, so that the time required in order to reach an assigned stage in the process of diffusion is greater the higher the electric conductivity. This statement will not appear paradoxical if we remember the result of Art. 655, that a medium of infinite conductivity forms a complete barrier to the process of diffusion of magnetic force.

In the next place, the time requisite for the production of an assigned stage in the process of diffusion is proportional to the square of the linear dimensions of the system.

There is no determinate velocity which can be defined as the velocity of diffusion. If we attempt to measure this velocity by ascertaining the time requisite for the production of a given amount of disturbance at a given distance from the origin of disturbance, we find that the smaller the selected value of the disturbance the greater the velocity will appear to be, for however great the distance, and however small the time, the value of the disturbance will differ mathematically from zero.

This peculiarity of diffusion distinguishes it from wave-propagation, which takes place with a definite velocity. No disturbance takes place at a given point till the wave reaches that point, and when the wave has passed, the disturbance ceases for ever.

804.] Let us now investigate the process which takes place when an electric current begins and continues to flow through a linear circuit, the medium surrounding the circuit being of finite electric conductivity. (Compare with Art. 660.)

When the current begins, its first effect is to produce a current of induction in the parts of the medium close to the wire. The direction of this current is opposite to that of the original current, and in the first instant its total quantity is equal to that of the original current, so that the electromagnetic effect on more distant parts of the medium is initially zero, and only rises to its final value as the induction-current dies away on account of the electric resistance of the medium.

But as the induction-current close to the wire dies away, a new induction-current is generated in the medium beyond, so that the space occupied by the induction-current is continually becoming wider, while its intensity is continually diminishing.

This diffusion and decay of the induction-current is a phenomenon precisely analogous to the diffusion of heat from a part of the medium initially hotter or colder than the rest. We must remember, however, that since the current is a vector quantity, and since in a circuit the current is in opposite directions at opposite points of the circuit, we must, in calculating any given component of the induction-current, compare the problem with one in which equal quantities of heat and of cold are diffused from neighbouring places, in which case the effect on distant points will be of a smaller order of magnitude.

805.] If the current in the linear circuit is maintained constant, the induction-currents, which depend on the initial change of state, will gradually be diffused and die away, leaving the medium in its permanent state, which is analogous to the permanent state of the flow of heat. In this state we have

$$\nabla^2 F = \nabla^2 G = \nabla^2 H = 0 \tag{2}$$

throughout the medium, except at the part occupied by the circuit, in which {when $\mu = 1$}

$$\left.\begin{aligned} \nabla^2 F &= 4\pi u, \\ \nabla^2 G &= 4\pi v, \\ \nabla^2 H &= 4\pi w. \end{aligned}\right\} \tag{3}$$

These equations are sufficient to determine the values of $F$, $G$, $H$ throughout the medium. They indicate that there are no currents except in the circuit, and that the magnetic forces are simply those due to the current in the circuit according to the ordinary theory. The rapidity with which this permanent state is established is so great that it could not be measured by our experimental methods, except perhaps in the case of a very large mass of a highly conducting medium such as copper.

NOTE.—In a paper published in Poggendorff's *Annalen*, July

1867, pp. 243–263, M. Lorenz has deduced from Kirchhoff's equations of electric currents (Pogg. *Ann.* cii. 1857), by the addition of certain terms which do not affect any experimental result, a new set of equations, indicating that the distribution of force in the electromagnetic field may be conceived as arising from the mutual action of contiguous elements, and that waves, consisting of transverse electric currents, may be propagated, with a velocity comparable to that of light, in non-conducting media. He therefore regards the disturbance which constitutes light as identical with these electric currents, and he shews that conducting media must be opaque to such radiations.

These conclusions are similar to those of this chapter, though obtained by an entirely different method. The theory given in this chapter was first published in the *Phil. Trans.* for 1865, pp. 459–512.

# CHAPTER XXI.

## MAGNETIC ACTION ON LIGHT.

806.] THE most important step in establishing a relation between electric and magnetic phenomena and those of light must be the discovery of some instance in which the one set of phenomena is affected by the other. In the search for such phenomena we must be guided by any knowledge we may have already obtained with respect to the mathematical or geometrical form of the quantities which we wish to compare. Thus, if we endeavour, as Mrs. Somerville did, to magnetize a needle by means of light, we must remember that the distinction between magnetic north and south is a mere matter of direction, and would be at once reversed if we reversed certain conventions about the use of mathematical signs. There is nothing in magnetism analogous to those phenomena of electrolysis which enable us to distinguish positive from negative electricity, by observing that oxygen appears at one pole of a cell and hydrogen at the other.

Hence we must not expect that if we make light fall on one end of a needle, that end will become a pole of a certain name, for the two poles do not differ as light does from darkness.

We might expect a better result if we caused circularly-polarized light to fall on the needle, right-handed light falling on one end and left-handed on the other, for in some respects these kinds of light may be said to be related to each other in the same way as the poles of a magnet. The analogy, however, is faulty even here, for the two rays when combined do not neutralize each other, but produce a plane polarized ray.

Faraday, who was acquainted with the method of studying the strains produced in transparent solids by means of polarized light, made many experiments in hopes of detecting some action on polarized light while passing through a medium in which electrolytic conduction or dielectric induction exists *. He was not, however, able to detect any action of this kind, though the experiments were arranged in the way best adapted to discover effects of tension, the electric force or current being at right angles to the direction of the ray, and at an angle of forty-five degrees to the plane of polarization. Faraday varied these experiments in many ways without discovering any action on light due to electrolytic currents or to static electric induction.

He succeeded, however, in establishing a relation between light and magnetism, and the experiments by which he did so are described in the nineteenth series of his *Experimental Researches*. We shall take Faraday's discovery as our starting-point for further investigation into the nature of magnetism, and we shall therefore describe the phenomenon which he observed.

807.] A ray of plane-polarized light is transmitted through a transparent diamagnetic medium, and the plane of its polarization, when it emerges from the medium, is ascertained by observing the position of an analyser when it cuts off the ray. A magnetic force is then made to act so that the direction of the force within the transparent medium coincides with the direction of the ray. The light at once reappears, but if the analyser is turned round through a certain angle, the light is again cut off. This shews that the effect of the magnetic force is to turn the plane of polarization, round the direction of the ray as an axis, through a certain angle, measured by the angle through which the analyser must be turned in order to cut off the light.

808.] The angle through which the plane of polarization is turned is proportional—

(1) To the distance which the ray travels within the medium. Hence the plane of polarization changes continuously from its position at incidence to its position at emergence.

(2) To the intensity of the resolved part of the magnetic force in the direction of the ray.

(3) The amount of the rotation depends on the nature of the medium. No rotation has yet been observed when the medium is air or any other gas †.

* *Experimental Researches*, 951–954 and 2216–2220.

† {Since this was written the rotation in gases has been observed and measured by H. Becquerel, *Compt. Rendus*, 88, p. 709; 90, p. 1407; Kundt and Röntgen, *Wied. Ann.*, 6, p. 332; 8, p. 278; Bichat, *Compt. Rendus*, 88, p. 712; *Journal de Physique*, 9, p. 275, 1880.}

These three statements are included in the more general one, that the angular rotation is numerically equal to the amount by which the magnetic potential increases, from the point at which the ray enters the medium to that at which it leaves it, multiplied by a coefficient, which, for diamagnetic media, is generally positive.

809.] In diamagnetic substances, the direction in which the plane of polarization is made to rotate is {generally} the same as the direction in which a positive current must circulate round the ray in order to produce a magnetic force in the same direction as that which actually exists in the medium.

Verdet, however, discovered that in certain ferromagnetic media, as, for instance, a strong solution of perchloride of iron in wood-spirit or ether, the rotation is in the opposite direction to the current which would produce the magnetic force.

This shews that the difference between ferromagnetic and diamagnetic substances does not arise merely from the 'magnetic permeability' being in the first case greater, and in the second less, than that of air, but that the properties of the two classes of bodies are really opposite.

The power acquired by a substance under the action of magnetic force of rotating the plane of polarization of light is not exactly proportional to its diamagnetic or ferromagnetic magnetizability. Indeed there are exceptions to the rule that the rotation is positive for diamagnetic and negative for ferromagnetic substances, for neutral chromate of potash is diamagnetic, but produces a negative rotation.

810.] There are other substances, which, independently of the application of magnetic force, cause the plane of polarization to turn to the right or to the left, as the ray travels through the substance. In some of these the property is related to an axis, as in the case of quartz. In others, the property is independent of the direction of the ray within the medium, as in turpentine, solution of sugar, &c. In all these substances, however, if the plane of polarization of any ray is twisted within the medium like a right-handed screw, it will still be twisted like a right-handed screw if the ray is transmitted through the medium in the opposite direction. The direction in which the observer has to turn his analyser in order to extinguish the ray after introducing the medium into its path, is the same with reference to the observer whether the ray comes to him from the north or from the south. The direction of the rotation in space is of course reversed when the direction of the ray is reversed. But when the rotation is produced by magnetic action, its direction in space is the same whether the ray be travelling north or south. The rotation is always in the same direction as that of the electric current which produces, or would produce, the actual magnetic state of the field, if the medium belongs to the positive class, or in the opposite direction if the medium belongs to the negative class.

It follows from this, that if the ray of light, after passing through the medium from north to south, is reflected by a mirror, so as to return through the medium from south to north, the rotation will be doubled when it results from magnetic action. When the rotation depends on the nature of the medium alone, as in turpentine, &c., the ray, when reflected back through the medium, emerges polarized in the same plane as when it entered, the rotation during the first passage through the medium having been exactly reversed during the second.

811.] The physical explanation of the phenomenon presents considerable difficulties, which can hardly be said to have been hitherto overcome, either for the magnetic rotation, or for that which certain media exhibit of themselves. We may, however, prepare the way for such an explanation by an analysis of the observed facts.

It is a well-known theorem in kinematics that two uniform circular vibrations, of the same amplitude, having the same periodic time, and in the same plane, but revolving in opposite directions, are equivalent, when compounded together, to a rectilinear vibration. The periodic time of this vibration is equal to that of the circular vibrations, its amplitude is double, and its direction is in the line joining the points at which two particles, describing the circular vibrations in opposite directions round the same circle, would meet. Hence if one of the circular vibrations has its phase accelerated, the direction of the rectilinear vibration will be turned, in the same direction as that of the circular vibration, through an angle equal to half the acceleration of phase.

It can also be proved by direct optical experiment that two rays of light, circularly-polarized in opposite directions, and of the same intensity, become, when united, a plane-polarized ray, and that if by any means the phase of one of the circularly-polarized rays is accelerated, the plane of polarization of the resultant ray is turned round half the angle of acceleration of the phase.

812.] We may therefore express the phenomenon of the rotation of the plane of polarization in the following manner:— A plane-polarized ray falls on the medium. This is equivalent to two circularly-polarized rays, one right-handed, the other left-handed (as regards the observer). After passing through the medium the ray is still plane-polarized, but the plane of polarization is turned, say, to the right (as regards the observer). Hence, of the two circularly-polarized rays, that which is right-handed must have had its phase accelerated with respect to the other during its passage through the medium.

In other words, the right-handed ray has performed a greater number of vibrations, and therefore has a smaller wave-length, within the medium, than the left-handed ray which has the same periodic time.

This mode of stating what takes place is quite independent of any theory of light, for though we use such terms as wave-length, circular-polarization, &c., which may be associated in our minds with a particular form of the undulatory theory, the reasoning is independent of this association, and depends only on facts proved by experiment.

813.] Let us next consider the configuration of one of these rays at a given instant. Any undulation, the motion of which at each point is circular, may be represented by a helix or screw. If the screw is made to revolve about its axis without any longitudinal motion, each particle will describe a circle, and at the same time the propagation of the undulation will be represented by the apparent longitudinal motion of the similarly situated parts of the thread of the screw. It is easy to see that if the screw is right-handed, and the observer is placed at that end towards which the undulation travels, the motion of the screw will appear to him left-handed, that is to say, in the opposite direction to that of the hands of a watch. Hence such a ray has been called, originally by French writers, but now by the whole scientific world, a left-handed circularly-polarized ray.

A right-handed circularly-polarized ray is represented in like manner by a left-handed helix. In Fig. 68 the right-handed helix $A$, on the right-hand of the figure, represents a left-handed ray, and the left-handed helix $B$, on the left-hand, represents a right-handed ray.

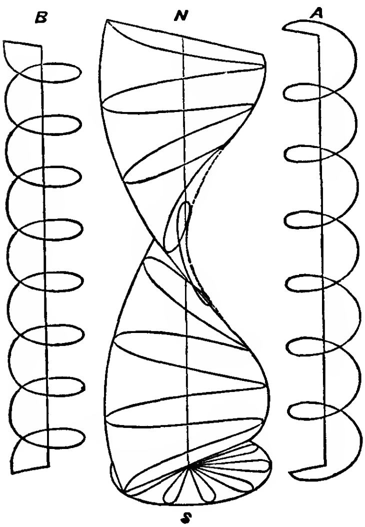


Fig. 68.

814.] Let us now consider two such rays which have the same wave-length within the medium. They are geometrically alike in all respects, except that one is the *perversion* of the other, like its image in a looking-glass. One of them, however, say $A$, has a shorter period of rotation than the other. If the motion is entirely due to the forces called into play by the displacement, this shews that greater forces are called into play by the same displacement when the configuration is like $A$ than when it is like $B$. Hence in this case the left-handed ray will be accelerated with respect to the right-handed ray, and this will be the case whether the rays are travelling from $N$ to $S$ or from $S$ to $N$.

This therefore is the explanation of the phenomenon as it is produced by turpentine, &c. In these media the displacement caused by a circularly-polarized ray calls into play greater forces of restitution when the configuration is like $A$ than when it is like $B$. The forces thus depend on the configuration alone, not on the direction of the motion.

But in a diamagnetic medium acted on by magnetism in the direction $SN$, of the two screws $A$ and $B$, that one always rotates with the greatest velocity whose motion, as seen by an eye looking from $S$ to $N$, appears like that of a watch. Hence for rays from $S$ to $N$ the right-handed ray $B$ will travel quickest, but for rays from $N$ to $S$ the left-handed ray $A$ will travel quickest.

815.] Confining our attention to one ray only, the helix $B$ has exactly the same configuration, whether it represents a ray from $S$ to $N$ or one from $N$ to $S$. But in the first instance the ray travels faster, and therefore the helix rotates more rapidly. Hence greater forces are called into play when the helix is going round one way than when it is going round the other way. The forces, therefore, do not depend solely on the configuration of the ray, but also on the direction of the motion of its individual parts.

816.] The disturbance which constitutes light, whatever its physical nature may be, is of the nature of a vector, perpendicular to the direction of the ray. This is proved from the fact of the interference of two rays of light, which under certain conditions produces darkness, combined with the fact of the non-interference of two rays polarized in planes perpendicular to each other. For since the interference depends on the angular position of the planes of polarization, the disturbance must be a directed quantity or vector, and since the interference ceases when the planes of polarization are at right angles, the vector representing the disturbance must be perpendicular to the line of intersection of these planes, that is, to the direction of the ray.

817.] The disturbance, being a vector, can be resolved into components parallel to $x$ and $y$, the axis of $z$ being parallel to the direction of the ray. Let $\xi$ and $\eta$ be these components, then, in the case of a ray of homogeneous circularly-polarized light,

$$\xi = r\cos\theta, \qquad \eta = r\sin\theta, \tag{1}$$

where

$$\theta = nt - qz + a. \tag{2}$$

In these expressions, $r$ denotes the magnitude of the vector, and $\theta$ the angle which it makes with the direction of the axis of $x$.

The periodic time, $\tau$, of the disturbance is such that

$$n\tau = 2\pi. \tag{3}$$

The wave-length, $\lambda$, of the disturbance is such that

$$q\lambda = 2\pi. \tag{4}$$

The velocity of propagation is $\frac{n}{q}\cdot$

The phase of the disturbance when $t$ and $z$ are both zero is $a$.

The circularly-polarized light is right-handed or left-handed according as $q$ is negative or positive.

Its vibrations are in the positive or the negative direction of rotation in the plane of $(x, y)$, according as $n$ is positive or negative.

The light is propagated in the positive or the negative direction of the axis of $z$, according as $n$ and $q$ are of the same or of opposite signs.

In all media $n$ varies when $q$ varies, and $\frac{dn}{dq}$ is always of the same sign with $\frac{n}{q}$.

Hence, if for a given numerical value of $n$ the value of $\frac{n}{q}$ is greater when $n$ is positive than when $n$ is negative, it follows that for a value of $q$, given both in magnitude and sign, the positive value of $n$ will be greater than the negative value.

Now this is what is {generally} observed in a diamagnetic medium, acted on by a magnetic force, $\gamma$, in the direction of $z$. Of the two circularly-polarized rays of a given period, that is accelerated of which the direction of rotation in the plane of $x$, $y$ is positive. Hence, of two circularly-polarized rays, both left-handed, whose wave-length within the medium is the same, that has the shortest period whose direction of rotation in the plane of $xy$ is positive, that is, the ray which is propagated in the positive direction of $z$ from south to north. We have therefore to account for the fact, that when in the equations of the system $q$ and $r$ are given, two values of $n$ will satisfy the equations, one positive and the other negative, the positive value being numerically greater than the negative.

818.] We may obtain the equations of motion from a consideration of the potential and kinetic energies of the medium. The potential energy, $V$, of the system depends on its configuration, that is, on the relative position of its parts. In so far as it depends on the disturbance due to circularly-polarized light, it must be a function of $r$, the amplitude, and $q$, the coefficient of torsion, only. It may be different for positive and negative values of $q$ of equal numerical value, and it probably is so in the case of media which of themselves rotate the plane of polarization.

The kinetic energy, $T$, of the system is a homogeneous function of the second degree of the velocities of the system, the coefficients of the different terms being functions of the coordinates.

819.] Let us consider the dynamical condition that the ray may be of constant intensity, that is, that $r$ may be constant.

Lagrange's equation for the force in $r$ becomes

$$\frac{d}{dt}\frac{dT}{d\dot{r}} - \frac{dT}{dr} + \frac{dV}{dr} = 0. \tag{5}$$

Since $r$ is constant, the first term vanishes. We have therefore the equation

$$-\frac{dT}{dr} + \frac{dV}{dr} = 0, \tag{6}$$

in which $q$ is supposed to be given, and we are to determine the value of the angular velocity $\dot{\theta}$, which we may denote by its actual value, $n$.

The kinetic energy, $T$, contains one term involving $n^2$; other terms may contain products of $n$ with other velocities, and the rest of the terms are independent of $n$. The potential energy, $V$, is entirely independent of $n$. The equation (6) is therefore of the form

$$An^2 + Bn + C = 0. \tag{7}$$

This being a quadratic equation, gives two values of $n$. It appears from experiment that both values are real, that one is positive and the other negative, and that the positive value is numerically the greater. Hence, if $A$ is positive, both $B$ and $C$ are negative, for, if $n_1$ and $n_2$ are the roots of the equation,

$$A(n_1 + n_2) + B = 0. \tag{8}$$

The coefficient, $B$, therefore, is not zero, at least when magnetic force acts on the medium. We have therefore to consider the expression $Bn$, which is the part of the kinetic energy involving the first power of $n$, the angular velocity of the disturbance.

820.] Every term of $T$ is of two dimensions as regards velocity. Hence the terms involving $n$ must involve some other velocity. This velocity cannot be $\dot{r}$ or $\dot{q}$, because, in the case we consider, $r$ and $q$ are constant. Hence it is a velocity which exists in the medium independently of that motion which constitutes light. It must also be a velocity related to $n$ in such a way that when it is multiplied by $n$ the result is a scalar quantity, for only scalar quantities can occur as terms in the value of $T$, which is itself scalar. Hence this velocity must be in the same direction as $n$, or in the opposite direction, that is, it must be an *angular velocity* about the axis of $z$.

Again, this velocity cannot be independent of the magnetic force, for if it were related to a direction fixed in the medium, the phenomenon would be different if we turned the medium end for end, which is not the case.

We are therefore led to the conclusion that this velocity is an invariable accompaniment of the magnetic force in those media which exhibit the magnetic rotation of the plane of polarization.

821.] We have been hitherto obliged to use language which is perhaps too suggestive of the ordinary hypothesis of motion in the undulatory theory. It is easy, however, to state our result in a form free from this hypothesis.

Whatever light is, at each point of space there is something going on, whether displacement, or rotation, or something not yet imagined, but which is certainly of the nature of a vector or directed quantity, the direction of which is normal to the direction of the ray. This is completely proved by the phenomena of interference.

In the case of circularly-polarized light, the magnitude of this vector remains always the same, but its direction rotates round the direction of the ray so as to complete a revolution in the periodic time of the wave. The uncertainty which exists as to whether this vector is in the plane of polarization or perpendicular to it, does not extend to our knowledge of the direction in which it rotates in right-handed and in left-handed circularly-polarized light respectively. The direction and the angular velocity of this vector are perfectly known, though the physical nature of the vector and its absolute direction at a given instant are uncertain.

When a ray of circularly-polarized light falls on a medium under the action of magnetic force, its propagation within the medium is affected by the relation of the direction of rotation of the light to the direction of the magnetic force. From this we conclude, by the reasoning of Art. 817, that in the medium, when under the action of magnetic force, some rotatory motion is going on, the axis of rotation being in the direction of the magnetic force; and that the rate of propagation of circularly-polarized light, when the direction of its vibratory rotation and the direction of the magnetic rotation of the medium are the same, is different from the rate of propagation when these directions are opposite.

The only resemblance which we can trace between a medium through which circularly-polarized light is propagated, and a medium through which lines of magnetic force pass, is that in both there is a motion of rotation about an axis. But here the resemblance stops, for the rotation in the optical phenomenon is that of the vector which represents the disturbance. This vector is always perpendicular to the direction of the ray, and rotates about it a known number of times in a second. In the magnetic phenomenon, that which rotates has no properties by which its sides can be distinguished, so that we cannot determine how many times it rotates in a second.

There is nothing, therefore, in the magnetic phenomenon which corresponds to the wave-length and the wave-propagation in the optical phenomenon. A medium in which a constant magnetic force is acting is not, in consequence of that force, filled with waves travelling in one direction, as when light is propagated through it. The only resemblance between the optical and the magnetic phenomenon is, that at each point of the medium something exists of the nature of an angular velocity about an axis in the direction of the magnetic force.

### *On the Hypothesis of Molecular Vortices.*

822.] The consideration of the action of magnetism on polarized light leads, as we have seen, to the conclusion that in a medium under the action of magnetic force something belonging to the same mathematical class as an angular velocity, whose axis is in the direction of the magnetic force, forms a part of the phenomenon.

This angular velocity cannot be that of any portion of the medium of sensible dimensions rotating as a whole. We must therefore conceive the rotation to be that of very small portions of the medium, each rotating on its own axis. This is the hypothesis of molecular vortices.

The motion of these vortices, though, as we have shewn (Art. 575), it does not sensibly affect the visible motions of large bodies, may be such as to affect that vibratory motion on which the propagation of light, according to the undulatory theory, depends. The displacements of the medium, during the propagation of light, will produce a disturbance of the vortices, and the vortices when so disturbed may react on the medium so as to affect the mode of propagation of the ray.

823.] It is impossible, in our present state of ignorance as to the nature of the vortices, to assign the form of the law which connects the displacement of the medium with the variation of the vortices. We shall therefore assume that the variation of the vortices caused by the displacement of the medium is subject to the same conditions which Helmholtz, in his great memoir on Vortex-motion *, has shewn to regulate the variation of the vortices of a perfect liquid.

Helmholtz's law may be stated as follows:—Let $P$ and $Q$ be two neighbouring particles in the axis of a vortex, then, if in consequence of the motion of the fluid these particles arrive at the points $P'$, $Q'$, the line $P'Q'$ will represent the new direction of the axis of the vortex, and its strength will be altered in the ratio of $P'Q'$ to $PQ$.

Hence if $\alpha$, $\beta$, $\gamma$ denote the components of the strength of a vortex, and if $\xi$, $\eta$, $\zeta$ denote the displacements of the medium, the values of $\alpha$, $\beta$, $\gamma$ will become

$$\left.\begin{aligned} \alpha' &= \alpha + \alpha\frac{d\xi}{dx} + \beta\frac{d\xi}{dy} + \gamma\frac{d\xi}{dz}, \\ \beta' &= \beta + \alpha\frac{d\eta}{dx} + \beta\frac{d\eta}{dy} + \gamma\frac{d\eta}{dz}, \\ \gamma' &= \gamma + \alpha\frac{d\zeta}{dx} + \beta\frac{d\zeta}{dy} + \gamma\frac{d\zeta}{dz}. \end{aligned}\right\} \qquad (1)$$

We now assume that the same condition is satisfied during the small displacements of a medium in which $\alpha$, $\beta$, $\gamma$ represent, not the components of the strength of an ordinary vortex, but the components of magnetic force.

824.] The components of the angular velocity of an element of the medium are

$$\left.\begin{aligned} \omega_1 &= \tfrac{1}{2}\frac{d}{dt}\left(\frac{d\zeta}{dy} - \frac{d\eta}{dz}\right), \\ \omega_2 &= \tfrac{1}{2}\frac{d}{dt}\left(\frac{d\xi}{dz} - \frac{d\zeta}{dx}\right), \\ \omega_3 &= \tfrac{1}{2}\frac{d}{dt}\left(\frac{d\eta}{dx} - \frac{d\xi}{dy}\right). \end{aligned}\right\} \qquad (2)$$

The next step in our hypothesis is the assumption that the kinetic energy of the medium contains a term of the form

$$2C(\alpha\omega_1 + \beta\omega_2 + \gamma\omega_3). \qquad (3)$$

* *Crelle's Journal*, vol. lv. (1858), pp. 25–55. Translated by Tait, *Phil. Mag.*, June, pp. 485–511, 1867.

This is equivalent to supposing that the angular velocity acquired by the element of the medium during the propagation of light is a quantity which may enter into combination with that motion by which magnetic phenomena are explained.

In order to form the equations of motion of the medium, we must express its kinetic energy in terms of the velocity of its parts, the components of which are $\dot{\xi}$, $\dot{\eta}$, $\dot{\zeta}$. We therefore integrate by parts, and find

$$2C\iiint(\alpha\omega_1+\beta\omega_2+\gamma\omega_3)\,dx\,dy\,dz$$

$$= C\iint(\gamma\dot{\eta}-\beta\dot{\zeta})\,dy\,dz + C\iint(\alpha\dot{\zeta}-\gamma\dot{\xi})\,dz\,dx + C\iint(\beta\dot{\xi}-\alpha\dot{\eta})\,dx\,dy$$

$$+C\iiint\left\{\dot{\xi}\left(\frac{d\gamma}{dy}-\frac{d\beta}{dz}\right)+\dot{\eta}\left(\frac{d\alpha}{dz}-\frac{d\gamma}{dx}\right)+\dot{\zeta}\left(\frac{d\beta}{dx}-\frac{d\alpha}{dy}\right)\right\}dx\,dy\,dz. \quad (4)$$

The double integrals refer to the bounding surface, which may be supposed at an infinite distance. We may therefore, while investigating what takes place in the interior of the medium, confine our attention to the triple integral.

825.] The part of the kinetic energy in unit of volume, expressed by this triple integral, may be written

$$4\pi C(\dot{\xi}u+\dot{\eta}v+\dot{\zeta}w), \quad (5)$$

where $u$, $v$, $w$ are the components of the electric current as given in equations (E), Art. 607.

It appears from this that our hypothesis is equivalent to the assumption that the velocity of a particle of the medium whose components are $\dot{\xi}$, $\dot{\eta}$, $\dot{\zeta}$ is a quantity which may enter into combination with the electric current whose components are $u$, $v$, $w$.

826.] Returning to the expression under the sign of triple integration in (4), substituting for the values of $\alpha$, $\beta$, $\gamma$, those of $\alpha'$, $\beta'$, $\gamma'$, as given by equations (1), and writing

$$\frac{d}{dh} \text{ for } \alpha\frac{d}{dx}+\beta\frac{d}{dy}+\gamma\frac{d}{dz}, \quad (6)$$

the expression under the sign of integration becomes

$$C\left\{\dot{\xi}\frac{d}{dh}\left(\frac{d\zeta}{dy}-\frac{d\eta}{dz}\right)+\dot{\eta}\frac{d}{dh}\left(\frac{d\xi}{dz}-\frac{d\zeta}{dx}\right)+\dot{\zeta}\frac{d}{dh}\left(\frac{d\eta}{dx}-\frac{d\xi}{dy}\right)\right\}. \quad (7)$$

In the case of waves in planes normal to the axis of $z$ the displacements are functions of $z$ and $t$ only, so that $\frac{d}{dh} = \gamma\frac{d}{dz}$, and this expression is reduced to

$$C\gamma\left(\frac{d^2\xi}{dz^2}\dot{\eta} - \frac{d^2\eta}{dz^2}\dot{\xi}\right). \tag{8}$$

The kinetic energy per unit of volume, so far as it depends on the velocities of displacement, may now be written

$$T = \tfrac{1}{2}\rho(\dot{\xi}^2+\dot{\eta}^2+\dot{\zeta}^2)+C\gamma\left(\frac{d^2\xi}{dz^2}\dot{\eta} - \frac{d^2\eta}{dz^2}\dot{\xi}\right), \tag{9}$$

where $\rho$ is the density of the medium.

827.] The components, $X$ and $Y$, of the impressed force, referred to unit of volume, may be deduced from this by Lagrange's equations, Art. 564. We observe that by two successive integrations by parts in regard to $z$, and the omission of the double integrals at the bounding surface, it may be shewn that

$$\iiint \frac{d^2\xi}{dz^2}\dot{\eta}\,dx\,dy\,dz = \iiint \xi\frac{d^3\eta}{dz^2\,dt}\,dx\,dy\,dz.$$

Hence
$$\frac{dT}{d\xi} = C\gamma\frac{d^3\eta}{dz^2\,dt}.$$

The expressions for the forces are therefore given by

$$X = \rho\frac{d^2\xi}{dt^2} - 2C\gamma\frac{d^3\eta}{dz^2dt}, \tag{10}$$

$$Y = \rho\frac{d^2\eta}{dt^2} + 2C\gamma\frac{d^3\xi}{dz^2dt}. \tag{11}$$

These forces arise from the action of the remainder of the medium on the element under consideration, and must in the case of an isotropic medium be of the form indicated by Cauchy,

$$X = A_0\frac{d^2\xi}{dz^2} + A_1\frac{d^4\xi}{dz^4} + \&c., \tag{12}$$

$$Y = A_0\frac{d^2\eta}{dz^2} + A_1\frac{d^4\eta}{dz^4} + \&c. \tag{13}$$

828.] If we now take the case of a circularly-polarized ray for which
$$\xi = r\cos(nt-qz), \qquad \eta = r\sin(nt-qz), \tag{14}$$
we find for the kinetic energy in unit of volume
$$T = \tfrac{1}{2}\rho r^2 n^2 - C\gamma r^2 q^2 n; \tag{15}$$
and for the potential energy in unit of volume
$$V = \tfrac{1}{2}r^2(A_0q^2 - A_1q^4 + \&c.)$$
$$= \tfrac{1}{2}r^2 Q, \tag{16}$$
where $Q$ is a function of $q^2$.

The condition of free propagation of the ray given in Art. 819, equation (6), is
$$\frac{dT}{dr} = \frac{dV}{dr}, \tag{17}$$
which gives
$$\rho n^2 - 2C\gamma q^2 n = Q, \tag{18}$$
whence the value of $n$ may be found in terms of $q$.

But in the case of a ray of given wave-period, acted on by magnetic force, what we want to determine is the value of $\frac{dq}{d\gamma}$, when $n$ is constant, in terms of $\frac{dq}{dn}$, when $\gamma$ is constant. Differentiating (18)
$$(2\rho n - 2C\gamma q^2)dn - \left(\frac{dQ}{dq} + 4C\gamma q n\right)dq - 2Cq^2 n\, d\gamma = 0. \tag{19}$$
We thus find
$$\frac{dq}{d\gamma} = -\frac{Cq^2 n}{\rho n - C\gamma q^2}\frac{dq}{dn}. \tag{20}$$

829.] If $\lambda$ is the wave-length in air, $v$ the velocity in air, and $i$ the corresponding index of refraction in the medium,
$$q\lambda = 2\pi i, \qquad n\lambda = 2\pi v. \tag{21}$$
$$\left\{\text{Hence} \qquad \frac{dq}{dn} = \frac{1}{v}\left(i - \lambda\frac{di}{d\lambda}\right)\right\}.$$

The change in the value of $q$, due to magnetic action, is in every case an exceedingly small fraction of its own value, so that we may write
$$q = q_0 + \frac{dq}{d\gamma}\gamma, \tag{22}$$
where $q_0$ is the value of $q$ when the magnetic force is zero. The angle, $\theta$, through which the plane of polarization is turned in passing through a thickness $c$ of the medium, is half the sum of the positive and negative values of $qc$, the sign of the result being changed, because the sign of $q$ is negative in equations (14). We thus obtain
$$\theta = -c\gamma\frac{dq}{d\gamma}, \tag{23}$$
$$= \frac{4\pi^2 C}{v\rho}c\gamma\frac{i^2}{\lambda^2}\left(i - \lambda\frac{di}{d\lambda}\right)\frac{1}{1 - 2\pi C\gamma\frac{i^2}{v\rho\lambda}}. \tag{24}$$

The second term of the denominator of this fraction is approximately equal to the angle of rotation of the plane of polarization during the passage of the light through a thickness of the medium equal to $\left\{\frac{1}{\pi}\text{ times}\right\}$ half a wave-length {in the medium}. It is therefore in all actual cases a quantity which we may neglect in comparison with unity.

Writing
$$\frac{4\pi^2 C}{v\rho} = m, \tag{25}$$
we may call $m$ the coefficient of magnetic rotation for the medium, a quantity whose value must be determined by observation. It is found to be positive for most diamagnetic, and negative for some paramagnetic media. We have therefore as the final result of our theory
$$\theta = mc\gamma\frac{i^2}{\lambda^2}\left(i-\lambda\frac{di}{d\lambda}\right), \tag{26}$$
where $\theta$ is the angular rotation of the plane of polarization, $m$ a constant determined by observation of the medium, $\gamma$ the intensity of the magnetic force resolved in the direction of the ray, $c$ the length of the ray within the medium, $\lambda$ the wave-length of the light in air, and $i$ its index of refraction in the medium*.

830.] The only test to which this theory has hitherto been subjected is that of comparing the values of $\theta$ for different kinds of light passing through the same medium and acted on by the same magnetic force.

This has been done for a considerable number of media by M. Verdet †, who has arrived at the following results:—

(1) The magnetic rotations of the planes of polarization of the rays of different colours follow approximately the law of the inverse square of the wave-length.

(2) The exact law of the phenomena is always such that the product of the rotation by the square of the wave-length increases from the least refrangible to the most refrangible end of the spectrum.

(3) The substances for which this increase is most sensible are also those which have the greatest dispersive power.

He also found that in the solution of tartaric acid, which of itself produces a rotation of the plane of polarization, the magnetic rotation is by no means proportional to the natural rotation.

In an addition to the same memoir ‡ Verdet has given the results of very careful experiments on bisulphide of carbon and on creosote, two substances in which the departure from the

* {Rowland (*Phil. Mag.* xi. p. 254, 1881) has shown that magnetic rotation of the plane of polarization would be produced if the Hall effect (Vol. I. p. 423) existed in dielectrics.}

† Recherches sur les propriétés optiques développées dans les corps transparents par l'action du magnétisme, 4^me partie. *Comptes Rendus*, t. lvi. p. 630 (6 April, 1863).

‡ *Comptes Rendus*, lvii. p. 670 (19 Oct., 1863).

law of the inverse square of the wave-length was very apparent. He has also compared these results with the numbers given by three different formulæ,

$$\text{(I)} \quad \theta = mc\gamma \frac{i^2}{\lambda^2}\left(i - \lambda\frac{di}{d\lambda}\right);$$

$$\text{(II)} \quad \theta = mc\gamma \frac{1}{\lambda^2}\left(i - \lambda\frac{di}{d\lambda}\right);$$

$$\text{(III)} \quad \theta = mc\gamma \quad \left(i - \lambda\frac{di}{d\lambda}\right).$$

The first of these formulæ, (I), is that which we have already obtained in Art. 829, equation (26). The second, (II), is that which results from substituting in the equations of motion, Art. 827, equations (10), (11), terms of the form $\frac{d^3\eta}{dt^3}$ and $-\frac{d^3\xi}{dt^3}$, instead of $\frac{d^3\eta}{dz^2dt}$ and $-\frac{d^3\xi}{dz^2dt}$. I am not aware that this form of the equation has been suggested by any physical theory. The third formula, (III), results from the physical theory of M. C. Neumann*, in which the equations of motion contain terms of the form $\frac{d\eta}{dt}$ and $-\frac{d\xi}{dt}$†.

It is evident that the values of $\theta$ given by the formula (III) are not even approximately proportional to the inverse square of the wave-length. Those given by the formulæ (I) and (II) satisfy this condition, and give values of $\theta$ which agree tolerably well with the observed values for media of moderate dispersive power. For bisulphide of carbon and creosote, however, the values given by (II) differ very much from those observed. Those given by (I) agree better with observation, but, though the agreement is somewhat close for bisulphide of carbon, the numbers for creosote still differ by quantities much greater than can be accounted for by any errors of observation.

* 'Explicare tentatur quomodo fiat ut lucis planum polarizationis per vires electricas vel magneticas declinetur.' *Halis Saxonum*, 1858.

† These three forms of the equations of motion were first suggested by Sir G. B. Airy (*Phil. Mag.*, June 1846, p. 477) as a means of analysing the phenomenon then recently discovered by Faraday. Mac Cullagh had previously suggested equations containing terms of the form $\frac{d^3}{dz^3}$ in order to represent mathematically the phenomena of quartz. These equations were offered by Mac Cullagh and Airy, 'not as giving a mechanical explanation of the phaenomena, but as showing that the phaenomena may be explained by equations, which equations appear to be such as might possibly be deduced from some plausible mechanical assumption, although no such assumption has yet been made.'

*Magnetic Rotation of the Plane of Polarization (from Verdet).*

Bisulphide of Carbon at 24°·9 C.

| Lines of the spectrum | *C* | *D* | *E* | *F* | *G* |
|---|---|---|---|---|---|
| Observed rotation | 592 | 768 | 1000 | 1234 | 1704 |
| Calculated by I. | 589 | 760 | 1000 | 1234 | 1713 |
| „ II. | 606 | 772 | 1000 | 1216 | 1640 |
| „ III. | 943 | 967 | 1000 | 1034 | 1091 |

Rotation of the ray $E = 25°·28'$.

Creosote at 24°·3 C.

| Lines of the spectrum | *C* | *D* | *E* | *F* | *G* |
|---|---|---|---|---|---|
| Observed rotation | 573 | 758 | 1000 | 1241 | 1723 |
| Calculated by I. | 617 | 780 | 1000 | 1210 | 1603 |
| „ II. | 623 | 789 | 1000 | 1200 | 1565 |
| „ III. | 976 | 993 | 1000 | 1017 | 1041 |

Rotation of the ray $E = 21°·58'$.

We are so little acquainted with the details of the molecular constitution of bodies, that it is not probable that any satisfactory theory can be formed relating to a particular phenomenon, such as that of the magnetic action on light, until, by an induction founded on a number of different cases in which visible phenomena are found to depend upon actions in which the molecules are concerned, we learn something more definite about the properties which must be attributed to a molecule in order to satisfy the conditions of observed facts.

The theory proposed in the preceding pages is evidently of a provisional kind, resting as it does on unproved hypotheses relating to the nature of molecular vortices, and the mode in which they are affected by the displacement of the medium. We must therefore regard any coincidence with observed facts as of much less scientific value in the theory of the magnetic rotation of the plane of polarization than in the electromagnetic theory of light, which, though it involves hypotheses about the electric properties of media, does not speculate as to the constitution of their molecules.

831.] NOTE.—The whole of this chapter may be regarded as an expansion of the exceedingly important remark of Sir William Thomson in the *Proceedings of the Royal Society*, June 1856:—'the magnetic influence on light discovered by Faraday depends on the direction of motion of moving particles. For instance, in a medium possessing it, particles in a straight line parallel to the lines of magnetic force, displaced to a helix round this line as axis, and then projected tangentially with such velocities as to describe circles, will have different velocities according as their motions are round in one direction (the same as the nominal direction of the galvanic current in the magnetizing coil), or in the contrary direction. But the elastic reaction of the medium must be the same for the same displacements, whatever be the velocities and directions of the particles; that is to say, the forces which are balanced by centrifugal force of the circular motions are equal, while the luminiferous motions are unequal. The absolute circular motions being therefore either equal or such as to transmit equal centrifugal forces to the particles initially considered, it follows that the luminiferous motions are only components of the whole motion; and that a less luminiferous component in one direction, compounded with a motion existing in the medium when transmitting no light, gives an equal resultant to that of a greater luminiferous motion in the contrary direction compounded with the same non-luminous motion. I think it is not only impossible to conceive any other than this dynamical explanation of the fact that circularly-polarized light transmitted through magnetized glass parallel to the lines of magnetizing force, with the same quality, right-handed always, or left-handed always, is propagated at different rates according as its course is in the direction or is contrary to the direction in which a north magnetic pole is drawn; but I believe it can be demonstrated that no other explanation of that fact is possible. Hence it appears that Faraday's optical discovery affords a demonstration of the reality of Ampère's explanation of the ultimate nature of magnetism; and gives a definition of magnetization in the dynamical theory of heat. The introduction of the principle of moments of momenta ("the conservation of areas") into the mechanical treatment of Mr. Rankine's hypothesis of "molecular vortices," appears to indicate a line perpendicular to the plane of resultant rotatory momentum ("the invariable plane") of the thermal motions as the magnetic axis of a magnetized body, and suggests the resultant moment of momenta of these motions as the definite measure of the "magnetic moment." The explanation of all phenomena of electro-magnetic attraction or repulsion, and of electro-magnetic induction, is to be looked for simply in the inertia and pressure of the matter of which the motions constitute heat. Whether this matter is or is not electricity, whether it is a continuous fluid interpermeating the spaces between molecular nuclei, or is itself molecularly grouped; or whether all matter is continuous, and molecular heterogeneousness consists in finite vortical or other relative motions of contiguous parts of a body; it is impossible to decide, and perhaps in vain to speculate, in the present state of science.'

A theory of molecular vortices, which I worked out at considerable length, was published in the *Phil. Mag.* for March, April, and May, 1861, Jan. and Feb. 1862.

I think we have good evidence for the opinion that some phenomenon of rotation is going on in the magnetic field, that this rotation is performed by a great number of very small portions of matter, each rotating on its own axis, this axis being parallel to the direction of the magnetic force, and that the rotations of these different vortices are made to depend on one another by means of some kind of mechanism connecting them.

The attempt which I then made to imagine a working model of this mechanism must be taken for no more than it really is, a demonstration that mechanism may be imagined capable of producing a connexion mechanically equivalent to the actual connexion of the parts of the electromagnetic field. The problem of determining the mechanism required to establish a given species of connexion between the motions of the parts of a system always admits of an infinite number of solutions. Of these, some may be more clumsy or more complex than others, but all must satisfy the conditions of mechanism in general.

The following results of the theory, however, are of higher value:—

(1) Magnetic force is the effect of the centrifugal force of the vortices.

(2) Electromagnetic induction of currents is the effect of the forces called into play when the velocity of the vortices is changing.

(3) Electromotive force arises from the stress on the connecting mechanism.

(4) Electric displacement arises from the elastic yielding of the connecting mechanism.

# CHAPTER XXII.

## FERROMAGNETISM AND DIAMAGNETISM EXPLAINED BY MOLECULAR CURRENTS.

### *On Electromagnetic Theories of Magnetism.*

832.] WE have seen (Art. 380) that the action of magnets on one another can be accurately represented by the attractions and repulsions of an imaginary substance called 'magnetic matter.' We have shewn the reasons why we must not suppose this magnetic matter to move from one part of a magnet to another through a sensible distance, as at first sight it appears to do when we magnetize a bar, and we were led to Poisson's hypothesis that the magnetic matter is strictly confined to single molecules of the magnetic substance, so that a magnetized molecule is one in which the opposite kinds of magnetic matter are more or less separated towards opposite poles of the molecule, but so that no part of either can ever be actually separated from the molecule (Art. 430).

These arguments completely establish the fact, that magnetization is a phenomenon, not of large masses of iron, but of molecules, that is to say, of portions of the substance so small that we cannot by any mechanical method cut one of them in two, so as to obtain a north pole separate from a south pole. But the nature of a magnetic molecule is by no means determined without further investigation. We have seen (Art. 442) that there are strong reasons for believing that the act of magnetizing iron or steel does not consist in imparting magnetization to the molecules of which it is composed, but that these molecules are already magnetic, even in unmagnetized iron, but with their axes placed indifferently in all directions, and that the act of magnetization consists in turning the molecules so that their axes are either rendered all parallel to one direction, or at least are deflected towards that direction.

833.] Still, however, we have arrived at no explanation of the nature of a magnetic molecule, that is, we have not recognized its likeness to any other thing of which we know more. We have therefore to consider the hypothesis of Ampère, that the magnetism of the molecule is due to an electric current constantly circulating in some closed path within it.

It is possible to produce an exact imitation of the action of any magnet on points external to it, by means of a sheet of electric currents properly distributed on its outer surface. But the action of the magnet on points in the interior is quite different from the action of the electric currents on corresponding points. Hence Ampère concluded that if magnetism is to be explained by means of electric currents, these currents must circulate within the molecules of the magnet, and must not flow from one molecule to another. As we cannot experimentally measure the magnetic action at a point in the interior of a molecule, this hypothesis cannot be disproved in the same way that we can disprove the hypothesis of currents of sensible extent within the magnet.

Besides this, we know that an electric current, in passing from one part of a conductor to another, meets with resistance and generates heat; so that if there were currents of the ordinary kind round portions of the magnet of sensible size, there would be a constant expenditure of energy required to maintain them, and a magnet would be a perpetual source of heat. By confining the circuits to the molecules, within which nothing is known about resistance, we may assert, without fear of contradiction, that the current, in circulating within the molecule, meets with no resistance.

According to Ampère's theory, therefore, all the phenomena of magnetism are due to electric currents, and if we could make observations of the magnetic force in the interior of a magnetic molecule, we should find that it obeyed exactly the same laws as the force in a region surrounded by any other electric circuit.

834.] In treating of the force in the interior of magnets, we have supposed the measurements to be made in a small crevasse hollowed out of the substance of the magnet, Art. 395. We were thus led to consider two different quantities, the magnetic force and the magnetic induction, both of which are supposed to be observed in a space from which the magnetic matter is removed. We were not supposed to be able to penetrate into the interior of a magnetic molecule and to observe the force within it.

If we adopt Ampère's theory, we consider a magnet, not as a continuous substance, the magnetization of which varies from point to point according to some easily conceived law, but as a multitude of molecules, within each of which circulates a system of electric currents, giving rise to a distribution of magnetic force of extreme complexity, the direction of the force in the interior of a molecule being generally the reverse of that of the average force in its neighbourhood, and the magnetic potential, where it exists at all, being a function of as many degrees of multiplicity as there are molecules in the magnet.

835.] But we shall find, that, in spite of this apparent complexity, which, however, arises merely from the coexistence of a multitude of simpler parts, the mathematical theory of magnetism is greatly simplified by the adoption of Ampère's theory, and by extending our mathematical vision into the interior of the molecules.

In the first place, the two definitions of magnetic force are reduced to one, both becoming the same as that for the space outside the magnet. In the next place, the components of the magnetic force everywhere satisfy the condition to which those of induction are subject, namely,

$$\frac{d\alpha}{dx}+\frac{d\beta}{dy}+\frac{d\gamma}{dz}=0. \tag{1}$$

In other words, the distribution of magnetic force is of the same nature as that of the velocity of an incompressible fluid, or, as we have expressed it in Art. 25, the magnetic force has no convergence.

Finally, the three vector functions—the electromagnetic momentum, the magnetic force, and the electric current—become more simply related to each other. They are all vector functions of no convergence, and they are derived one from the other in order, by the same process of taking the space-variation which is denoted by Hamilton by the symbol $\nabla$.

836.] But we are now considering magnetism from a physical point of view, and we must enquire into the physical properties of the molecular currents. We assume that a current is circulating in a molecule, and that it meets with no resistance. If $L$ is the coefficient of self-induction of the molecular circuit, and $M$ the coefficient of mutual induction between this circuit and some other circuit, then if $\gamma$ is the current in the molecule, and $\gamma'$ that in the other circuit, the equation of the current $\gamma$ is

$$\frac{d}{dt}(L\gamma + M\gamma') = -R\gamma; \tag{2}$$

and since by the hypothesis there is no resistance, $R = 0$, and we get by integration

$$L\gamma + M\gamma' = \text{constant}, = L\gamma_0, \text{ say.} \tag{3}$$

Let us suppose that the area of the projection of the molecular circuit on a plane perpendicular to the axis of the molecule is A, this axis being defined as the normal to the plane on which the projection is greatest. If the action of other currents produces a magnetic force, $X$, in a direction whose inclination to the axis of the molecule is $\theta$, the quantity $M\gamma'$ becomes $XA\cos\theta$, and we have as the equation of the current

$$L\gamma + XA\cos\theta = L\gamma_0, \tag{4}$$

where $\gamma_0$ is the value of $\gamma$ when $X = 0$.

It appears, therefore, that the strength of the molecular current depends entirely on its primitive value $\gamma_0$, and on the intensity of the magnetic force due to other currents.

837.] If we suppose that there is no primitive current, but that the current is entirely due to induction, then

$$\gamma = -\frac{XA}{L}\cos\theta. \tag{5}$$

The negative sign shews that the direction of the induced current is opposite to that of the inducing current, and its magnetic action is such that in the interior of the circuit it acts in the opposite direction to the magnetic force. In other words, the molecular current acts like a small magnet whose poles are turned towards the poles of the same name of the inducing magnet.

Now this is an action the reverse of that of the molecules of iron under magnetic action. The molecular currents in iron, therefore, are not excited by induction. But in diamagnetic substances an action of this kind is observed, and in fact this is the explanation of diamagnetic polarity which was first given by Weber.

*Weber's Theory of Diamagnetism.*

838.] According to Weber's theory, there exist in the molecules of diamagnetic substances certain channels round which an electric current can circulate without resistance. It is manifest that if we suppose these channels to traverse the molecule in every direction, this amounts to making the molecule a perfect conductor.

Beginning with the assumption of a linear circuit within the molecule, we have the strength of the current given by equation (5).

The magnetic moment of the current is the product of its strength by the area of the circuit, or $\gamma A$, and the resolved part of this in the direction of the magnetizing force is $\gamma A \cos\theta$, or, by (5),

$$-\frac{XA^2}{L}\cos^2\theta. \tag{6}$$

If there are $n$ such molecules in unit of volume, and if their axes are distributed indifferently in all directions, then the average value of $\cos^2\theta$ will be $\frac{1}{3}$, and the intensity of magnetization of the substance will be

$$-\frac{1}{3}\frac{nXA^2}{L}. \tag{7}$$

Neumann's coefficient of magnetization is therefore

$$\kappa = -\frac{1}{3}\frac{nA^2}{L}. \tag{8}$$

The magnetization of the substance is therefore in the opposite direction to the magnetizing force, or, in other words, the substance is diamagnetic. It is also exactly proportional to the magnetizing force, and does not tend to a finite limit, as in the case of ordinary magnetic induction. See Arts. 442, &c.

839.] If the directions of the axes of the molecular channels are arranged, not indifferently in all directions, but with a preponderating number in certain directions, then the sum

$$\Sigma\frac{A^2}{L}\cos^2\theta$$

extended to all the molecules will have different values according to the direction of the line from which $\theta$ is measured, and the distribution of these values in different directions will be similar to the distribution of the values of moments of inertia about axes in different directions through the same point.

Such a distribution will explain the magnetic phenomena related to axes in the body, described by Plücker, which Faraday has called Magne-crystallic phenomena. See Art. 435.

840.] Let us now consider what would be the effect, if, instead of the electric current being confined to a certain channel within the molecule, the whole molecule were supposed a perfect conductor.

Let us begin with the case of a body the form of which is acyclic, that is to say, which is not in the form of a ring or perforated body, and let us suppose that this body is everywhere surrounded by a thin shell of perfectly conducting matter.

We have proved in Art. 654, that a closed sheet of perfectly conducting matter of any form, originally free from currents, becomes, when exposed to external magnetic force, a current-sheet, the action of which on every point of the interior is such as to make the magnetic force zero.

It may assist us in understanding this case if we observe that the distribution of magnetic force in the neighbourhood of such a body is similar to the distribution of velocity in an incompressible fluid in the neighbourhood of an impervious body of the same form.

It is obvious that if other conducting shells are placed within the first, since they are not exposed to magnetic force, no currents will be excited in them. Hence, in a solid of perfectly conducting material, the effect of magnetic force is to generate a system of currents which are entirely confined to the surface of the body.

841.] If the conducting body is in the form of a sphere of radius $r$, its magnetic moment may be shewn {by the method given in Art. 672} to be

$$-\tfrac{1}{2}r^3 X,$$

and if a number of such spheres are distributed in a medium, so that in unit of volume the volume of the conducting matter is $k'$, then, by putting $k_1 = \infty$, $k_2 = 1$, and $p = k'$ in equation (17), Art. 314, we find the coefficient of magnetic permeability, taking it as the reciprocal of the resistance in that article, viz.

$$\mu = \frac{2-2k'}{2+k'}, \tag{9}$$

whence we obtain for Poisson's magnetic coefficient

$$k = -\tfrac{1}{2}k', \tag{10}$$

and for Neumann's coefficient of magnetization by induction

$$\kappa = -\frac{3}{4\pi}\frac{k'}{2+k'}. \tag{11}$$

Since the mathematical conception of perfectly conducting bodies leads to results exceedingly different from any phenomena which we can observe in ordinary conductors, let us pursue the subject somewhat further.

842.] Returning to the case of the conducting channel in the form of a closed curve of area $A$, as in Art. 836, we have, for the moment of the electromagnetic force tending to increase the angle $\theta$,

$$\gamma\gamma'\frac{dM}{d\theta} = -\gamma XA\sin\theta \tag{12}$$

$$= \frac{X^2A^2}{L}\sin\theta\cos\theta. \tag{13}$$

This force is positive or negative according as $\theta$ is less or greater than a right angle. Hence the effect of magnetic force on a perfectly conducting channel tends to turn it with its axis at right angles to the line of magnetic force, that is, so that the plane of the channel becomes parallel to the lines of force.

An effect of a similar kind may be observed by placing a penny or a copper ring between the poles of an electromagnet. At the instant that the magnet is excited the ring turns its plane towards the axial direction, but this force vanishes as soon as the currents are deadened by the resistance of the copper *.

843.] We have hitherto considered only the case in which the molecular currents are entirely excited by the external magnetic force. Let us next examine the bearing of Weber's theory of the magneto-electric induction of molecular currents on Ampère's theory of ordinary magnetism. According to Ampère and Weber, the molecular currents in magnetic substances are not excited by the external magnetic force, but are already there, and the molecule itself is acted on and deflected by the electromagnetic action of the magnetic force on the conducting circuit in which the current flows. When Ampère devised this hypothesis, the induction of electric currents was not known, and he made no

* See Faraday, *Exp. Res.*, 2310, &c.

hypothesis to account for the existence, or to determine the strength, of the molecular currents.

We are now, however, bound to apply to these currents the same laws that Weber applied to his currents in diamagnetic molecules. We have only to suppose that the primitive value of the current $\gamma$, when no magnetic force acts, is not zero but $\gamma_0$. The strength of the current when a magnetic force, $X$, acts on a molecular current of area $A$, whose axis is inclined at an angle $\theta$ to the line of magnetic force, is

$$\gamma = \gamma_0 - \frac{XA}{L}\cos\theta, \tag{14}$$

and the moment of the couple tending to turn the molecule so as to increase $\theta$ is

$$-\gamma_0 XA\sin\theta + \frac{X^2A^2}{2L}\sin 2\theta. \tag{15}$$

Hence, putting

$$A\gamma_0 = m, \quad \frac{A}{L\gamma_0} = B, \tag{16}$$

in the investigation in Art. 443, the equation of equilibrium becomes

$$X\sin\theta - BX^2\sin\theta\cos\theta = D\sin(\alpha-\theta). \tag{17}$$

The resolved part of the magnetic moment of the current in the direction of $X$ is

$$\gamma A\cos\theta = \gamma_0 A\cos\theta - \frac{XA^2}{L}\cos^2\theta, \tag{18}$$

$$= m\cos\theta\,(1-BX\cos\theta). \tag{19}$$

844.] These conditions differ from those in Weber's theory of magnetic induction by the terms involving the coefficient $B$. If $BX$ is small compared with unity, the results will approximate to those of Weber's theory of magnetism. If $BX$ is large compared with unity, the results will approximate to those of Weber's theory of diamagnetism.

Now the greater $\gamma_0$, the primitive value of the molecular current, the smaller will $B$ become, and if $L$ is also large, this will also diminish $B$. Now if the current flows in a ring channel, the value of $L$ depends on $\log\frac{R}{r}$, where $R$ is the radius of the mean line of the channel, and $r$ that of its section. The smaller therefore the section of the channel compared with its area, the greater will be $L$, the coefficient of self-induction, and the more nearly will the phenomena agree with Weber's original theory. There will be this difference, however, that as $X$, the magnetizing force, increases, the temporary magnetic moment will not only reach a maximum, but will afterwards diminish as $X$ increases.

If it should ever be experimentally proved that the temporary magnetization of any substance first increases, and then diminishes as the magnetizing force is continually increased, the evidence of the existence of these molecular currents would, I think, be raised almost to the rank of a demonstration *.

845.] If the molecular currents in diamagnetic substances are confined to definite channels, and if the molecules are capable of being deflected like those of magnetic substances, then, as the magnetizing force increases, the diamagnetic polarity will always increase, but, when the force is great, not quite so fast as the magnetizing force. The small absolute value of the diamagnetic coefficient shews, however, that the deflecting force on each molecule must be small compared with that exerted on a magnetic molecule, so that any result due to this deflexion is not likely to be perceptible.

If, on the other hand, the molecular currents in diamagnetic bodies are free to flow through the whole substance of the molecules, the diamagnetic polarity will be strictly proportional to the magnetizing force, and its amount will lead to a determination of the whole space occupied by the perfectly conducting masses, and, if we know the number of the molecules, to the determination of the size of each.

* {No indication of this effect has as yet been found, though Prof. Ewing has sought for it in very intense magnetic fields. See Ewing and Low 'On the Magnetisation of Iron and other Magnetic Metals in very Strong Fields,' *Phil. Trans.* 1889, A. p. 221.}

# CHAPTER XXIII.

## THEORIES OF ACTION AT A DISTANCE.

### *On the Explanation of Ampère's Formula given by Gauss and Weber.*

846.] THE attraction between the elements $ds$ and $ds'$ of two circuits, carrying electric currents of intensity $i$ and $i'$, is, by Ampère's formula,

$$\frac{ii'ds\,ds'}{r^2}\left(2\cos\epsilon + 3\frac{dr}{ds}\frac{dr}{ds'}\right); \tag{1}$$

or

$$-\frac{ii'ds\,ds'}{r^2}\left(2r\frac{d^2r}{ds\,ds'} - \frac{dr}{ds}\frac{dr}{ds'}\right); \tag{2}$$

the currents being estimated in electromagnetic units. See Art. 526.

The quantities, whose meaning as they appear in these expressions we have now to interpret, are

$$\cos\epsilon,\ \frac{dr}{ds}\frac{dr}{ds'},\ \text{and}\ \frac{d^2r}{ds\,ds'};$$

and the most obvious phenomenon in which to seek for an interpretation founded on a direct relation between the currents is the relative velocity of the electricity in the two elements.

847.] Let us therefore consider the relative motion of two particles, moving with constant velocities $v$ and $v'$ along the elements $ds$ and $ds'$ respectively. The square of the relative velocity of these particles is

$$u^2 = v^2 - 2vv'\cos\epsilon + v'^2; \tag{3}$$

and if we denote by $r$ the distance between the particles,

$$\frac{\partial r}{\partial t} = v\frac{dr}{ds} + v'\frac{dr}{ds'}, \tag{4}$$

$$\left(\frac{\partial r}{\partial t}\right)^2 = v^2\left(\frac{dr}{ds}\right)^2 + 2\,vv'\frac{dr}{ds}\frac{dr}{ds'} + v'^2\left(\frac{dr}{ds'}\right)^2, \tag{5}$$

$$\frac{\partial^2 r}{\partial t^2} = v^2\frac{d^2 r}{ds^2} + 2\,vv'\frac{d^2 r}{ds\,ds'} + v'^2\frac{d^2 r}{ds'^2}, \tag{6}$$

where the symbol $\partial$ indicates that, in the quantity differentiated, the coordinates of the particles are to be expressed in terms of the time.

It appears, therefore, that the terms involving the product $vv'$ in the equations (3), (5), and (6) contain the quantities occurring in (1) and (2) which we have to interpret. We therefore endeavour to express (1) and (2) in terms of $u^2$, $\overline{\frac{\partial r}{\partial t}}\Big|^2$, and $\frac{\partial^2 r}{\partial t^2}$. But in order to do so we must get rid of the first and third terms of each of these expressions, for they involve quantities which do not appear in the formula of Ampère. Hence we cannot explain the electric current as a transfer of electricity in one direction only, but we must combine two opposite streams in each current, so that the combined effect of the terms involving $v^2$ and $v'^2$ may be zero.

848.] Let us therefore suppose that in the first element, $ds$, we have one electric particle, $e$, moving with velocity $v$, and another, $e_1$, moving with velocity $v_1$, and in the same way two particles, $e'$ and $e'_1$, in $ds'$, moving with velocities $v'$ and $v'_1$ respectively.

The term involving $v^2$ for the combined action of these particles is

$$\Sigma(v^2 ee') = (v^2 e + v_1^2 e_1)(e' + e'_1). \tag{7}$$

Similarly

$$\Sigma(v'^2 ee') = (v'^2 e' + v'^2_1 e'_1)(e + e_1); \tag{8}$$

and

$$\Sigma(vv'ee') = (ve + v_1 e_1)(v'e' + v'_1 e'_1). \tag{9}$$

In order that $\Sigma(v^2 ee')$ may be zero, we must have either

$$e' + e'_1 = 0, \quad \text{or} \quad v^2 e + v_1^2 e_1 = 0. \tag{10}$$

According to Fechner's hypothesis, the electric current consists of a current of positive electricity in the positive direction, combined with a current of negative electricity in the negative direction, the two currents being exactly equal in numerical magnitude, both as respects the quantity of electricity in motion and the velocity with which it is moving. Hence both the conditions of (10) are satisfied by Fechner's hypothesis.

But it is sufficient for our purpose to assume, either—

That the quantity of positive electricity in each element is numerically equal to the quantity of negative electricity; or—

That the quantities of the two kinds of electricity are inversely as the squares of their velocities.

Now we know that by charging the second conducting wire as a whole, we can make $e'+e'_1$ either positive or negative. Such a charged wire, even without a current, according to this formula, would act on the first wire carrying a current in which $v^2e+v_1^{\,2}e_1$ has a value differing from zero. Such an action has never been observed.

Therefore, since the quantity $e'+e'_1$ may be shewn experimentally not to be always zero, and since the quantity $v^2e+v^2{}_1e_1$ is not capable of being experimentally tested, it is better for these speculations to assume that it is the latter quantity which invariably vanishes.

849.] Whatever hypothesis we adopt, there can be no doubt that the total transfer of electricity, reckoned algebraically, along the first circuit, is represented by

$$ve+v_1e_1 = c\,i\,ds,$$

where $c$ is the number of units of statical electricity which are transmitted by the unit electric current in the unit of time; so that we may write equation (9)

$$\Sigma\,(vv'ee') = c^2 ii'ds\,ds'. \tag{11}$$

Hence the sums of the four values of (3), (5), and (6) become

$$\Sigma\,(ee'u^2) = -2\,c^2 ii'ds\,ds'\cos\epsilon, \tag{12}$$

$$\Sigma\left(ee'\left(\frac{\partial r}{\partial t}\right)^2\right) = 2\,c^2 ii'ds\,ds'\frac{dr}{ds}\frac{dr}{ds'}, \tag{13}$$

$$\Sigma\left(ee'r\frac{\partial^2 r}{\partial t^2}\right) = 2\,c^2 ii'ds\,ds'\,r\frac{d^2 r}{ds\,ds'}, \tag{14}$$

and we may write the two expressions (1) and (2) for the attraction between $ds$ and $ds'$

$$-\frac{1}{c^2}\,\Sigma\left[\frac{ee'}{r^2}\left(u^2-\tfrac{3}{2}\left(\frac{\partial r}{\partial t}\right)^2\right)\right], \tag{15}$$

and

$$-\frac{1}{c^2}\,\Sigma\left[\frac{ee'}{r^2}\left(r\frac{\partial^2 r}{\partial t^2}-\tfrac{1}{2}\left(\frac{\partial r}{\partial t}\right)^2\right)\right]. \tag{16}$$

850.] The ordinary expression, in the theory of statical electricity, for the repulsion of two electrical particles $e$ and $e'$ is $\dfrac{ee'}{r^2}$, and $$\Sigma\left(\frac{ee'}{r^2}\right) = \frac{(e+e_1)(e'+e'_1)}{r^2}, \tag{17}$$

which gives the electrostatic repulsion between the two elements if they are charged as wholes.

Hence, if we assume for the repulsion of the two particles either of the modified expressions

$$\frac{ee'}{r^2}\left[1+\frac{1}{c^2}\left(u^2-\tfrac{3}{2}\left(\frac{\partial r}{\partial t}\right)^2\right)\right], \tag{18}$$

or $$\frac{ee'}{r^2}\left[1+\frac{1}{c^2}\left(r\frac{\partial^2 r}{\partial t^2}-\tfrac{1}{2}\left(\frac{\partial r}{\partial t}\right)^2\right)\right],^* \tag{19}$$

we may deduce from them both the ordinary electrostatic forces, and the forces acting between currents as determined by Ampère.

851.] The first of these expressions, (18), was discovered by Gauss † in July 1835, and interpreted by him as a fundamental law of electrical action, that 'Two elements of electricity in a state of relative motion attract or repel one another, but not in the same way as if they are in a state of relative rest.' This discovery was not, so far as I know, published in the lifetime of Gauss, so that the second expression, which was discovered independently by W. Weber, and published in the first part of his celebrated *Elektrodynamische Maasbestimmungen* ‡, was the first result of the kind made known to the scientific world.

852.] The two expressions lead to precisely the same result when they are applied to the determination of the mechanical force between two electric currents, and this result is identical with that of Ampère. But when they are considered as expressions of the physical law of the action between two electrical particles, we are led to enquire whether they are consistent with other known facts of nature.

Both of these expressions involve the relative velocity of the particles. Now, in establishing by mathematical reasoning the well-known principle of the conservation of energy, it is generally assumed that the force acting between two particles is a function of the distance only, and it is commonly stated

* {For an account of other theories of this kind see *Report on Electrical Theories*, by J. J. Thomson, *B. A. Report*, 1885, pp. 97–155.}

† *Werke* (Göttingen edition, 1867), vol. v. p. 616.

‡ *Abh. Leibnizens Ges.*, Leipzig (1846), p. 316.

that if it is a function of anything else, such as the time, or the velocity of the particles, the proof would not hold.

Hence a law of electrical action, involving the velocity of the particles, has sometimes been supposed to be inconsistent with the principle of the conservation of energy.

853.] The formula of Gauss is inconsistent with this principle, and must therefore be abandoned, as it leads to the conclusion that energy might be indefinitely generated in a finite system by physical means. This objection does not apply to the formula of Weber, for he has shewn * that if we assume as the potential energy of a system consisting of two electric particles,

$$\psi = \frac{ee'}{r}\left[1 - \frac{1}{2c^2}\left(\frac{\partial r}{\partial t}\right)^2\right], \tag{20}$$

the repulsion between them, which is found by differentiating this quantity with respect to $r$, and changing the sign, is that given by the formula (19).

Hence the work done on a moving particle by the repulsion of a fixed particle is $\psi_0 - \psi_1$, where $\psi_0$ and $\psi_1$ are the values of $\psi$ at the beginning and at the end of its path. Now $\psi$ depends only on the distance, $r$, and on the velocity resolved in the direction of $r$. If, therefore, the particle describes any closed path, so that its position, velocity, and direction of motion are the same at the end as at the beginning, $\psi_1$ will be equal to $\psi_0$, and no work will be done on the whole during the cycle of operations.

Hence an indefinite amount of work cannot be generated by a particle moving in a periodic manner under the action of the force assumed by Weber.

854.] But Helmholtz, in his very powerful memoir on the 'Equations of Motion of Electricity in Conductors at Rest' †, while he shews that Weber's formula is not inconsistent with the principle of the conservation of energy, as regards only the work done during a complete cyclical operation, points out that it leads to the conclusion, that two electrified particles, which move according to Weber's law, may have at first finite velocities, and yet, while still at a finite distance from each other, they may acquire an infinite kinetic energy, and may perform an infinite amount of work.

* *Pogg. Ann.*, lxxiii. p. 229 (1848).
† *Crelle's Journal*, 72. pp. 57–129 (1870).

To this Weber * replies, that the initial relative velocity of the particles in Helmholtz's example, though finite, is greater than the velocity of light; and that the distance at which the kinetic energy becomes infinite, though finite, is smaller than any magnitude which we can perceive, so that it may be physically impossible to bring two molecules so near together. The example, therefore, cannot be tested by any experimental method.

Helmholtz † has therefore stated a case in which the distances are not too small, nor the velocities too great, for experimental verification. A fixed non-conducting spherical surface, of radius $a$, is uniformly charged with electricity to the surface-density $\sigma$. A particle, of mass $m$ and carrying a charge $e$ of electricity, moves within the sphere with velocity $v$. The electrodynamic potential calculated from the formula (20) is

$$4\pi a\sigma e\left(1-\frac{v^2}{6c^2}\right), \tag{21}$$

and is independent of the position of the particle within the sphere. Adding to this $V$, the remainder of the potential energy arising from the action of other forces, and $\frac{1}{2}mv^2$, the kinetic energy of the particle, we find as the equation of energy

$$\tfrac{1}{2}\left(m-\tfrac{4}{3}\frac{\pi a\sigma e}{c^2}\right)v^2+4\pi a\sigma e+V=\text{const.} \tag{22}$$

Since the second term of the coefficient of $v^2$ may be increased indefinitely by increasing $a$, the radius of the sphere, while the surface-density $\sigma$ remains constant, the coefficient of $v^2$ may be made negative. Acceleration of the motion of the particle would then correspond to diminution of its *vis viva*, and a body moving in a closed path and acted on by a force like friction, always opposite in direction to its motion, would continually increase in velocity, and that without limit. This impossible result is a necessary consequence of assuming any formula for the potential which introduces negative terms into the coefficient of $v^2$.

855.] But we have now to consider the application of Weber's theory to phenomena which can be realised. We have seen how it gives Ampère's expression for the force of attraction between

* *Elektr. Maasb. inbesondere über das Princip der Erhaltung der Energie.*

† *Berlin Monatsbericht*, April 1872, pp. 247–256 ; *Phil. Mag.*, Dec. 1872, *Supp.*, pp. 530–537.

two elements of electric currents. The potential of one of these elements on the other is found by taking the sum of the values of the potential $\psi$ for the four combinations of the positive and negative currents in the two elements. The result is, by equation (20), taking the sum of the four values of $\left.\frac{\partial r}{\partial t}\right|^2$,

$$-ii'ds\,ds'\frac{1}{r}\frac{dr}{ds}\frac{dr}{ds'}, \tag{23}$$

and the potential of one closed current on another is

$$-ii'\iint\frac{1}{r}\frac{dr}{ds}\frac{dr}{ds'}ds\,ds' = ii'M, \tag{24}$$

where $\qquad M = \iint\frac{\cos\epsilon}{r}ds\,ds'$, as in Arts. 423, 524.

In the case of closed currents, this expression agrees with that which we have already (Art. 524) obtained *.

*Weber's Theory of the Induction of Electric Currents.*

856.] After deducing from Ampère's formula for the action between the elements of currents, his own formula for the action between moving electric particles, Weber proceeded to apply his formula to the explanation of the production of electric currents by magneto-electric induction. In this he was eminently successful, and we shall indicate the method by which the laws of induced currents may be deduced from Weber's formula. But we must observe, that the circumstance that a law deduced from the phenomena discovered by Ampère is able also to account for the phenomena afterwards discovered by Faraday does not give so much additional weight to the evidence for the physical truth of the law as we might at first suppose.

For it has been shewn by Helmholtz and Thomson (see Art. 543), that if the phenomena of Ampère are true, and if the principle of the conservation of energy is admitted, then the phenomena of induction discovered by Faraday follow of necessity. Now Weber's law, with the various assumptions about the nature of electric currents which it involves, leads by mathematical transformations to the formula of Ampère.

* In the whole of this investigation Weber adopts the electrodynamic system of units. In this treatise we always use the electromagnetic system. The electromagnetic unit of current is to the electrodynamic unit in the ratio of $\sqrt{2}$ to 1. Art. 526.

Weber's law is also consistent with the principle of the conservation of energy in so far that a potential exists, and this is all that is required for the application of the principle by Helmholtz and Thomson. Hence we may assert, even before making any calculations on the subject, that Weber's law will explain the induction of electric currents. The fact, therefore, that it is found by calculation to explain the induction of currents, leaves the evidence for the physical truth of the law exactly where it was.

On the other hand, the formula of Gauss, though it explains the phenomena of the attraction of currents, is inconsistent with the principle of the conservation of energy, and therefore we cannot assert that it will explain all the phenomena of induction. In fact, it fails to do so, as we shall see in Art. 859.

857.] We must now consider the electromotive force tending to produce a current in the element $ds'$, due to the current in $ds$, when $ds$ is in motion, and when the current in it is variable.

According to Weber, the action on the material of the conductor of which $ds'$ is an element, is the *sum* of all the actions on the electricity which it carries. The electromotive force, on the other hand, on the electricity in $ds'$, is the *difference* of the electric forces acting on the positive and the negative electricity within it. Since all these forces act in the line joining the elements, the electromotive force on $ds'$ is also in this line, and in order to obtain the electromotive force in the direction of $ds'$ we must resolve the force in that direction. To apply Weber's formula, we must calculate the various terms which occur in it, on the supposition that the element $ds$ is in motion relatively to $ds'$, and that the currents in both elements vary with the time. The expressions thus found will contain terms involving $v^2$, $vv'$, $v'^2$, $v$, $v'$, and terms not involving $v$ or $v'$, all of which are multiplied by $ee'$. Examining, as we did before, the four values of each term, and considering first the mechanical force which arises from the sum of the four values, we find that the only term which we must take into account is that involving the product $vv'ee'$.

If we then consider the force tending to produce a current in the second element, arising from the difference of the action of the first element on the positive and the negative electricity of the second element, we find that the only term which we have to examine is that which involves $vee'$. We may write the four terms included in $\Sigma(vee')$, thus

$$e'(ve+v_1e_1) \quad \text{and} \quad e'_1(ve+v_1e_1).$$

Since $e'+e'_1=0$, the mechanical force arising from these terms is zero, but the electromotive force acting on the positive electricity $e'$ is $(ve+v_1e_1)$, and that acting on the negative electricity $e'_1$ is equal and opposite to this.

858.] Let us now suppose that the first element $ds$ is moving relatively to $ds'$ with velocity $V$ in a certain direction, and let us denote by $\widehat{Vds}$ and $\widehat{Vds'}$, the angles between the direction of $V$ and those of $ds$ and of $ds'$ respectively, then the square of the relative velocity, $u$, of two electric particles is

$$u^2 = v^2+v'^2+V^2-2vv'\cos\epsilon+2Vv\cos\widehat{Vds}-2Vv'\cos\widehat{Vds'}. \tag{25}$$

The term in $vv'$ is the same as in equation (3). That in $v$, on which the electromotive force depends, is

$$2Vv\cos\widehat{Vds}.$$

We have also for the value of the time-variation of $r$ in this case

$$\frac{\partial r}{\partial t} = v\frac{dr}{ds}+v'\frac{dr}{ds'}+\frac{dr}{dt}, \tag{26}$$

where $\frac{\partial r}{\partial t}$ refers to the motion of the electric particles, and $\frac{dr}{dt}$ to that of the material conductor. If we form the square of this quantity, the term involving $vv'$, on which the mechanical force depends, is the same as before, in equation (5), and that involving $v$, on which the electromotive force depends, is

$$2v\frac{dr}{ds}\frac{dr}{dt}.$$

Differentiating (26) with respect to $t$, we find

$$\begin{aligned}\frac{\partial^2 r}{\partial t^2} &= v^2\frac{d^2r}{ds^2}+2vv'\frac{d^2r}{ds\,ds'}+v'^2\frac{d^2r}{ds'^2}+\frac{dv}{dt}\frac{dr}{ds}+\frac{dv'}{dt}\frac{dr}{ds'} \\ &\quad +v\frac{dv}{ds}\frac{dr}{ds}+v'\frac{dv'}{ds}\frac{dr}{ds'}+2v\frac{d}{ds}\frac{dr}{d}+2v'\frac{d}{ds'}\frac{dr}{dt}+\frac{d^2r}{dt^2}.*\end{aligned} \tag{27}$$

We find that the term involving $vv'$ is the same as before in (6). The terms whose sign alters with that of $v$ are $\frac{dv}{dt}\frac{dr}{ds}$ and $2v\frac{d}{ds}\frac{dr}{dt}$.

* {In the 1st and 2nd editions the terms $2v\frac{d}{ds}\frac{dr}{dt}+2v'\frac{d}{ds'}\frac{dr}{dt}$ were omitted; since however $\frac{\partial^2}{\partial t} = \left\{v\frac{d}{ds}+v'\frac{d}{ds'}+\frac{d}{dt}\right\}^2$ it would seem that they ought to be included, they do not however affect the result when the circuits are closed.}

859.] If we now calculate by the formula of Gauss (equation (18)), the resultant electrical force in the direction of the second element $ds'$, arising from the action of the first element $ds$, we obtain

$$\frac{1}{r^2} ds\, ds'\, i\, V(2\cos \widehat{Vds} - 3\cos \widehat{Vr} \cos \widehat{rds}) \cos \widehat{rds'}. \qquad (28)$$

As in this expression there is no term involving the rate of variation of the current $i$, and since we know that the variation of the primary current produces an inductive action on the secondary circuit, we cannot accept the formula of Gauss as a true expression of the action between electric particles.

860.] If, however, we employ the formula of Weber, (19), we obtain

$$\frac{1}{r^2} ds\, ds' \left(r \frac{dr}{ds}\frac{di}{dt} + 2i \frac{d}{ds}\frac{dr}{dt} - i \frac{dr}{ds}\frac{dr}{dt}\right)\frac{dr}{ds'}, \qquad (29)$$

or

$$\frac{d}{dt}\left(\frac{i}{r}\frac{dr}{ds}\frac{dr}{ds'}\right) ds\, ds' + \frac{i}{r}\left(\frac{d^2r}{ds\,dt}\frac{dr}{ds'} - \frac{d^2r}{ds'dt}\frac{dr}{ds}\right) ds\, ds'. \quad (30)$$

If we integrate this expression with respect to $s$ and $s'$, we obtain for the electromotive force on the second circuit

$$\frac{d}{dt} i \iint \frac{1}{r}\frac{dr}{ds}\frac{dr}{ds'} ds ds' + i \iint \frac{1}{r}\left(\frac{d^2r}{ds\,dt}\frac{dr}{ds'} - \frac{d^2r}{ds'dt}\frac{dr}{ds}\right) ds ds'. \quad (31)$$

Now, when the first circuit is closed,

$$\int \frac{d^2r}{ds\,ds'} ds = 0.$$

Hence $$\int \frac{1}{r}\frac{dr}{ds}\frac{dr}{ds'} ds = \int \left(\frac{1}{r}\frac{dr}{ds}\frac{dr}{ds'} + \frac{d^2r}{ds ds'}\right) ds = -\int \frac{\cos\epsilon}{r} ds. \quad (32)$$

But $$\iint \frac{\cos\epsilon}{r} ds\, ds' = M, \text{ by Arts. 423, 524.} \qquad (33)$$

Since the second term in equation (31) vanishes if both circuits are closed, we may write the electromotive force on the second circuit

$$-\frac{d}{dt}(iM), \qquad (34)$$

which agrees with what we have already established by experiment; Art. 539.

*On Weber's Formula, considered as resulting from an Action transmitted from one Electric Particle to the other with a Constant Velocity.*

861.] In a very interesting letter of Gauss to W. Weber * he

* March 19, 1845, *Werke*, bd. v. 629.

refers to the electrodynamic speculations with which he had been occupied long before, and which he would have published if he could then have established that which he considered the real keystone of electrodynamics, namely, the deduction of the force acting between electric particles in motion from the consideration of an action between them, not instantaneous, but propagated in time, in a similar manner to that of light. He had not succeeded in making this deduction when he gave up his electrodynamic researches, and he had a subjective conviction that it would be necessary in the first place to form a consistent representation of the manner in which the propagation takes place.

Three eminent mathematicians have endeavoured to supply this keystone of electrodynamics.

862.] In a memoir presented to the Royal Society of Göttingen in 1858, but afterwards withdrawn, and only published in Poggendorff's *Annalen*, bd. cxxxi. pp. 237-263, in 1867, after the death of the author, Bernhard Riemann deduces the phenomena of the induction of electric currents from a modified form of Poisson's equation

$$\frac{d^2V}{dx^2}+\frac{d^2V}{dy^2}+\frac{d^2V}{dz^2}+4\pi\rho=\frac{1}{a^2}\frac{d^2V}{dt^2},$$

where $V$ is the electrostatic potential, and $a$ a velocity.

This equation is of the same form as those which express the propagation of waves and other disturbances in elastic media. The author, however, seems to avoid making explicit mention of any medium through which the propagation takes place.

The mathematical investigation given by Riemann has been examined by Clausius *, who does not admit the soundness of the mathematical processes, and shews that the hypothesis that potential is propagated like light does not lead either to the formula of Weber, or to the known laws of electrodynamics.

863.] Clausius has also examined a far more elaborate investigation by C. Neumann on the 'Principles of Electrodynamics' †. Neumann, however, has pointed out ‡ that his theory of the transmission of potential from one electric particle to another is quite different from that proposed by Gauss, adopted by Riemann, and criticized by Clausius, in which the propagation is like that of light. There is, on the contrary, the greatest

* Pogg., bd. cxxxv. p. 612.

† Tübingen, 1868.

‡ *Mathematische Annalen*, i. 317.

possible difference between the transmission of potential, according to Neumann, and the propagation of light.

A luminous body sends forth light in all directions, the intensity of which depends on the luminous body alone, and not on the presence of the body which is enlightened by it.

An electric particle, on the other hand, sends forth a potential, the value of which, $\frac{ee'}{r}$, depends not only on $e$, the emitting particle, but on $e'$, the receiving particle, and on the distance $r$ between the particles *at the instant of emission.*

In the case of light the intensity diminishes as the light is propagated further from the luminous body; the emitted potential flows to the body on which it acts without the slightest alteration of its original value.

The light received by the illuminated body is in general only a fraction of that which falls on it; the potential as received by the attracted body is identical with, or equal to, the potential which arrives at it.

Besides this, the velocity of transmission of the potential is not, like that of light, constant relative to the æther or to space, but rather like that of a projectile, constant relative to the velocity of the emitting particle at the instant of emission.

It appears, therefore, that in order to understand the theory of Neumann, we must form a very different representation of the process of the transmission of potential from that to which we have been accustomed in considering the propagation of light. Whether it can ever be accepted as the 'construirbar Vorstellung' of the process of transmission, which appeared necessary to Gauss, I cannot say, but I have not myself been able to construct a consistent mental representation of Neumann's theory.

864.] Professor Betti *, of Pisa, has treated the subject in a different way. He supposes the closed circuits in which the electric currents flow to consist of elements each of which is polarized periodically, that is, at equidistant intervals of time. These polarized elements act on one another as if they were little magnets whose axes are in the direction of the tangent to the circuits. The periodic time of this polarization is the same in all electric circuits. Betti supposes the action of one polarized

* *Nuovo Cimento,* xxvii (1868).

element on another at a distance to take place, not instantaneously, but after a time proportional to the distance between the elements. In this way he obtains expressions for the action of one electric circuit on another, which coincide with those which are known to be true. Clausius, however, has, in this case also, criticized some parts of the mathematical calculations into which we shall not here enter.

865.] There appears to be, in the minds of these eminent men, some prejudice, or *à priori* objection, against the hypothesis of a medium in which the phenomena of radiation of light and heat and the electric actions at a distance take place. It is true that at one time those who speculated as to the causes of physical phenomena were in the habit of accounting for each kind of action at a distance by means of a special æthereal fluid, whose function and property it was to produce these actions. They filled all space three and four times over with æthers of different kinds, the properties of which were invented merely to 'save appearances,' so that more rational enquirers were willing rather to accept not only Newton's definite law of attraction at a distance, but even the dogma of Cotes *, that action at a distance is one of the primary properties of matter, and that no explanation can be more intelligible than this fact. Hence the undulatory theory of light has met with much opposition, directed not against its failure to explain the phenomena, but against its assumption of the existence of a medium in which light is propagated.

866.] We have seen that the mathematical expressions for electrodynamic action led, in the mind of Gauss, to the conviction that a theory of the propagation of electric action in time would be found to be the very keystone of electrodynamics. Now we are unable to conceive of propagation in time, except either as the flight of a material substance through space, or as the propagation of a condition of motion or stress in a medium already existing in space. In the theory of Neumann, the mathematical conception called Potential, which we are unable to conceive as a material substance, is supposed to be projected from one particle to another, in a manner which is quite independent of a medium, and which, as Neumann has himself pointed out, is extremely different from that of the propagation

* Preface to Newton's *Principia*, 2nd edition.

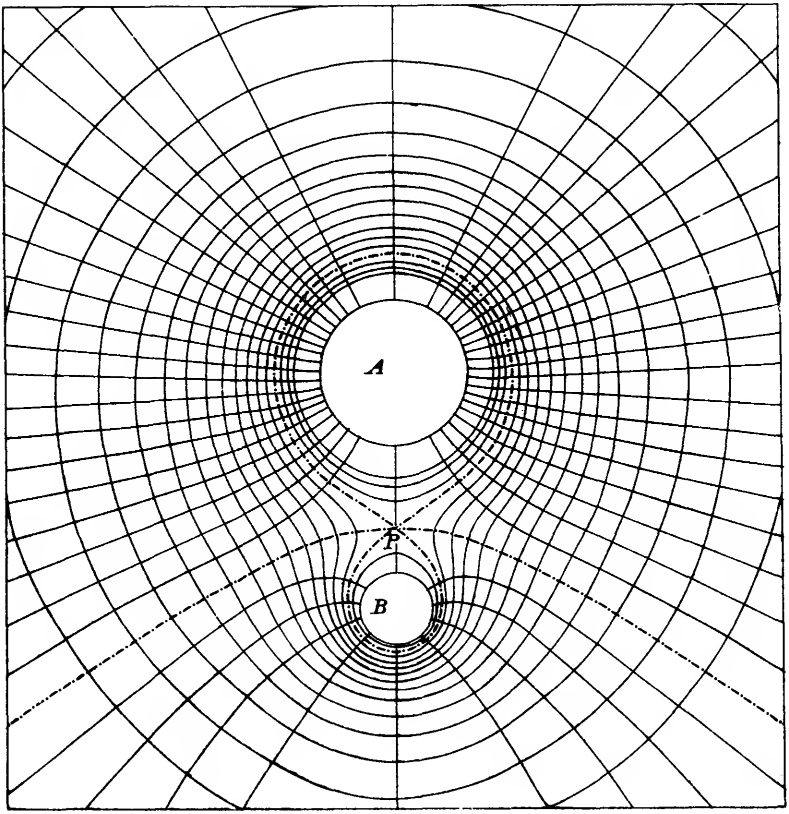
of light. In the theories of Riemann and Betti it would appear that the action is supposed to be propagated in a manner somewhat more similar to that of light.

But in all of these theories the question naturally occurs:— If something is transmitted from one particle to another at a distance, what is its condition after it has left the one particle and before it has reached the other? If this something is the potential energy of the two particles, as in Neumann's theory, how are we to conceive this energy as existing in a point of space, coinciding neither with the one particle nor with the other? In fact, whenever energy is transmitted from one body to another in time, there must be a medium or substance in which the energy exists after it leaves one body and before it reaches the other, for energy, as Torricelli * remarked, 'is a quintessence of so subtile a nature that it cannot be contained in any vessel except the inmost substance of material things.' Hence all these theories lead to the conception of a medium in which the propagation takes place, and if we admit this medium as an hypothesis, I think it ought to occupy a prominent place in our investigations, and that we ought to endeavour to construct a mental representation of all the details of its action, and this has been my constant aim in this treatise.

* *Lezioni Accademiche* (Firenze, 1715), p. 25.

FIG. I.

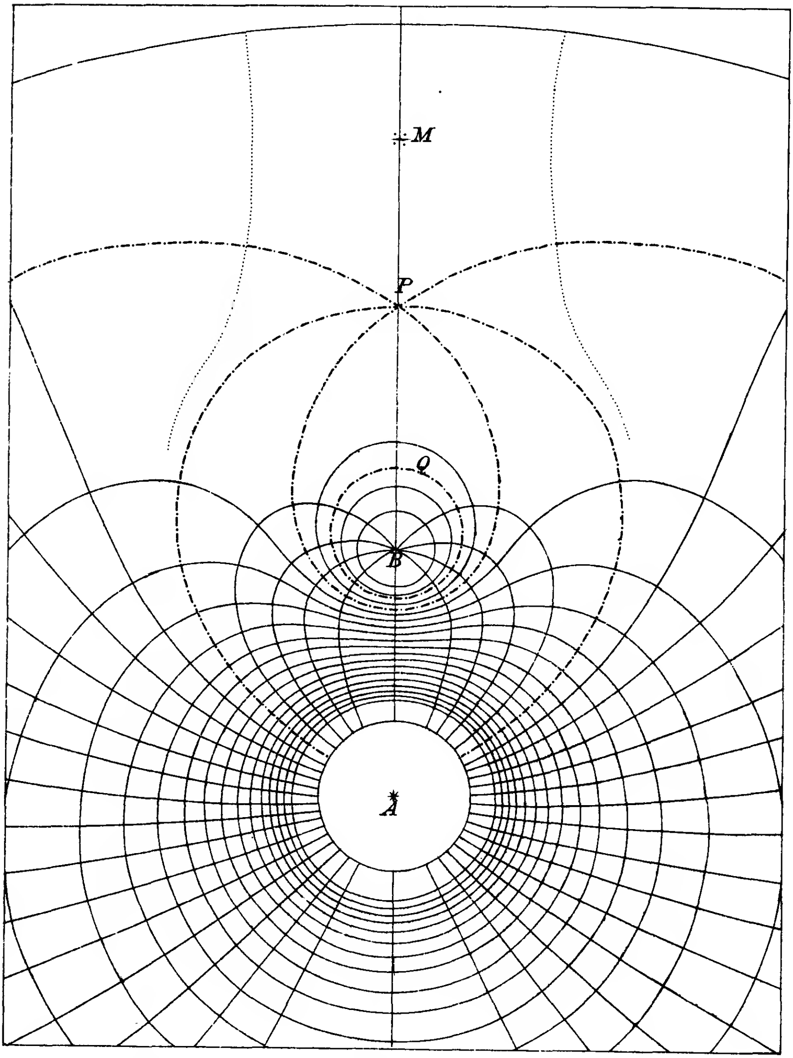
Art. 118.




*Lines of Force and Equipotential Surfaces.*

*A* = 20 *B* = 5. *P*, *Point of Equilibrium.* $AP = \frac{2}{3}AB$.

FIG. II
Art. 119




*Lines of Force and Equipotential Surfaces.*

$A = 20$ $B = -5$ P. Point of Equilibrium. $AP = 2AB$

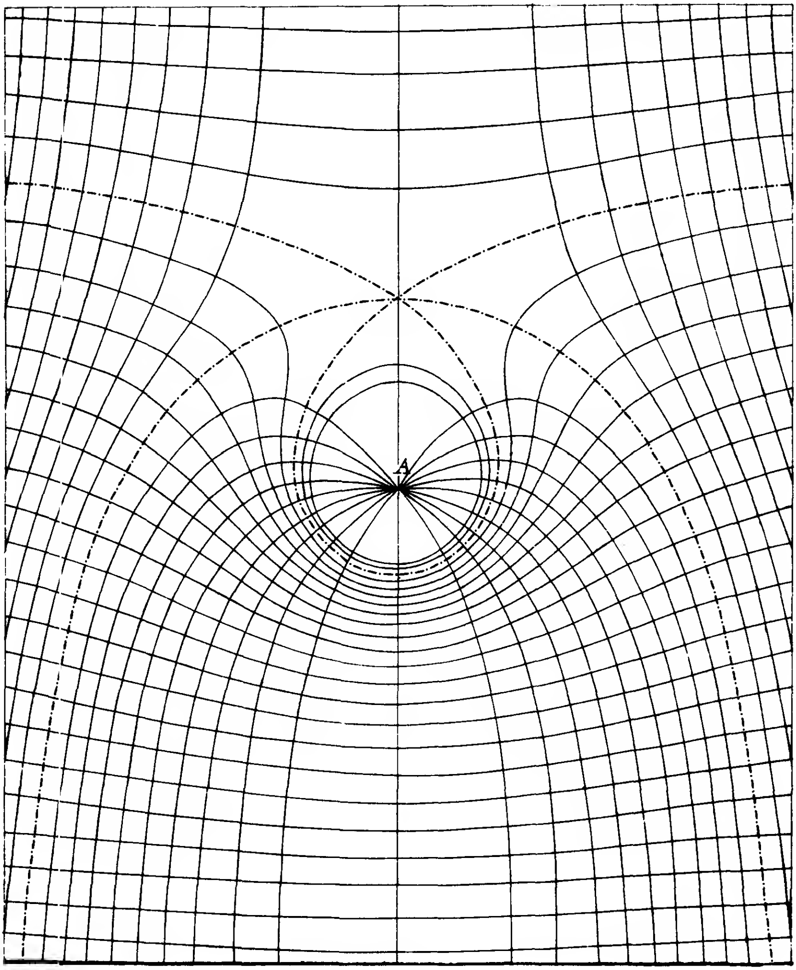
Q, Spherical surface of Zero potential.

M. Point of Maximum Force along the axis.

The dotted line is the Line of Force $\Psi = 0.1$. thus ..........

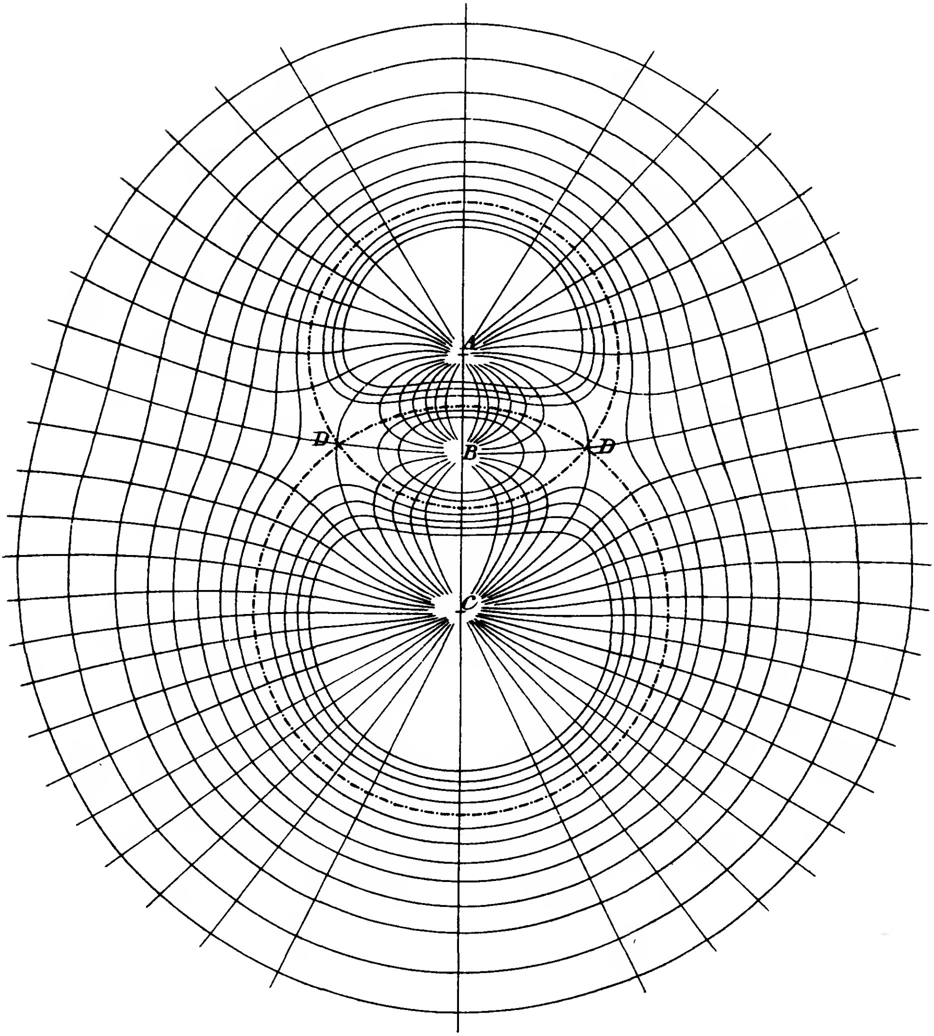
FIG. III.

Art. 120.




*Lines of Force and Equipotential surfaces*

FIG. IV.
Art. 121.

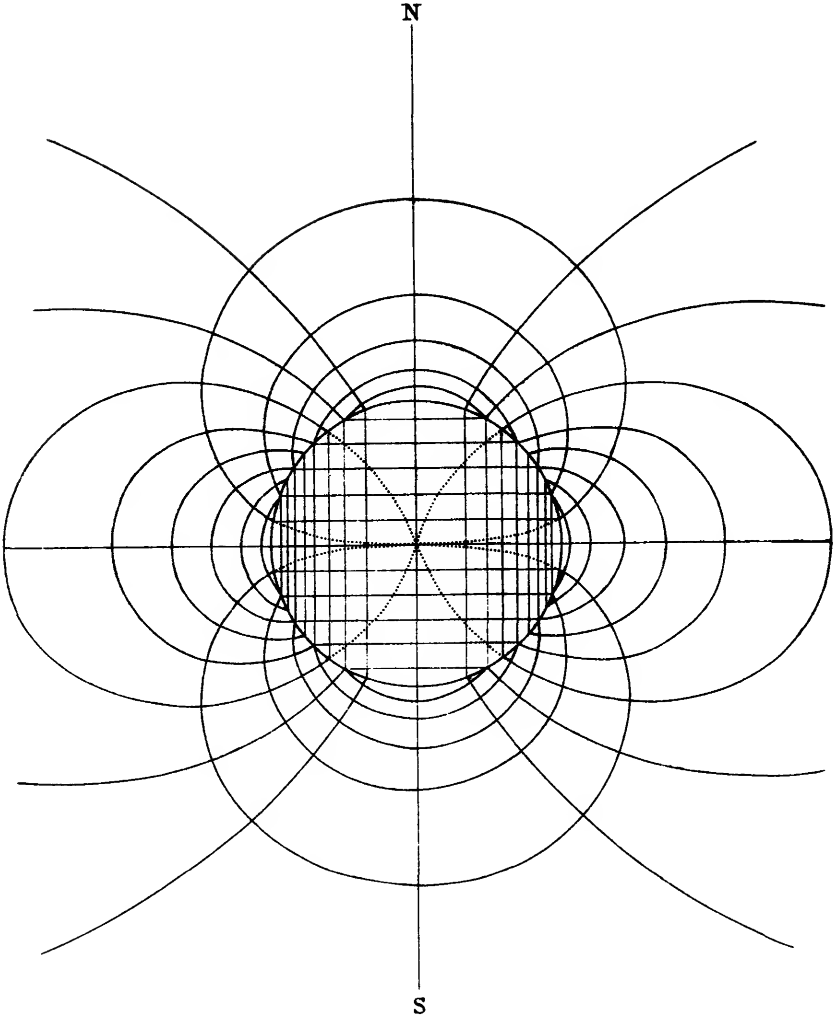



*Lines of Force and Equipotential Surfaces.*

A-15. B-12. C-20.

FIG. V.

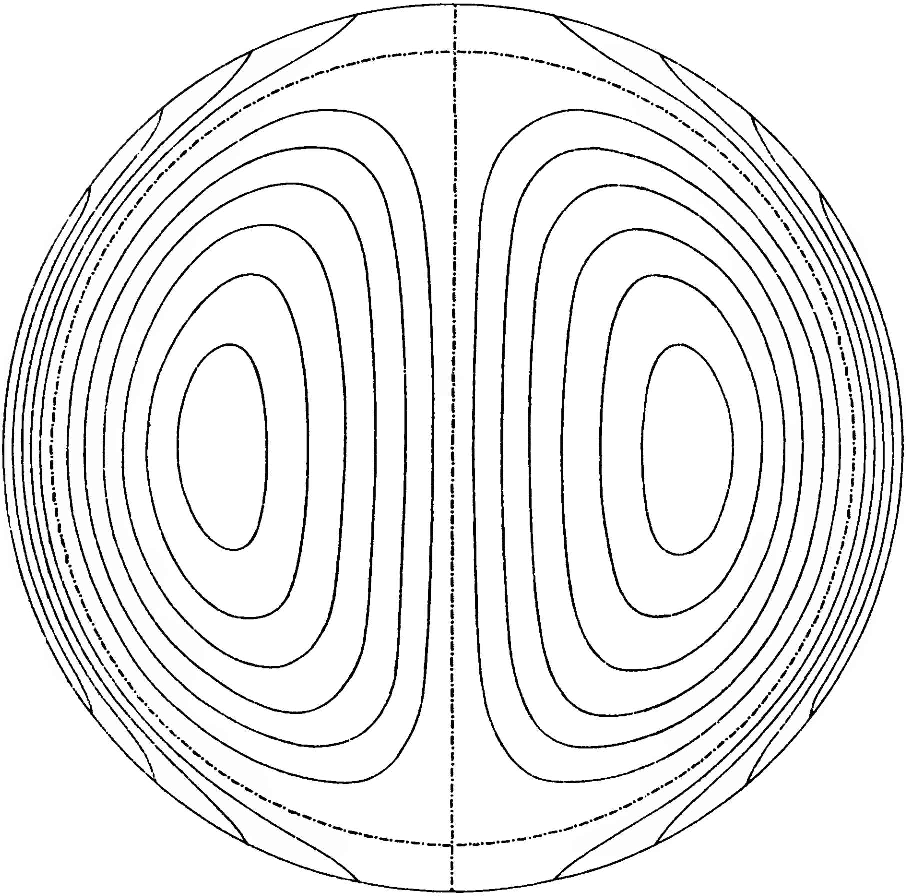
Art 143.




*Lines of Force and Equipotential Surfaces in a diametral section of a spherical Surface in which the superficial density is a harmonic of the first degree.*

FIG. VI.

Art. 143.

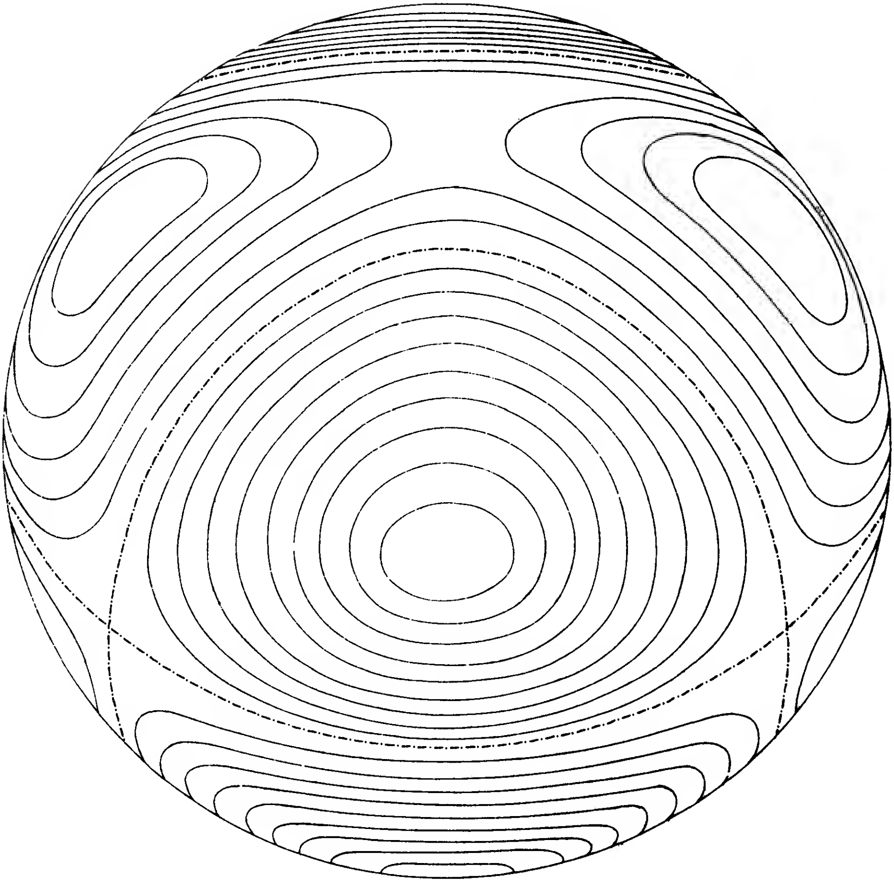


*Spherical Harmonic of the third order.*

$n = 3 \qquad \sigma = 1.$

FIG. VII.

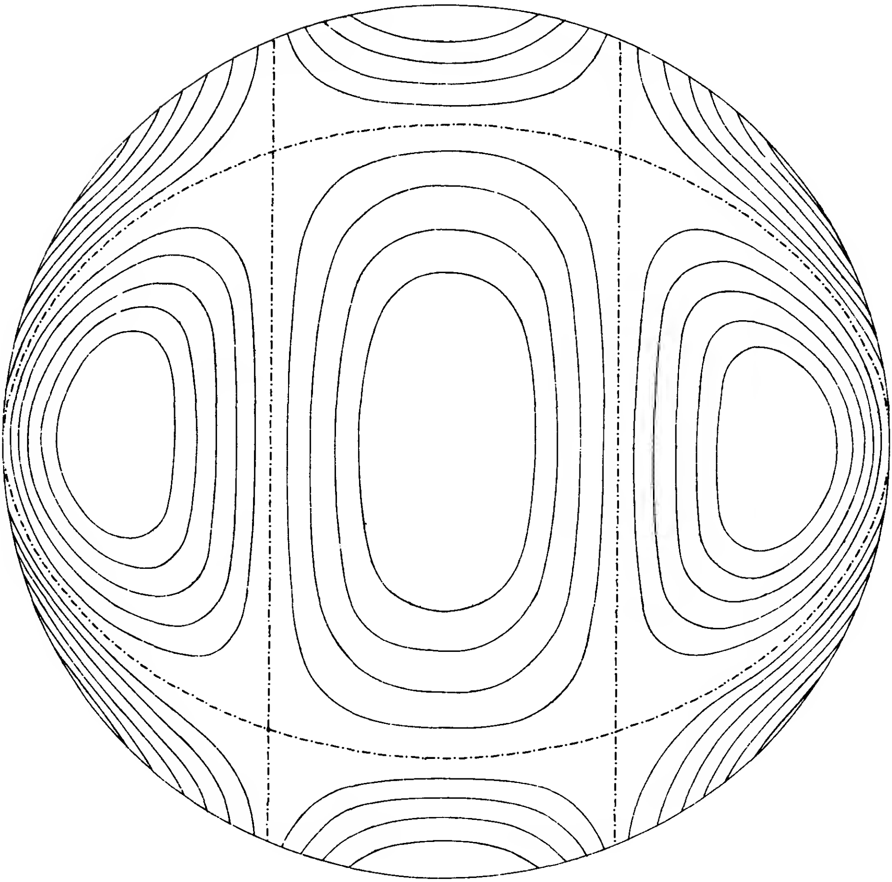
Art. 143.



*Spherical Harmonic of the third order.*

$n = 3$

FIG. VIII.
Art. 143.

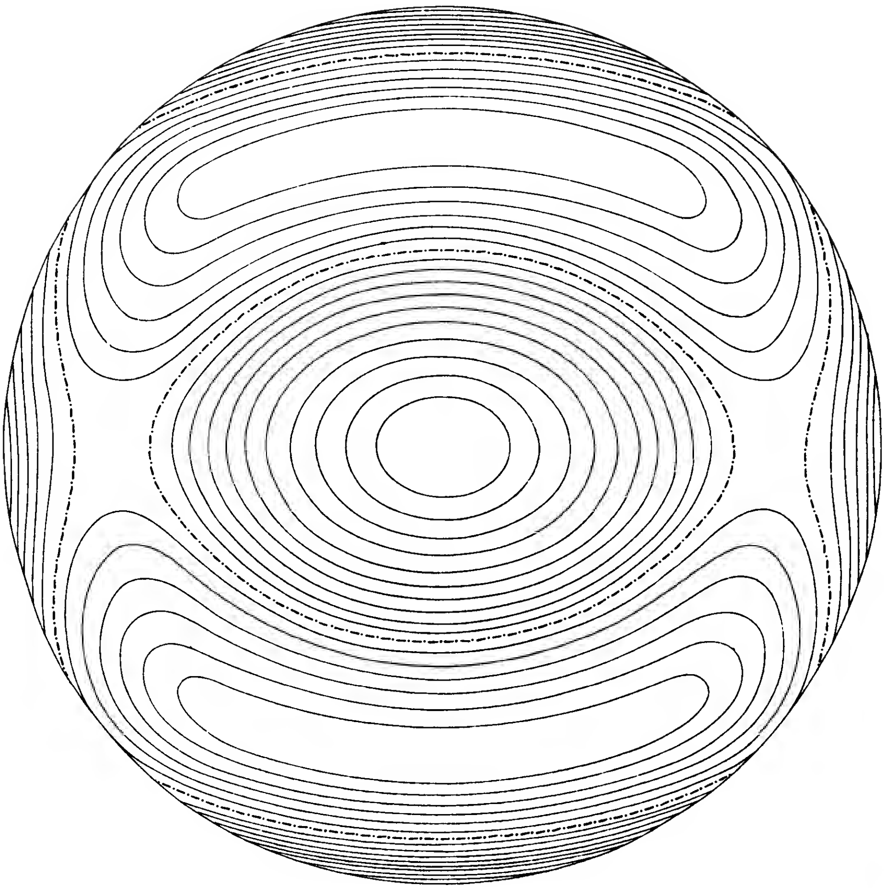


*Spherical Harmonic of the fourth order*

$n = 4 \quad \sigma = 2$

FIG. IX.

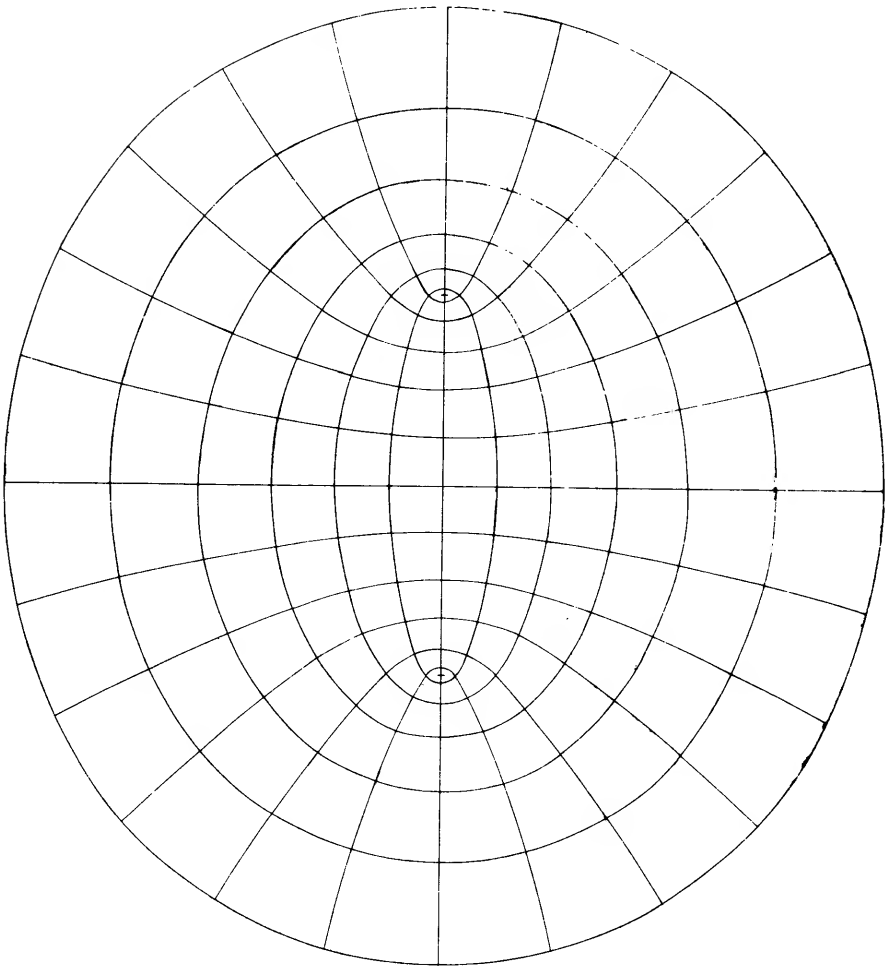
Art. 143



*Spherical Harmonic of the fourth order.*

FIG. X.

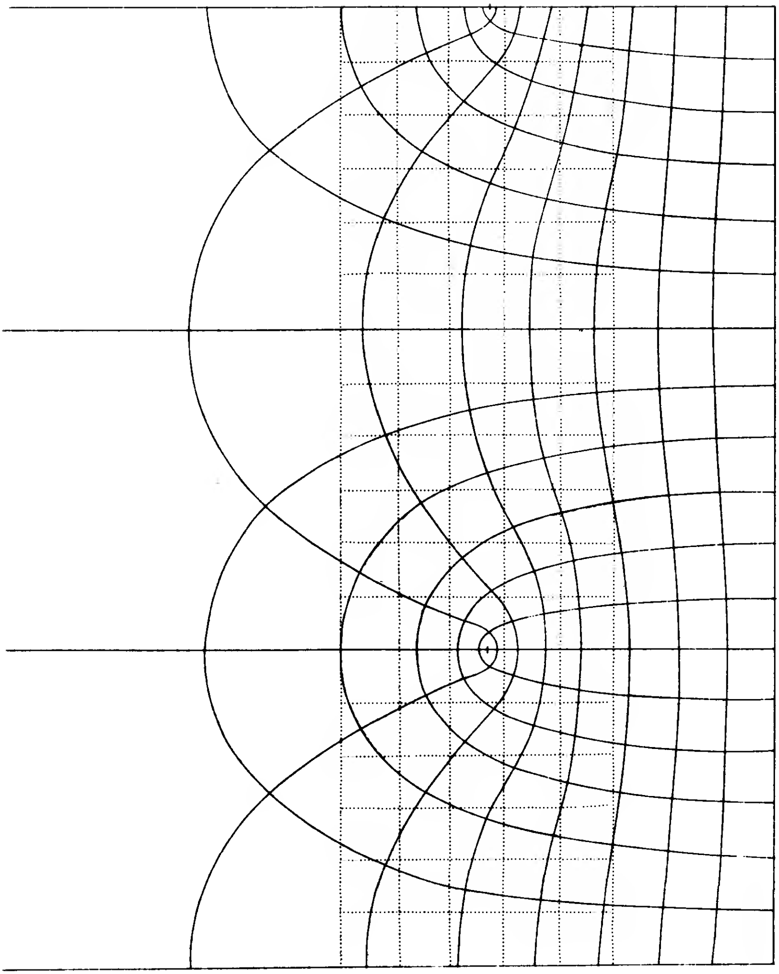
Art. 192.



*Confocal Ellipses and Hyperbolas*

FIG. XI.

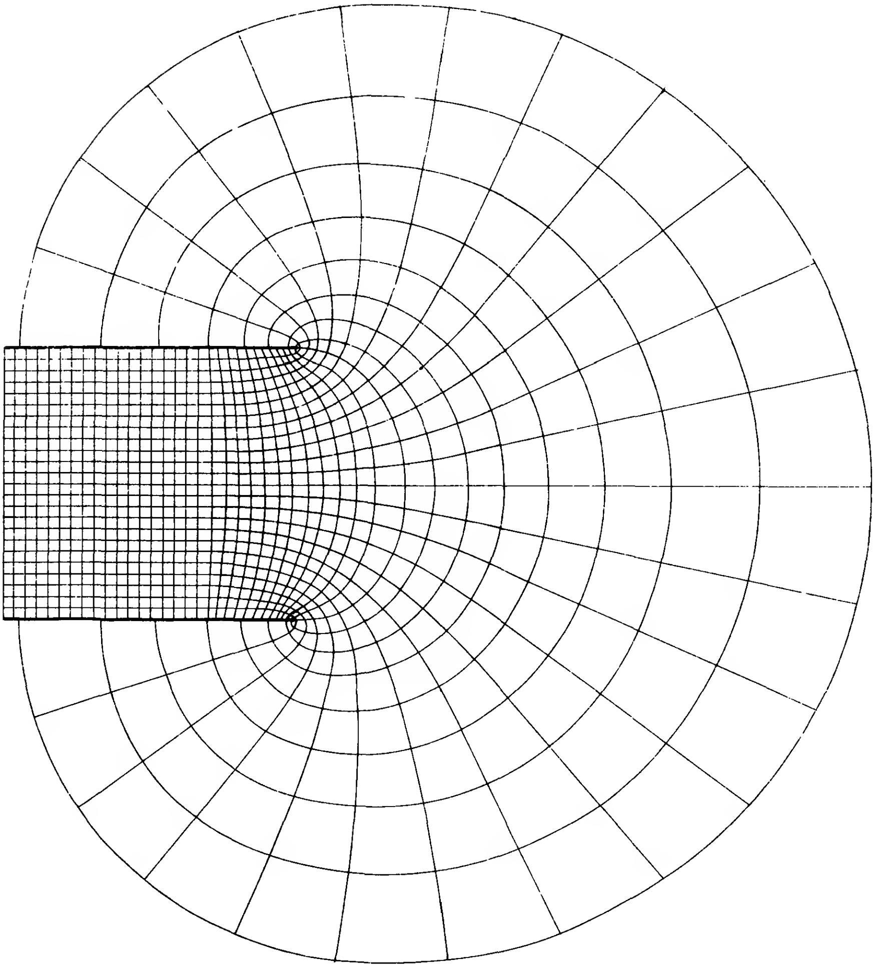
Art. 193.



*Lines of Force near the edge of a Plate.*

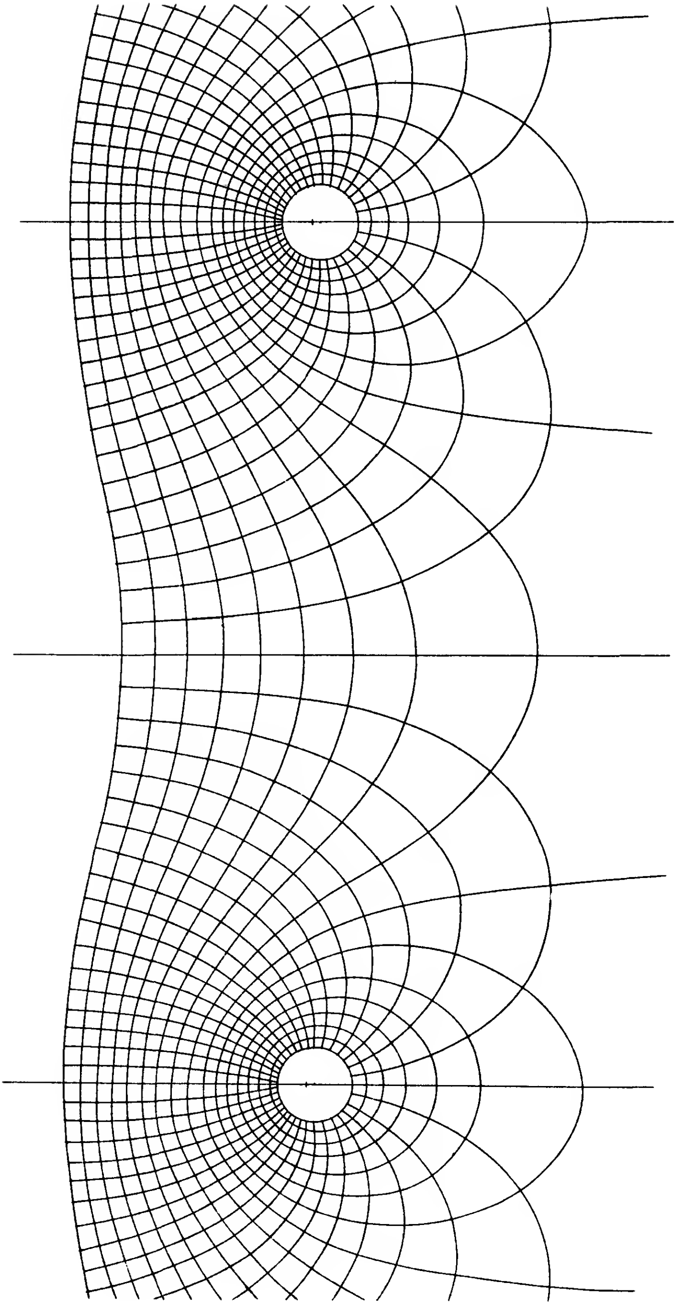
FIG. XII.

Art. 202.



*Lines of Force between two Plates*

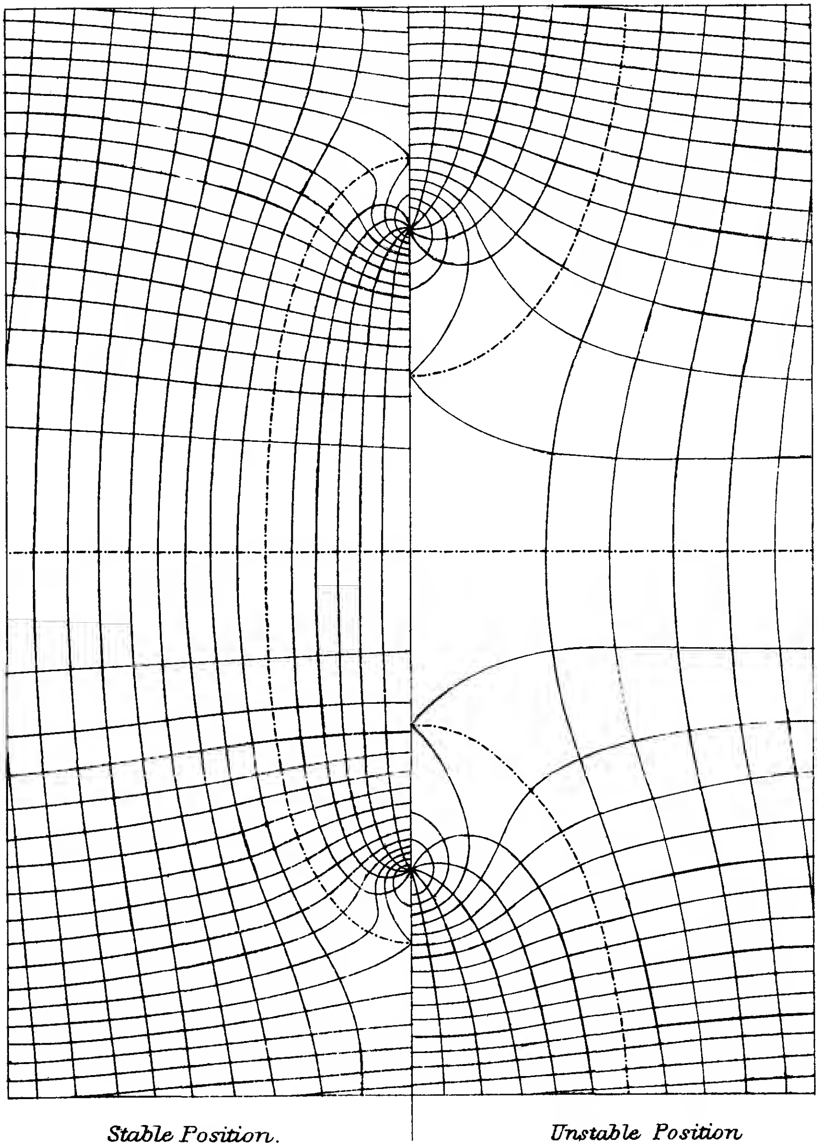
FIG. XIII.
Art. 203



*Lines of Force near a Grating.*

**FIG. XIII A.**

Art. 225.



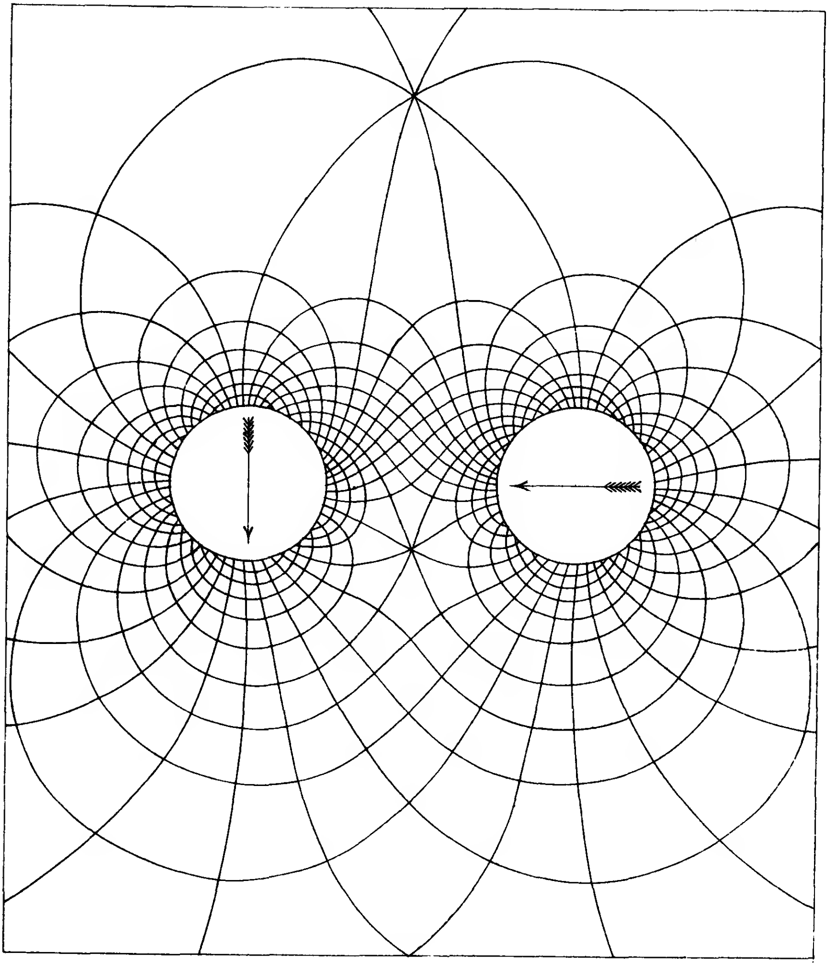
*Stable Position.*

*Unstable Position*

*Circular Current in uniform field of Force.*

FIG. XIV.

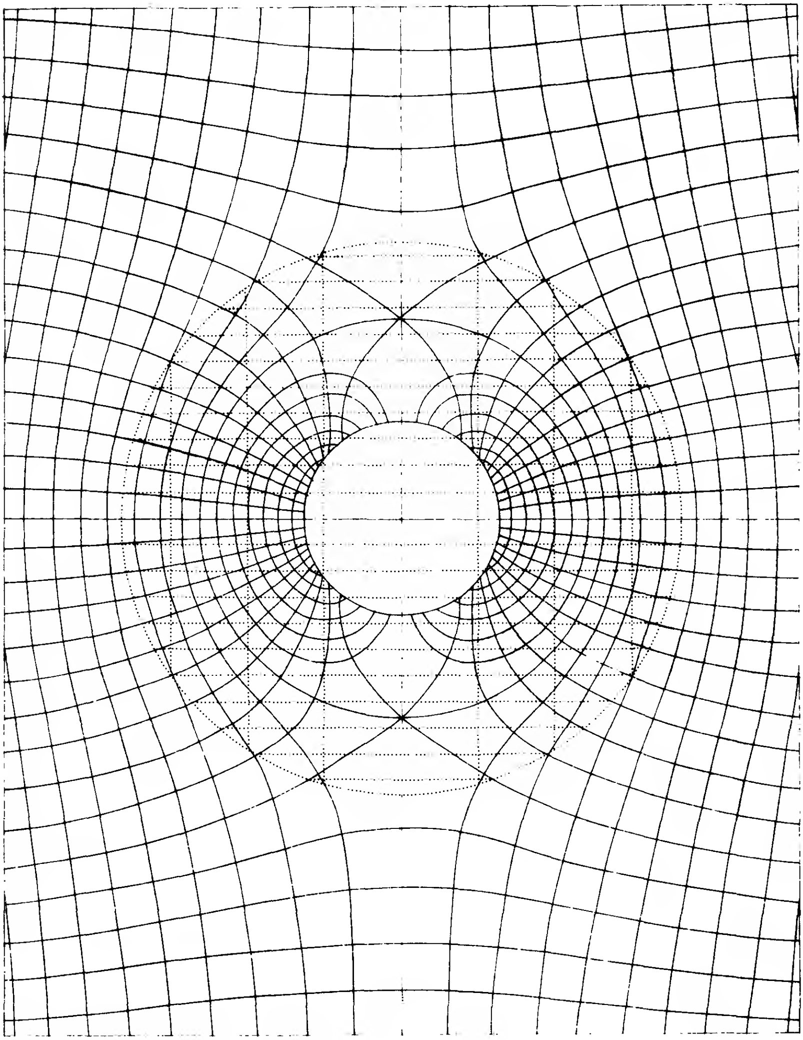
Art. 388.



*Two Cylinders magnetized transversely.*

FIG. XV.

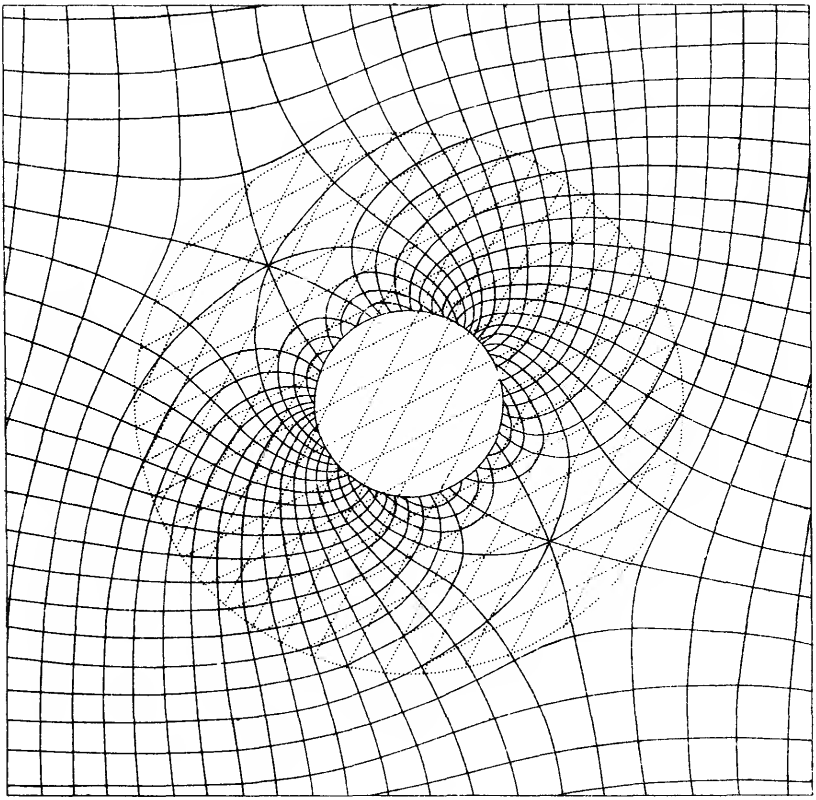
Art. 434



*Cylinder magnetized transversely, placed North and South in a uniform magnetic field.*

FIG. XVI.

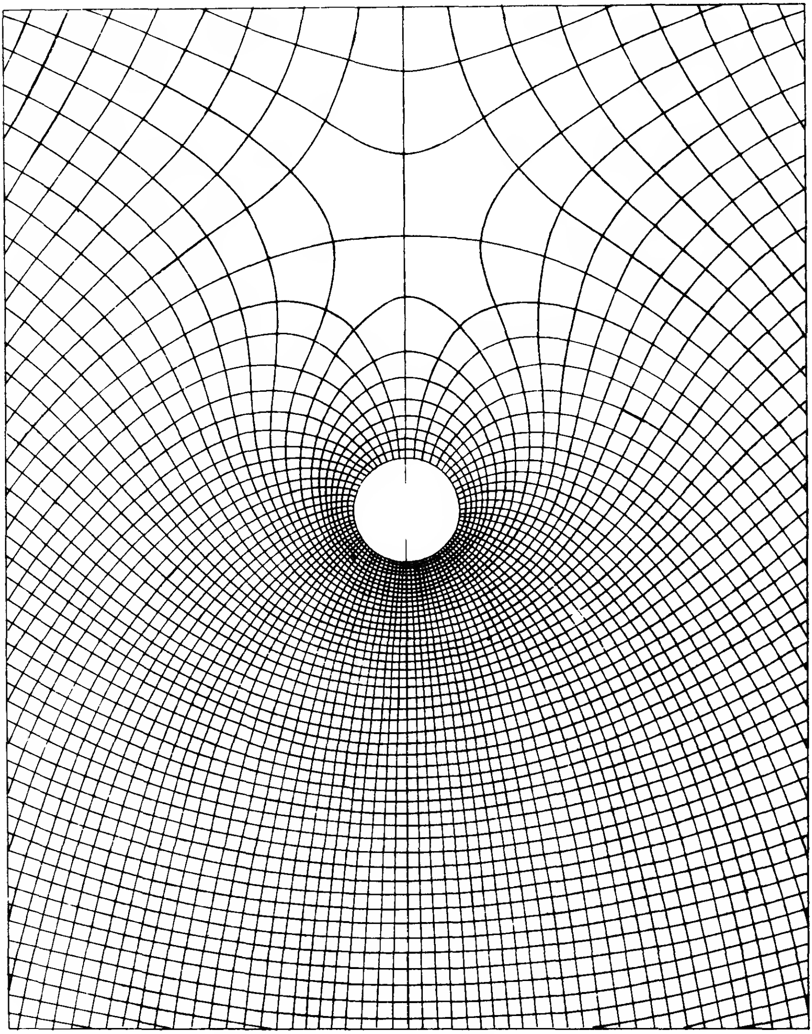
Art. 436.



*Cylinder magnetized transversely, placed East and West in a uniform magnetic field.*

FIG. XVII.

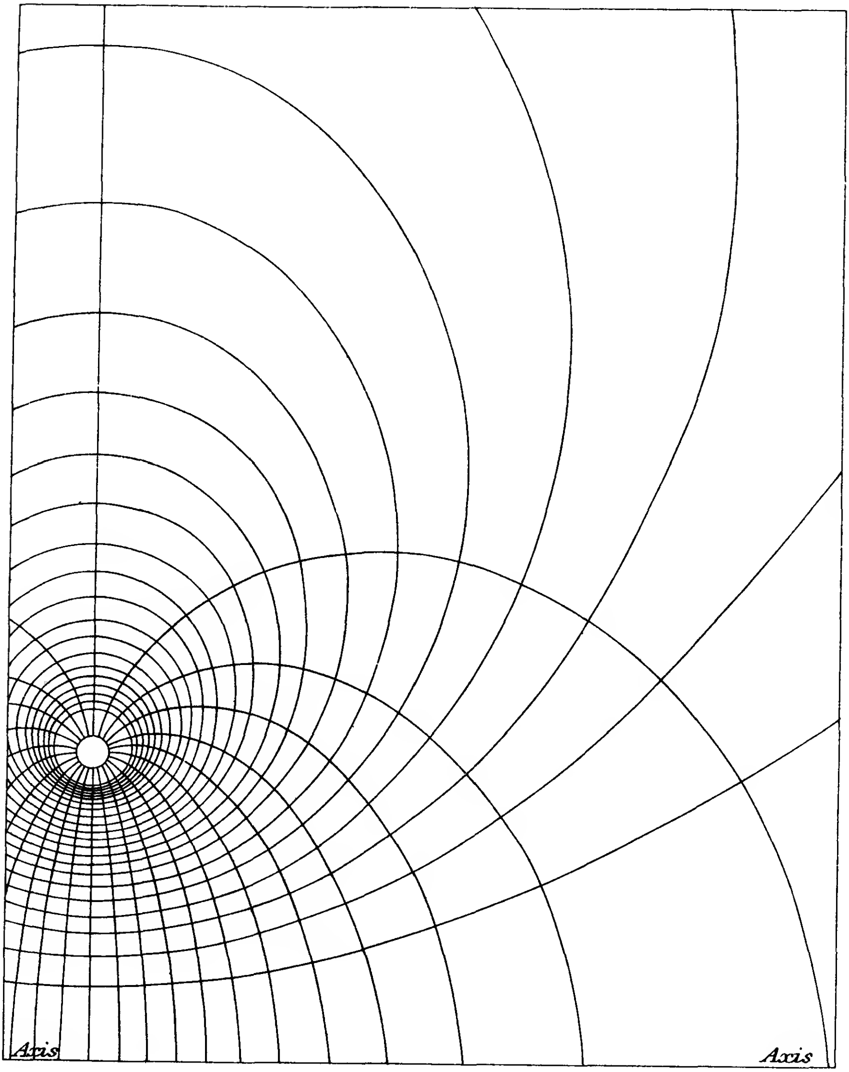
Art. 496



*Uniform magnetic field disturbed by an Electric Current in a straight conductor.*

FIG. XVIII.

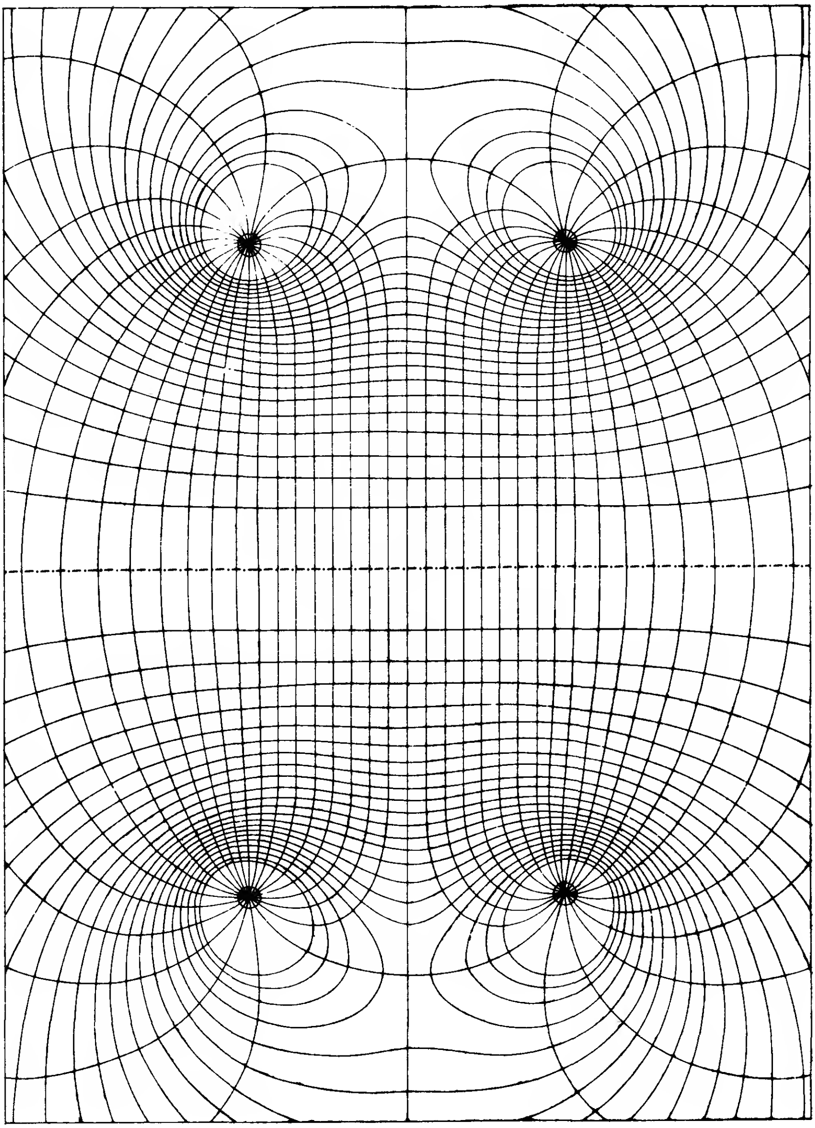
Art. 487, 702




*Circular Current*

FIG. XIX

Art. 713



*Two Circular Currents*

# INDEX.

*[This comprehensive Index covers both volumes of the work. The References are to Articles: Volume One contains Articles 1 through 370; Volume Two contains Articles 371 through 866.]*



* Sir Charles Wheatstone, in his paper on 'New Instruments and Processes,' *Phil. Trans.*, 1843, brought this arrangement into public notice, with due acknowledgment of the original inventor, Mr. S. Hunter Christie, who had described it in his paper on 'Induced Currents,' *Phil Trans.*, 1833, under the name of a Differential Arrangement. See the remarks of Mr. Latimer Clark in the *Society of Telegraph Engineers*, May 8, 1872.

END OF VOL. II.

Maxwell

A TREATISE ON ELECTRICITY & MAGNETISM

Vol.2

DOVER

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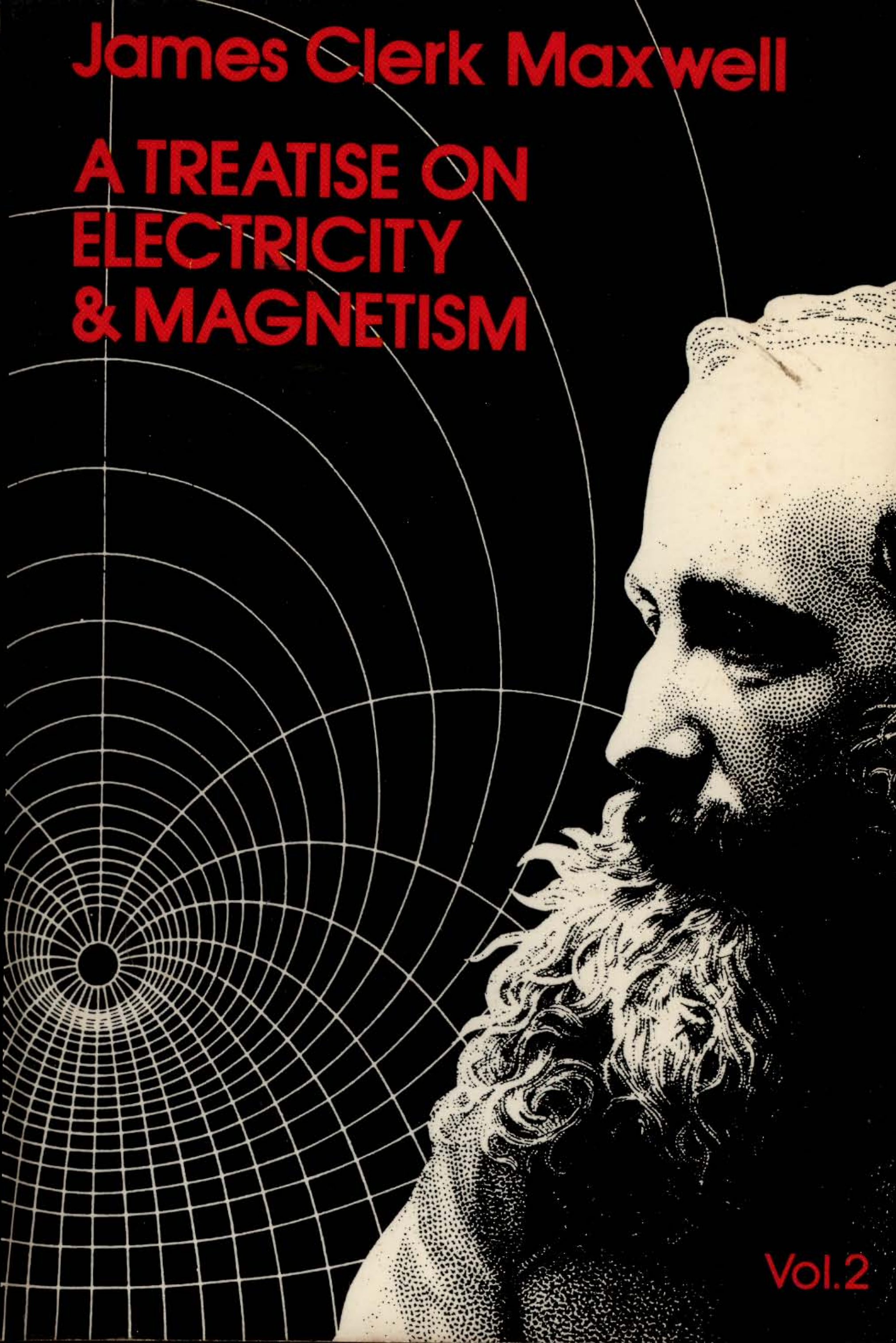
# James Clerk Maxwell

# A TREATISE ON ELECTRICITY & MAGNETISM



Vol.2